Lecture 11. Relative, rubber stable maps

- Expanding the target

- Predeformability

- An example of Double Ramification cycles

§1. Relative stable maps

Let X = nonsingular projective variety / $\mathbb{C}$.

$D \subset X$: nonsingular effective divisor

For $\beta \in H_2(X, \mathbb{Z})$, $(\beta . D) = m$, let

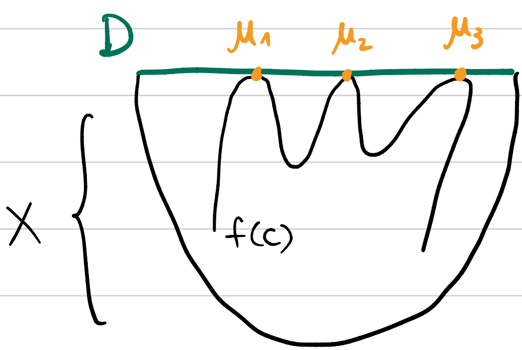
$$\mu = (\mu_1, \cdots \mu_n), \quad \mu_i \geq 1$$

be the partition of m. We want to consider stable maps with the incident condition along the divisor D

$$\mathcal{M}(X/D, \mu) = \left\{ f : (C, p_1, \cdots p_n) \longrightarrow X \;\middle|\; \begin{array}{l} f(C) \cap D = \{p_1 \cdots p_n\} \text{ and} \\ f^*D = \sum \mu_i p_i \end{array} \right\}$$

$$\cap$$

$$\overline{\mathcal{M}}_{g,n}(X, \beta)$$



Our incidence condition is **not** a closed condition.
In the limit, components of C can fall into D.

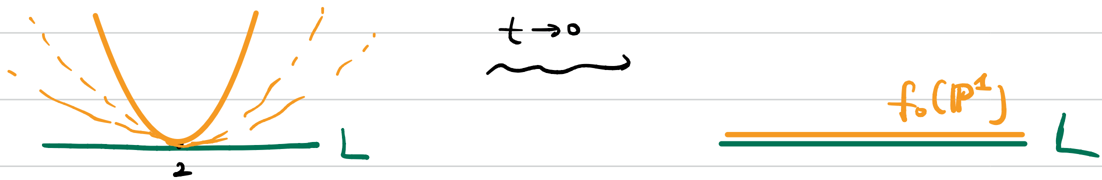
Example Let $L \subset \mathbb{P}^2$: line inside $\mathbb{P}^2$. Consider a one parameter family of quadrics inside $\mathbb{P}^2$

$$f_t : (\mathbb{P}^1, 0) \longrightarrow (\mathbb{P}^2, L)$$

s.t. $t \neq 0 \Rightarrow f_t(\mathbb{P}^1)$ is tangent to L

$t = 0 \Rightarrow f_0(\mathbb{P}^1) = L$

On an affine chart : $f_t(z) = tz^2$



Q: How to compactify $\mathcal{M}(X/D, \mu)$?

One can take the closure inside $\overline{\mathcal{M}}_{g,n}(X, \beta)$.

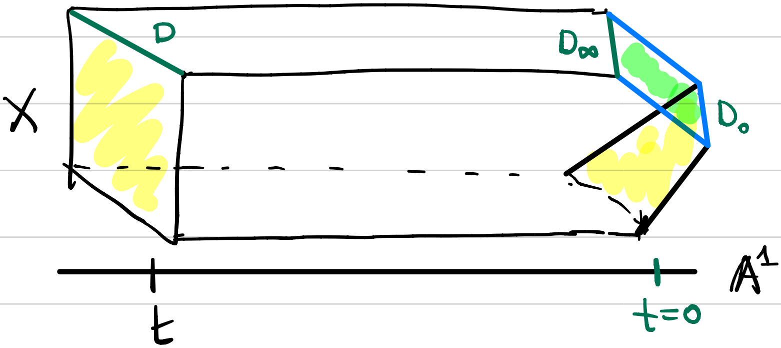
$\Rightarrow$ In most cases, this is wrong thing to do (no virtual fund. class, not appropriate for our purpose ...)

