Liouville Sectors

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Def: A symplectic mfld (M, ω), $\omega \in \Omega^2(M)$, non-degenerate and closed.
If ω is exact $(M, \omega = d\lambda)$ is an exact symplectic mfld,
λ - Liouville form.
A symplectic vector field $X \in \mathfrak{X}(M)$, $\mathcal{L}_X \omega = 0$.
A Hamiltonian v.f. X_H for $H \in C^\infty(M)$, $dH = \omega(\cdot, X_H)$
A Liouville v.f. $Z \in \mathfrak{X}(M)$, $d(i_Z \omega) = \omega$.
Consider a hypersurface $\Sigma \subseteq (M, \omega)$
Characteristic line bundle $\mathcal{L}(\Sigma) := \ker \omega|_{T\Sigma}$
Foliation of $\mathcal{L}(\Sigma)$ is called characteristic foliation.
If $\partial M \neq \phi$ it gives an orientation on $\mathcal{L}(\partial M)$
$\mathcal{C} := \{v \in \mathcal{L}(\partial M) \mid \omega(v, \eta) > 0$ for η inward pointing$\}$

Def: A contact manifold $(N, \xi = \ker \alpha)$, with $\alpha \wedge (d\alpha)^n \neq 0$.
A contact v.f. $Y \in \mathfrak{X}(N)$ s.t. $\mathcal{L}_Y \alpha = g\alpha$ $\quad g \in C^\infty(N)$
A convex hypersurface $\Sigma \subseteq (N, \xi)$ s.t. there exists a contact v.f transverse to Σ.
$CV(N, \xi)$ - the set of contact. v.f.

Def: A contact type hypersurface $\Sigma \subseteq (M, \omega)$, s.t. in a nbhd of Σ there exists a Liouville v.f transverse to Σ.
Given a contact mfld $(N, \xi = \ker \alpha)$ then we define its symplectization $(M, \omega) = (N \times \mathbb{R}, d(e^r \alpha))$.

Def: A Liouville manifold (M, ω) is a symplectic mfld satisfying the following:
1. There exists a contact type hypersurface $\partial_\infty M \subseteq (M, \omega)$ with the associated Liouville v.f. Z transverse to it.

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Def: A <u>Liouville manifold</u> (M, ω) is a symplectic mfld satisfying the following:

1. There exists a contact type hypersurface $\partial_\infty M \subseteq (M, \omega)$ with the associated Liouville v.f. Z transverse to it.
2. The flow ϕ^t of Z is well defined for all $(p, t) \in \partial_\infty M \times \mathbb{R}_+$ and the map $\Phi : \partial_\infty M \times \mathbb{R}_+ \to M$, $\Phi(p, t) := \phi^t(p)$ is a diffeomorphism onto its image and $\Phi^* \omega = d(e^t \alpha)$.
3. The set $A := M \setminus \Phi(\partial_\infty M \times \mathbb{R}_+)$ is pre-compact.

A <u>Liouville manifold with boundary</u> is a Liouville mfld with $\partial M \neq \emptyset$ and $\partial(\partial_\infty M) \neq \emptyset$ and $Z \in T(\partial M \setminus A)$.

Lemma: Let $(N, \xi = \ker\alpha)$ be a contact mfld and let $(M, \omega) = (N \times \mathbb{R}, d(e^r \alpha))$ be its symplectization. Denote $Z = \partial_r$. Then the following sets are in bijection:

1. $HF(M, Z) := \{ H \in C^\infty(M) \mid dH(Z) = H \}$;
2. $SV(M, Z) := \{ X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0 \text{ and } [X, Z] = 0 \}$;
3. $CV(N, \xi)$.

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Proof: The map $HF(M,Z) \to CV(N,\xi)$

$H \mapsto Y$, where $X_H|_N = Y + gZ$

is well defined

$Y \in \mathfrak{X}(N)$, $g \in C^\infty(N)$

$TM|_N = TN \oplus \langle Z \rangle$ $\qquad \lambda := i_Z \omega$

For $H \in HF(M,Z)$

$$\lambda(X_H) = \omega(Z, X_H) = dH(Z) = H$$

$$\mathcal{L}_{X_H} \lambda = i_{X_H} d\lambda + d i_{X_H} \lambda = i_{X_H} \omega + dH = 0$$

$$\mathcal{L}_Y \alpha = \mathcal{L}_{X_H} \lambda - \mathcal{L}_{gZ} \lambda = 0 - g i_Z d\lambda - d(g \underbrace{i_Z \lambda}_{0}) = -g\lambda|_N = -g\alpha.$$

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