A RECENT DEVELOPMENT IN ABEL-JACOBI THEORY

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Abstract. Abel-Jacobi theory studies when a divisor on a smooth curve is linearly equivalent to zero. When the curve degenerates over a family, the locus of the Abel-Jacobi question is related to the double ramification cycle. In this talk, we vary curves and line bundles simultaneously. The closure of the Abel-Jacobi section defines a locus on the universal Picard stack. We will see the closed formula of this locus in terms of tautological classes on the universal Picard stack. If time permits, we will discuss two interesting applications of the Abel-Jacobi problem on the universal Picard stack: (i) the multiple cover formula for GW invariants of K3 surfaces and (ii) the locus of meromorphic differentials.

1. Introduction

1.1. **Abel-Jacobi theory on $\mathcal{M}_{g,n}$.** Let $k = \mathbb{C}$ be the base field. We first consider the Abel-Jacobi problem over a family of smooth curves. Let $\mathcal{M}_{g,n}$ be the moduli space of smooth curves of genus g with n markings ($g \geqslant 1$). This space has a universal curve $\pi : \mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$. Consider the relative Jacobian

$$\mathrm{Jac} \to \mathcal{M}_{g,n}$$

where the fiber over $[C]$ is the Abelian variety of degree zero line bundles on C. Let

$$A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$$

be a vector of integers such that $\sum_i a_i = 0$.

Question 1.1. *What is the locus corresponding to*

$$\mathrm{DR}^{\mathrm{sm}}_{g,A} = \{(C, p_1, \ldots, p_n) : \sum a_i p_i \sim 0\} \subset \mathcal{M}_{g,n}?$$

This condition is equivalent to define a map

eqn:maps (1) $$f : C \to \mathbb{P}^1 \text{ where } f^{-1}[0] = \sum_{i:a_i>0} a_i p_i \,, f^{-1}[\infty] = \sum_{i:a_i<0} |a_i| p_i$$

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upto $\mathbb{C}^*$ scaling on $\mathbb{P}^1$ ($a_i = 0$ corresponds to a normal marked point). This is why this locus is called a 'double ramification' (DR) cycle locus. There are two sections

eqn:ajsection
$$0, \sigma_A : \mathcal{M}_{g,n} \to \mathrm{Jac} \tag{2}$$

corresponding to $\mathcal{O}_C$ and $\mathcal{O}_C(\sum_i a_i p_i)$ respectively. Then $\mathrm{DR}^{\mathrm{sm}}_{g,A} = 0^{-1}[\sigma_A]$ and the cycle class is

$$\mathrm{DR}^{\mathrm{sm}}_{g,A} = 0^![\sigma_A] \in \mathrm{CH}^g(\mathcal{M}_{g,n}).$$

The same construction is well-defined for the moduli space of curves of compact type $\mathcal{M}^{\mathrm{ct}}_{g,n}$.

Theorem 1.2 (Hain)**.** *We have*

$$\mathrm{DR}^{\mathrm{sm}}_{g,A} = \frac{1}{g!}\Theta^g \text{ in } \mathrm{CH}^g(\mathcal{M}^{\mathrm{ct}}_{g,n})$$

where Θ *is the normalized theta divisor.*

1.2. **Extending Abel-Jacobi locus.** It is a natural question to ask if it is possible to extend $\mathrm{DR}^{\mathrm{sm}}_{g,A}$ to the moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$. Most naive attempt: consider the relative Picard scheme

$$\mathrm{Pic}(\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}) \to \overline{\mathcal{M}}_{g,n}$$

and do (2). This is not possible because the image of σ_A is no longer closed.