Idea : (Jun Li) Expand the target X along D !

Degeneration to the normal cone

Consider $D \times 0 \subset X \times \mathbb{A}^1$.

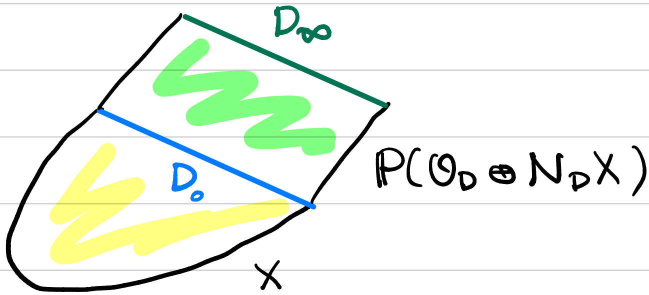
$$\begin{array}{c} Bl_{D\times 0}(X \times \mathbb{A}^1) = X[1] \\ \downarrow \rho \\ \mathbb{A}^1 \end{array}$$



We consider a flat family of targets $X[1]$.

$$X[1]_t = \begin{cases} X & \text{if } t \neq 0 \\ \mathbb{P}_D(\mathcal{O}_D \oplus N_D X) \cup_D X & \text{if } t = 0. \end{cases}$$

$X[1]_0$



Let's go back to our example & see why expanding X helps us.

Example We consider the family of maps $\{f_t\}$ as

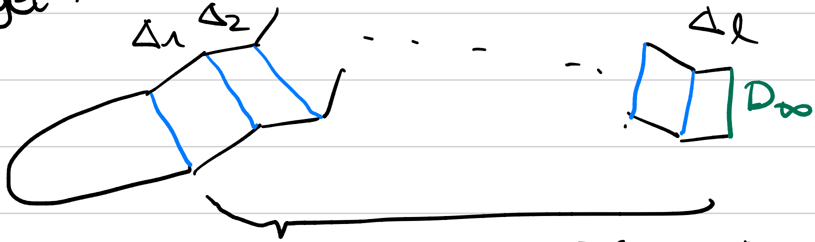
$$F : \mathbb{P}^1 \times (\mathbb{A}^1 - \{0\}) \longrightarrow \mathbb{P}^2 \times \mathbb{A}^1 \subset Bl_{L\times 0}(\mathbb{P}^2 \times \mathbb{A}^1)$$
$$z,\ t \longmapsto (f_t(z), t)$$

On an affine chart:

$$\mathbb{A}^1 \times (\mathbb{A}^1 - \{0\}) \longrightarrow \mathbb{A}^2 \times \mathbb{A}^1 \subset Bl_{L\times 0}(\mathbb{A}^2 \times \mathbb{A}^1)$$
$$(z, t) \longmapsto (z, tz^2, t)$$

If we take the closure inside $Bl_{L\times 0}(\mathbb{P}^2 \times \mathbb{A}^1)$, the limit f_0 meets D_∞ property!

In general we allow further degeneration of the target:



$:= X_\ell$

ℓ copies of $\mathbb{P}_D(\mathcal{O}_D \oplus N_D X)$.

$\rightsquigarrow$ $\exists$ moduli space of expansions of X along D.

□ Predeformability condition

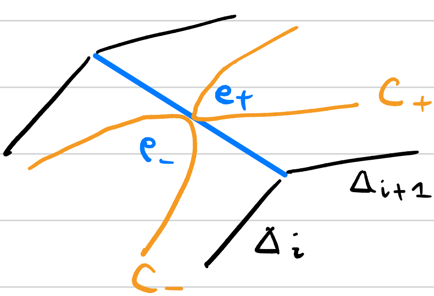
$$X_\ell := X \cup_D \underbrace{\mathbb{P}(\mathcal{O}_D \oplus N_D X) \cup_D \cdots \cup_D \mathbb{P}(\mathcal{O}_D \oplus N_D X)}_{\ell \text{ copies}}$$

$$\mathrm{Sing}(X_\ell) = \bigcup_{i=1}^{\ell} D_i \quad : \text{singular locus of } X_\ell$$

