Revisiting persistent private information

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Abstract

We revisit the contracting problem in the persistent private information model of Williams (2011) [Wil11], in which an agent provides a report y of a privately observed path b to the principal, who in turn pays the agent in the least expensive way that induces truthful reporting. In this note we argue that, under the restrictions on the reporting strategies imposed in that paper, the only available examples are such that: a) in the case of persistent information, the contract in [Wil11] does not induce truthful reporting; b) in the case of non-persistent information, the contract makes the agent indifferent, hence it is also optimal with no restrictions. We show that the contract becomes incentive compatible in case a), if one imposes an additional bound on under-reporting. We also find the optimal contract that does induce truth-telling under the conditions of [Wil11].

1 Introduction

This work is, at the same time, inspired by, and a critique of Williams [Wil11]. That paper is the first, to the best of our knowledge, that considers a continuous-time principal-agent problem in which there is an output process, denoted B, that is unobserved by the principal, but observed by the agent. Later work that considers related problems includes Prat and Jovanovic [PJ14] and He et al. [HWYG17]. However, those two papers consider a model in which there is a drift component unobserved by both the principal and the agent, and they learn about it over time.

All three papers solve the problem for the agent with CARA utility, and the risk-neutral principal. Mathematically, they all encounter the same main difficulties: (i) there is an additional state variable that depends on the cumulative agent's action, unobserved by the principal; (ii) there are technical issues in the usual weak formulation of the problem, due to the infinite horizon. These complicate the usual first order approach, applied, for example, in Sannikov [San08], Cvitanić et al. [CPT17], [CPT18], and Cvitanić and Xing [CX18]. The first order condition now depends additionally on the expected value of a functional of the future action. Still, [Wil11], [PJ14] and [HWYG17] provide the first

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order necessary conditions for the optimal contract, and also conditions which are not necessary, but are sufficient, together with the first-order conditions, for a contract to be optimal.

In [Wil11], as well as in the current paper, the principal minimizes the expected cost among the contracts that are incentive compatible, that is, under which the agent would report truthfully. In its main example (Section 6), [Wil11] provides an "optimal" contract, and derives the main economic insights of the paper based on such a contract. However, we show that, if the agent is allowed to under-report with unbounded deviations, as is the case in [Wil11], the stated contract is not incentive compatible in the most interesting case in which the persistency parameter λ (the rate of mean reversion) is strictly positive. Hence, it cannot be optimal, as it is outside of the set of admissible contracts.[1] Next, we find the contract that is optimal in that framework. We also show that the contract in [Wil11] becomes incentive compatible if the difference between the misreported process and the true process is constrained to stay sufficiently bounded.

The above is the main critique we present. In addition, we comment on the assumptions made in [Wil11]. In particular, the agent's report y (realization of a process Y) for the realized path b of B has to be such that $y \leq b$ (i.e., $Y \leq B$). [Wil11] justifies this by stating that, at least in some applications, one can imagine that the agent deposits the value y in an account, and cannot deposit more than the actual value b. However, in addition to assuming $Y \leq B$, [Wil11] assumes much more, i.e., that we have

$$Y_t - B_t = \int_0^t \Delta_s ds \ , \quad \text{with} \quad \Delta_s \leq 0. \tag{1.1}$$

This assumption not only requires that $Y - B$ is non-positive, but also that its rate of change Δ is nonpositive. [2] Also, for Girsanov theorem to be applied, as [Wil11] does, one needs at least that $\int_0^t \Delta_s^2 ds < \infty$, for all $t \leq T$, where T is the time horizon. While this may not be a serious restriction, with $T = \infty$ (which is the situation considered in the examples in [Wil11]), one needs stronger assumptions to be able to apply Girsanov theorem.

Suppose now that we do impose the same restrictions as [Wil11], that is, $Y \leq B$ and (1.1). There are two examples that [Wil11] discusses. In the so-called "permanent shocks" case, corresponding to the persistency parameter $\lambda = 0$, we study the same examples using a simpler approach based on [CPT17] and [CPT18]. We get the same results in that case, and we recognize that the contract can be written in a completely deterministic form.[3] Since the contract is deterministic in this case, the agent is indifferent with respect to misreporting, and it is assumed that he does not misreport. This case is, arguably, not very interesting, since the contract provides no incentives to the agent – in this case, misreporting can only be avoided by a non-incentive contract.