We use alternative description of (1) - it defines a map to $\mathbb{P}^1$ with prescribed ramification data up to scaling. In the relative GW theory there is a natural compactification of this space - the *rubber* space. Consider the following moduli space

$$\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1)^\sim = \{f : C \to (\mathbb{P}^1)^\sim : f^{-1}([0] - [\infty]) \sim \sum_i a_i[p_i]\}. \tag{3}$$

By [Li, Graber-Vakil] this space is proper over $\mathbb{C}$ and has a virtual fundamental class of dimension $2g - 2 + n$. Consider the forgetful map

$$p : \overline{\mathcal{M}}_{g,A}(\mathbb{P}^1)^\sim \to \overline{\mathcal{M}}_{g,n}.$$

Definition 1.3.

$$\mathrm{DR}_{g,A} = p_*[\overline{\mathcal{M}}_{g,A}(\mathbb{P}^1)^\sim]^{\mathrm{vir}} \in \mathrm{CH}^g(\overline{\mathcal{M}}_{g,n}).$$

This is our definition of DR cycle. Alternatively, Holmes defined the DR cycle by resolving th Abel-Jacobi section for the multi-degree zero Picard scheme $\mathrm{Pic}^{\underline{0}}_{g,n}$.

Now we want to write the class $\mathrm{DR}_{g,A}$ explicitly. Let $\mathrm{R}^*(\overline{\mathcal{M}}_{g,n}) \subset \mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ be the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

Theorem 1.4 (Faber-Pandharipande). *For all g, A we have*

$$\mathrm{DR}_{g,A} \in \mathrm{R}^g(\overline{\mathcal{M}}_{g,n}).$$

The proof requires relative/absolute correspondence in GW theory and the virtual localization. They gave an explicit algorithm to compute the class $\mathrm{DR}_{g,A}$ but this is highly complicated - it is not clear why $\mathrm{DR}_{g,A}$ restricted to $\mathcal{M}^{\mathrm{ct}}_{g,n}$ has Hain's formula.

Example 1.5. Let $\mathbb{E}_g$ be the Hodge bundle over $\overline{\mathcal{M}}_g$ and let $\lambda_g = e(\mathbb{E})$. Then

$$\mathrm{DR}_{g,\varnothing} = (-1)^g \lambda_g.$$

1.3. **Pixton's formula.** Pixton suggested the following formula. Let Γ be a stable graph of genus g with n markings. Let r be a positive integer. A weighting mod r of Γ is a function,

$$w : H(\Gamma) \to \{0, \dots, r-1\}$$

which satisfies the following three properties:

(1) for all h_i, $i \in [n]$
$$w(h_i) \equiv a_i \mod r,$$
(2) for $e = (h, h')$
$$w(h) + w(h') \mod r,$$
(3) for $v \in V(\Gamma)$
$$\sum_{v(h)=v} w(h) \equiv 0 \mod r.$$

Let $W_{\Gamma,r}$ be the set of weightings mod r on Γ. Then $|W_{\Gamma,r}| = r^{h^1(\Gamma)}$.

We denote $\mathsf{P}^{c,r}_{g,A}$ be the degree c component of the following tautological class

(4)
$$\sum_{\Gamma} \sum_{w \in W_{\Gamma,r}} \frac{1}{r^{h^1(\Gamma)}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{i=1}^{n} \exp(\frac{1}{2} a_i^2 \psi_i) \prod_{e=(h,h')} \frac{1 - \exp(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}}$$

Proposition 1.6. *When $r \gg 0$ $\mathsf{P}^{c,r}_{g,A}$ is polynomial in r.*

Let $\mathsf{P}^{c}_{g,A}$ be the constant term of $\mathsf{P}^{c,r}_{g,A}$.

Theorem 1.7 (JPPZ). $\mathrm{DR}_{g,A} = \mathsf{P}^{g}_{g,A}$.

Example 1.8. put examples λ_2, $\mathrm{DR}(a, -a)$.

The proof requires the virtual localization for the relative/orbifold stable map spaces to the target $\mathbb{P}^1[r]/0$.