<u>Def</u> A morphism $f : C \longrightarrow X_\ell$ satisfies the predeformability condition if

(i) $f^{-1}(D_i) \subset C^{\text{nodes}}$

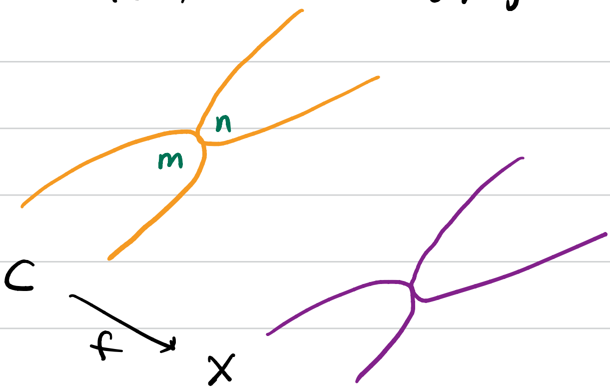
(ii) let $q \in f^{-1} D_i$ be an intersection of two irred components C_- & C_+ s.t $f(C_-) \subset \Delta_i$, $f(C_+) \subset \Delta_{i+1}$ and e_-, e_+ are contact orders of $f|_{C_-}$ & $f|_{C_+}$. Then $e_- = e_+$



<u>Example</u> The second condition happens when we try to smooth out the relative stable map. Let's look at the local picture when $\dim X = 1$

$$f: C \longrightarrow X \qquad x \mapsto \alpha u^n, \; y \mapsto \beta v^m, \; \alpha, \beta \in \mathbb{C}^\times$$

$$C \cong \operatorname{Spec} \mathbb{C}[u,v]/(uv) \qquad X \supseteq \operatorname{Spec} \mathbb{C}[x,y]/(xy) \qquad n, m > 0$$



Suppose we want to lift f to a family over arbitrary base, for simplicity we consider the base

$$S_d = \operatorname{Spec} R_d, \qquad R_d = \mathbb{C}[\varepsilon]/(\varepsilon^{d+1})$$

($(d+1)$ th neighborhood of $0 \in \mathbb{A}^1$)

$\mathrm{Spec}\, R_d[u,v]/(uv - a_d)$

$\cong$

$$\begin{tikzcd} C_d \arrow[r, "f_d"] \arrow[d, "\pi"] & X_d = \mathrm{Spec}\, R_d[x,y]/(xy - b_d) \arrow[dl] \\ S_d & \end{tikzcd}$$

$a_d, b_d \in (\varepsilon)$

s.t. $f_d|_{S_{d-1}} = f_{d-1}$ & $f_0 = f$

Exercise : If $n \neq m$, this is not possible to find systematic choice of $\{f_d\}$.

<u>Thm</u> (Jun Li) $\exists$ moduli space $\overline{\mathcal{M}}(X|D,\mu)$ of stable maps to X relative to D with the incidence condition μ. Moreover $\overline{\mathcal{M}}(X/D,\mu)$ is a proper DM stack with a virtual fundamental class

$\mathbb{C}$-points : $f : C \longrightarrow X_e$

- $f^*D_\infty = \sum_i \mu_i p_i$
- predeformability condition
- $|\mathrm{Aut}(f)| < \infty$

We have a map $\overline{\mathcal{M}}_g(X/D,\mu) \longrightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)$.

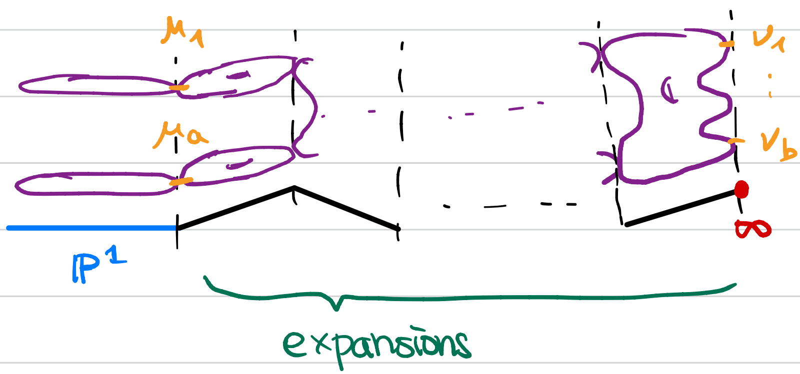
⚠ The predeformability condition is only locally closed condition inside all maps to X_e.