In the second, so-called "persistent shocks" case ($\lambda > 0$), as we already mention above, the contract found in [Wil11] is not incentive compatible, hence not optimal. We find the

[1] We explain in the main text where the verification of incentive compatibility in [Wil11] goes wrong.

[2] However, it is not clear that the latter assumption leads to loss of generality, meaning, it is not clear whether there are examples in which a different contract would be optimal without that restriction on agent's misreporting.

[3] [Wil11] presents the optimal contract in a form depending on the output process and the agent's continuation value, as opposed to presenting it in its deterministic form as we do here.

optimal contract also for this case, and show that it is also optimal if the only restriction on misreporting is that it be under-reporting, $Y \leq B$.

We also show how the other example considered in [Wil11] is easily reduced to the first example.

The paper is organized as follows. Section 2 recalls the persistent private information reporting problem from [Wil11] and clarifies the technical assumptions needed to rigorously pose the problem. In Section 3 we present our counterexample to the contract claimed to be incentive compatible and optimal in [Wil11], we elaborate on the reason why the argument presented there is erroneous, and we identify the additional condition that makes the contract incentive compatible. We define admissible contracts in Section 4, and find the optimal contract. Section 5 discusses the question of what would be the minimal set of restrictions on misreporting which would still result in non-trivial contracts. We conclude in Section 6.

Notation and Conventions. For a process X, we denote by $\mathbb{F}^X$ the raw filtration generated by X. With $\mathcal{C}(I; A)$ we denote the set of continuous functions $f : I \to A$, equipped with the topology of uniform convergence on compact sets. For a continuous local martingale M, we use the symbol $\mathscr{E}(M)$ to denote the stochastic exponential of M. Finally, let $\mathbf{P}$ and $\mathbf{Q}$ be two probability measures defined on a measurable space $(\Omega, \mathcal{F})$ equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$. We will say that $\mathbf{P}$ is locally absolutely continuous with respect to, resp. equivalent to $\mathbf{Q}$, if the restricted measures $\mathbf{P}|_{\mathcal{F}_t}$ and $\mathbf{Q}|_{\mathcal{F}_t}$ are absolutely continuous, resp. equivalent for each $t < \infty$. In this case we call the Radon-Nikodym derivative the local density.

2 The model and contracting problem

2.1 Framework

We recall first the framework of [Wil11]. We fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ supporting a Brownian motion W with its usual filtration $\mathbb{F}^W = \mathbb{F} = (\mathcal{F}_t : t \geq 0)$. The agent observes the realization of the process B, whose evolution is given by

$$dB_t = (\mu_0 - \lambda B_t)dt + \sigma dW_t, \quad B_0 = b_0, \tag{2.1}$$

where $\mu_0 \in \mathbb{R}, \lambda \geq 0, \sigma > 0$ and $b_0 \in \mathbb{R}$. We call λ the persistency parameter. If $\lambda > 0$, we say that information is persistent.

The principal knows the dynamics of the process B, that is, she knows μ_0, λ, σ and b_0 in (2.1), but does not observe the realization of B. In order to elicit B, she offers a contract to the agent, to receive a report on the observed realization. More formally, the agent chooses a *reporting strategy*

$$Y : [0, \infty) \times \Omega \to \mathbb{R}$$

adapted to $\mathbb{F}^B$. One easily verifies that up to completion and closure from the right, $\mathbb{F}^B = \mathbb{F}$. We shall thus assume that Y is actually $\mathbb{F}$-adapted.

Since B has a modification with continuous paths, it is natural to assume that also the reported path must be continuous: [4]

[4]See Section 5 below for a discussion of other properties that a principal might want to require.

Condition 2.1. The reporting strategy Y is continuous.