2. Universal DR

2.1. Universal Picard stacks.

We put additional complexity of the Abel-Jacobi problem. Now we also vary curves and the line bundle. Let $\mathfrak{M}_{g,n}$ be the moduli stack of nodal curves of genus g with n markings[1] and $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$ be the universal curve. Let

$$\mathfrak{Pic}_{g,n} = Pic(\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}) = \bigsqcup_{d\in\mathbb{Z}} \mathfrak{Pic}_{g,n,d}$$

be the universal Picard stack over $\mathfrak{M}_{g,n}$. This is a locally finite type, smooth stack of dimension $\dim \mathfrak{M}_{g,n} + g - 1$. On $\mathfrak{Pic}_{g,n}$, there are additional tautological classes coming from the universal line bundle. Let

$$\pi : \mathfrak{C}_{g,n} \to \mathfrak{Pic}_{g,n},\ p_i : \mathfrak{Pic}_{g,n} \to \mathfrak{C}_{g,n} \mathfrak{L} \to \mathfrak{C}_{g,n}$$

be the universal curve, universal section and the universal line bundle. Then the additional classes include

- $\xi_i = p_i^* c_1(\mathfrak{L})$
- $\eta_{a,b} = \pi_*(c_1(\omega_{\log})^a c_1(\mathfrak{L})^b)$ (twisted κ-classes).

For simplicity $\eta = \eta_{0,2}$. Let $\mathrm{R}^*(\mathfrak{Pic}_{g,n})$ be the tautological ring of $\mathfrak{Pic}_{g,n}$. This ring can be defined in CH^* or $\mathrm{CH}^*_{\mathrm{op}}$ (the operational Chow ring).

2.2. Main theorems.

Let $d \in \mathbb{Z}$ and let $A = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $\sum_i a_i = d$.

Question 2.1. *What is the proper compactification of*

$$\{(C, p_1, \ldots, p_n, L) : \mathcal{O}_C(\sum_i a_i p_i) \cong L\} \subset \mathfrak{Pic}_{g,n}?$$

Can we compute this locus in $\mathrm{CH}^g(\mathfrak{Pic}_{g,,n})$*?*

The goal of this talk is to answer this question. Let

$$\sigma_A : \mathfrak{M}_{g,n} \to \mathfrak{Pic}_{g,n,d}$$

be the universal Abel-Jacobi section.

Definition 2.2. Let $\mathsf{DR}^{\mathrm{op}}_{g,A}$ be the class of the scheme theoretic closure of σ_A[2] in $\mathrm{CH}^g_{\mathrm{op}}(\mathfrak{Pic}_{g,n})$.

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[1] If you are not comfortable, you can stick to $\overline{\mathcal{M}}_{g,n}$.

[2] The scheme theoretic image is the smallest reduced closed substack of $\mathfrak{Pic}$ where σ_A factors through.

We twisted Pixton's formula as follows. We denote $\mathsf{P}^{c,r}_{g,A,d}$ be the degree c component of the following tautological class

(5)
$$\sum_{\Gamma}\sum_{w\in W_{\Gamma,r}}\frac{1}{r^{h^1(\Gamma)}}\frac{1}{|\mathrm{Aut}(\Gamma)|}\prod_{v}\exp(-\frac{1}{2}\eta_v)\prod_{i=1}^{n}\exp(\frac{1}{2}a_i^2\psi_i+a_i\xi_i)$$
$$\prod_{e=(h,h')}\frac{1-\exp(-\frac{w(h)w(h')}{2}(\psi_h+\psi_{h'}))}{\psi_h+\psi_{h'}}$$

The following theorems are two main results of [BHPSS].

thm:A **Theorem 2.3** (BHPSS). $\mathsf{DR}^{\mathrm{op}}_{g,A}=P^{g}_{g,A,d}$ *in* $\mathrm{CH}^{g}_{\mathrm{op}}(\mathfrak{Pic}_{g,n,d})$.

thm:B **Theorem 2.4** (BHPSS). *If* $c>g$*, then* $P^{c}_{g,A,d}=0$ *in* $\mathrm{CH}^{c}_{\mathrm{op}}(\mathfrak{Pic}_{g,n,d})$.

The proof of the above is technically involved (around 120 pages). The main idea: the map

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^N,d)\to\mathfrak{Pic}_{g,n},\ [f:C\to\mathbb{P}^N]\mapsto(C,f^*\mathcal{O}(1))$$

defines an "atlas" as $N\to\infty$. Now we use the result already known for $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^N,d)$. Also log description of Abel-Jacobi section by [MW] and there comparison result to relative stable map spaces...