⇒ This creates substantial difficulties to study the geometry of $\overline{M}_g(X/D, \mu)$ & its virtual fundamental class.

⇝ logarithmic geometry

- Relative vs rubber stable maps.

$\overline{M}_g(\mathbb{P}^1/0.\infty)^{\sim}_{\mu\nu}$ is a ($\mathbb{C}^*$-fixed) component of $\overline{M}_g(\mathbb{P}^1/\infty, \mu)$.



§2. Double Ramification cycle

<u>Recall</u> $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$. $a_i \neq 0$ and $\sum a_i = 0$.

$\mu = (\mu_1, \dots)$: positive parts of A

$\nu = (\nu_1, \dots)$: negative parts of A.

Abel-Jacobi picture :

$$\mathcal{O}_C \mapsto 0 \qquad \begin{tikzcd} \mathrm{Jac}(\mathcal{C}_{g,n}/\mathcal{M}_{g,n}) \arrow[d] \\ \mathcal{M}_{g,n} \arrow[u, bend left=60] \arrow[u, bend right=60, "AJ_A"'] \end{tikzcd} \qquad \leftarrow \mathcal{O}_C\Big(\sum_{i=1}^{n} a_i p_i\Big)$$

$$DR^{\circ}_{g,A} = [AJ_A^{*}(0)] \in \underline{RH}^{2g}(\mathcal{M}_{g,n})$$

↳ Porteous formula + GRR

Q How to extend this construction to $\overline{\mathcal{M}}_{g,n}$

$\rightsquigarrow$ Stable maps to the rubber $\mathbb{P}^1/0,\infty$!

We have

$$\varepsilon : \overline{\mathcal{M}}_g(\mathbb{P}^1/0.\infty)^{\sim}_{\mu.\nu} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

$$\rotatebox{90}{$\in$}$$

$$[f:(C,\vec{p}) \to \mathbb{P}^1 \cup \cdots \cup \mathbb{P}^1] \longmapsto [(C,\vec{p})^{st}]$$

$Z = \mathrm{Im}(\varepsilon)$ ← the image has many irreducible components with different dimensions.

<u>Def</u> $DR_{g,A} := \varepsilon_* [\overline{\mathcal{M}}_g(\mathbb{P}^1/0.\infty)^{\sim}_{\mu\nu}]^{vir} \in H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$

Q1) Is $DR_{g,A} \in RH^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$? If so, how can we compute $DR_{g,A}$? (Eliashberg's question)

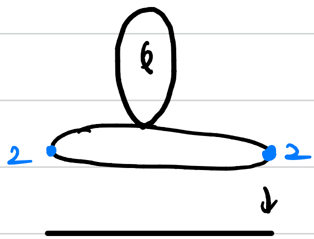
Q2) In which sense $DR_{g,A}$ extends the Abel-Jacobi construction ?

Example $g = 1$ $\quad A = (2, -2)$.

Let's see some possible configuration of $[f] \in \overline{\mathcal{M}}_1(\mathbb{P}^1/0,\infty)^\sim$

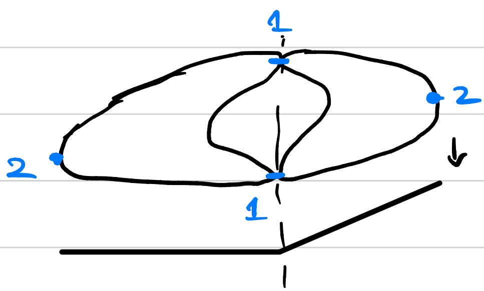
(i)

(ii)