Contracts

For a fixed reporting strategy Y, we denote by $y = (y_t)_{t\in[0,\infty)}$ its realization, which is the path reported to the principal by the agent as the realization $b = (b_t)_{t\in[0,\infty)}$ of B. We use $y_{s\le t}$ as a shorthand for $(y_s)_{s\in[0,t]}$, the reported path up to time t. In exchange for the report, the principal pays the agent a salary $s = (s_t)_{t\in[0,\infty)}$ continuously in time. Those payments depend, in an adapted way, only on the path revealed by the agent. This means that the principal chooses a jointly Borel measurable function $s : [0,\infty)\times\mathcal{C}([0,\infty),\mathbb{R}) \to \mathbb{R}$ that satisfies

$$s_t(y) = s_t\left(y_{s\le t}\right), \quad \text{for all } t \in [0,\infty).$$

We call $s = (s_t)_{t\in[0,\infty)}$ a *contract.*

Decision problems

As is standard in contract theory, the principal proposes a contract to the agent. Then, the agent may either accept or reject the contract. Once accepted, the agent continuously makes the report y_t to the principal and in turn receives the contractual compensation $s_t(y)$ from the principal. The principal, anticipating the agent's choices, will offer only contracts that maximize her objective, as we describe now.

For a given contract s, the agent aims to maximize

$$J_0(s(Y),B) := \mathbf{E}\left[\int_0^\infty e^{-\rho t}u(s_t(Y),B_t))\,dt\right] \tag{2.2}$$

over reporting strategies Y satisfying Condition 2.1, where u is the flow utility function, and $\rho > 0$ the discount factor. We call any strategy Y^s that maximizes (2.2), if it exists, an *optimal report* for s.

Definition 2.2. A contract s is said to satisfy the

- *Truth Revelation Constraint*, if there exists at least one optimal report Y^s which is truth-telling, that is, such that $Y^s = B$, $\mathbf{P}$-almost surely;
- *Participation Constraint*, if
$$J_0\left(s(B),B\right) \ge v_0,$$
for a given reservation utility v_0 of the agent.

We call a contract that satisfies both constraints *incentive compatible.*

As usual in contract theory, it is assumed that, if indifferent, the agent will report truthfully. Having formalized the agent's problem, we state the principal's objective as minimizing

$$\mathbf{E}\left[\int_0^\infty e^{-\rho t}s_t(Y^s)dt\right] \tag{2.3}$$

over all contracts s that are incentive compatible.

2.2 Additional restrictions on misreporting

So far, a reporting strategy is a general $\mathbb{F}$-adapted process with continuous paths. Optimizing over this set is too general, even simply to be able to apply stochastic calculus. In [Wil11] the following is imposed.

Condition 2.3. The agent can only under-report: $Y \leq B$.

The justification provided in [Wil11] is that for some applications one can envision that the agent deposits the reported values y_t in a savings account observable by the principal, and is unable to deposit more than the actual values b_t. However, for many applications this may not be a valid restriction. In Section 5 below we discuss various possible restrictions on reporting strategies.

The reporting rules satisfying Conditions 2.1 and 2.3 still include a lot of possibilites. For instance, $Y_t := \min_{s \leq t}\{B_s\}$ is such a reporting strategy, and so is $Y_t := \min\{B_t, Y_t^\circ\}$ for any continuous $\mathbb{F}$-adapted process Y°.

Denote the misreporting process by

$$m_t := Y_t - B_t. \tag{2.4}$$

[Wil11] assumes in addition the following

Condition 2.4. The misreporting process is of the form

$$m_t = \int_0^t \Delta_s ds \ ,$$

for some $\mathbb{F}$-adapted process $\Delta \leq 0$.

Clearly, this requirement is stronger than Condition 2.3. It is assumed for tractability. In the two examples that are considered in [Wil11] and in the present paper, we will see that the requirement that m is differentiable is not necessary, and that, when $\lambda = 0$, the requirement that m is no larger than zero is not necessary.

The requirement that m_t is differentiable is justified in [Wil11] by saying that "the agent's report Y must be absolutely continuous with respect to his true state B". Indeed, in theory, it could be argued that, given the reported path, the principal could keep performing, continuously in time, (uncountably) many tests for the likelihood ratio $\frac{dP^B}{0.5(dP^B+dQ)}$, where P^B is the law of the true process B, for all possible alternative laws Q. For a path that comes from a law that is singular with respect to P^B, there would be a positive probability that the tests would reject P^B in favor of some singular Q in finite time. [5] In practice, this would, of course, be very hard to do. Nevertheless, we also adopt this condition from [Wil11] in our general setup.