3. Applications

3.1. **Locus of meromorphic differentials.** Our main result is not only useful to study $\mathfrak{Pic}_{g,n}$ but it is also very useful to study $\overline{\mathcal{M}}_{g,n}$. Let $A=(a_1,\dots,a_n)\in\mathbb{Z}^n$ satisfying $\sum_i a_i=2g-2$. We assume that at least one a_i is strictly negative. Let

$$\mathcal{H}_{g,A}=\{(C,p_1,\dots,p_n):\mathcal{O}_C(\sum_i a_ip_i)\cong\omega_C\}\subset\mathcal{M}_{g,n}.$$

be the locus of meromorphic differential with prescribed zeros and poles. In [Farkas-P] they considered a compactification of $\widetilde{\mathcal{H}}_{g,A}\subset\overline{\mathcal{M}}_{g,n}$ which is pure of codimension g. Moreover we can assign a weight[3] on each irreducible component of $\widetilde{\mathcal{H}}_{g,A}$ and get a weighted fundamental class $\mathsf{H}_{g,A}$.

Theorem 3.1. *We have* $\mathsf{H}_{g,A}=P^{g}_{g,A,2g-2}\cap[\overline{\mathcal{M}}_{g,n}]$ *where the class* $P^{g}_{g,A,2g-2}$ *is pulled back from*

$$\overline{\mathcal{M}}_{g,n}\to\mathfrak{Pic}_{g,n,2g-2}\,,C\mapsto\omega_C\,.$$

[3]For example the multiplicity of the main component is one.

This result gives an algorithm to compute the class of the main component recursively.

Idea: For a sheaf of holomorphic differentials, we have

$$\sigma_\omega : \overline{\mathcal{M}}_{g,n} \to \mathfrak{Pic}_{g,n}, C \mapsto \omega_C.$$

3.2. **Multiple cover formula for the GW invariants of K3 surfaces.** Let S be a smooth projective K3 surface. Then the moduli space of stable maps to S has a reduced virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(S,\beta)]^{\mathrm{red}} \in \mathrm{CH}_{g+n}(\overline{\mathcal{M}}_{g,n}(S,\beta))\,.$$

When the curve class is not primitive[4], GW invariants of K3 surface are very hard to compute because of the reduced virtual fundamental class. Motivated by the multiple cover formula of genus 0 GW invariants of K3 surfaces, [MPT] [OP] conjectured:

conj:MCC **Conjecture 3.2** (MPT, OP)**.** *Let S be an elliptic fibered K3 surface.*

(1) *The generating series of GW invariants of S in class of β of multiplicity m is a level m quasi-modular form*[5].
(2) *There exists an explicit formula of computing imprimitive GW invariants of K3 surfaces from primitive GW invariants.*[6]

This conjecture was proven for Hodge integrals

$$\int_{[\overline{\mathcal{M}}_{g,n}(S,\beta)]^{\mathrm{red}}} \lambda_g$$

by Pandparipande-Thomas.

We go back to our main theorem. For a line bundle L on S we have

$$\overline{\mathcal{M}}_{g,n}(S,\beta) \to \mathfrak{Pic}_{g,n}, [f : C \to S] \mapsto f^*L.$$

By Theorem 2.4 we have a bunch of tautological relations on $\mathfrak{Pic}_{g,n}$. Pulling back relations to $\overline{\mathcal{M}}_{g,n}(S,\beta)$ and capping with reduced VFC, we get relations among GW invariants of S.

Theorem 3.3 (B-Buelles)**.** *When the multiplicity $m = 2$ the Conjecture 3.2 (i) is true and the generating series satisfies the holomorphic anomaly equation. Moreover Conjecture 3.2 (ii) is true for $g = 0, 1, 2$.*

References

[4] A curve class β is primitive if it is not a multiple of other integral class in $H_2(S,\mathbb{Z})$.
[5] It can have a pole at $q = 0$
[6] (ii) implies (i).