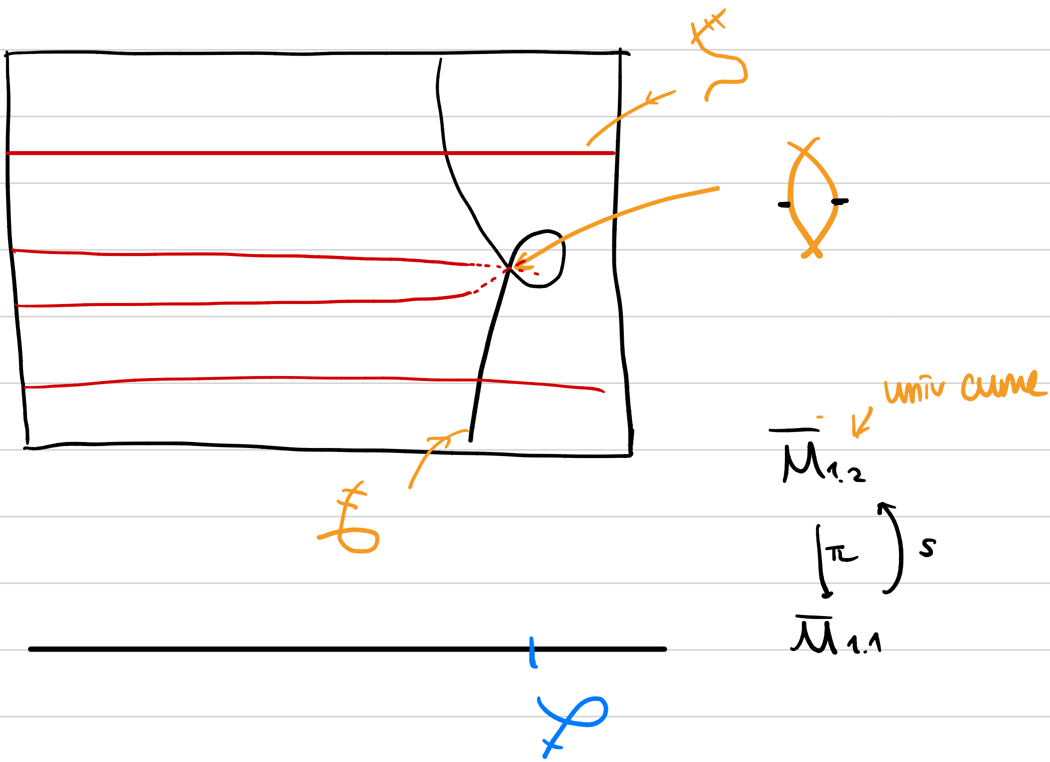


← 2:1 Galois cover

(iii)



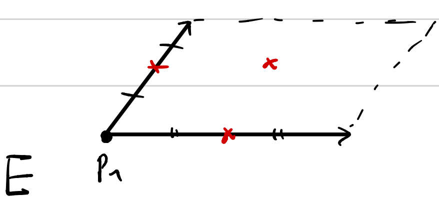
Let's consider the Abel-Jacobi side:



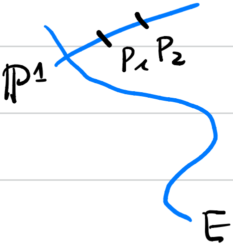
(A) on the locus $M_{1,2}$: for given $(E, p_1) \in M_{1,1}$.
$\mathrm{Pic}^0(E) \cong E$.

$$\mathcal{O}_E(2p_1 - 2p_2) \sim \mathcal{O}_E \iff p_2 - p_1 \text{ is 2 torsion point}$$

of $\mathrm{Pic}^0(E)$. & $p_1 \neq p_2$

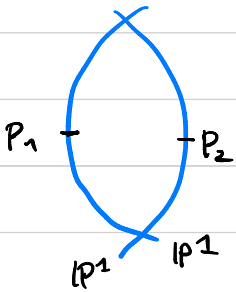


(B) On the locus $\operatorname{Im} s$:



$\mathcal{O}_{\mathbb{P}^1}(2p_1-2p_2) \sim \mathcal{O}_{\mathbb{P}^1}$ always holds

(C) On the locus  : NOT clear from naive AJ picture!



$\leftarrow$ $\mathcal{O}(2p_1-2p_2)$ has multidegree $(2,-2)$