For Y_t to be obtained by a locally absolutely continuous change of measure on $\mathcal{F}_t$, as [Wil11] does, Δ must satisfy the following integrability condition:

Condition 2.5. The process Δ in Condition 2.4 satisfies, for each $t < \infty$,

$$\int_0^t \Delta_s^2 \, ds < \infty \ , \qquad \mathbf{P}\text{-almost surely.}$$

[5] We thank Martin Larsson for this observation.

Note that this is a statement about the *local densities for each* $t < \infty$. If this condition is violated for some $t < \infty$, then the law of $\int_0^t \Delta_s \, ds + B_t$ and that of B_t are no longer equivalent on $\mathcal{F}_t$, see e.g. [JS13, Ch. IV.4b], [Lie86] or [LS13, Thm. 7.2 & Thm. 7.5]. In this case the principal can identify certain reports as being a lie, a.s. Indeed, there is a measurable set $L \subset \mathcal{C}([0,t),\mathbb{R})$ such that $\mathbf{P}[Y^{-1}(L)] > 0$ but $\mathbf{P}[B^{-1}(L)] = 0$. Thus, a path in L is a.s. not a realization of B and may be considered a.s. a lie, and such reports occur with positive probability under the strategy Y. If the principal punishes the agent with infinite penalty in case of a verified lie, the agent would not use strategies which do not correspond to local equivalence.

Thus, local equivalence for each $t < \infty$ is a natural restriction. Mathematically, it would help to have global equivalence. However, it is well-known that local equivalence of laws does not imply that the extension to $\vee_{t\geq 0}\mathcal{F}_t$ is still equivalent; see, e.g. [RY13, Prop. VIII.1.1, VIII.1.1'].

In [Wil11], global equivalence is implicitly assumed, by formally taking the limit when $T \to \infty$. In principle, global equivalence is needed for the weak formulation in which the agent changes the distribution of the output, but we will avoid it using a somewhat different approach.

3 A Counterexample to Incentive Compatibility

Before presenting results on our contracting problem, we show that the contract in the main example of [Wil11, Sec. 6] is not incentive compatible, under the assumptions of that paper. In the example, the agent's expected utility is

$$\mathbf{E}\left[\int_0^\infty e^{-\rho t} u(s_t(Y) + B_t) dt\right],$$

where

$$u(x) = -e^{-\theta x}$$

for some parameter $\theta > 0$. We consider here the persistent case $\lambda > 0$. The case $\lambda = 0$ is considered in Section 4.1 below, and in that case we get the same contract as [Wil11, Sec. 6], with a different approach.

Consider the contract

$$S_t := s_t(Y) = \alpha - \frac{1}{\theta}\log(-q_t) - Y_t, \tag{3.1}$$

from [Wil11, p. 1271]. Here α is an appropriate constant, q_t is the promised utility (i.e., value) process of the agent under truthful reporting, given by

$$dq_t = -\sigma\theta\beta q_t dW_t^Y \ ,$$

with W_t^Y given by

$$W_t^Y := \frac{1}{\sigma}\left(Y_t - b_0 - \int_0^t [\mu_0 - \lambda Y_s] ds\right), \tag{3.2}$$

and

$$\beta = \frac{\rho}{\rho + \lambda} \tag{3.3}$$

is a constant. Note that the functional s in (3.1) satisfies the measurability restrictions required in the definition of a contract from Section 2.1. Contract (3.1) may be written as

$$S_t + Y_t = \tilde{\alpha} + \frac{1}{2}\theta\beta^2\sigma^2 t + \beta\sigma W_t^Y. \tag{3.4}$$

It is claimed in [Wil11] that this contract is optimal and thus, in particular, incentive compatible. We now show that this is not the case.

Proposition 3.1. *Any contract that satisfies (3.4) for a constant $\beta \in (0,1)$ and an arbitrary constant $\tilde{\alpha}$ is not incentive compatible.*

The proof will show that contract (3.1) actually does not admit an optimal reporting strategy. Instead, the higher the misreporting, the higher the agent's utility.

Proof. Just as in [Wil11, p. 1272], for a process of accumulated lies $dm_t = \Delta_t dt$ satisfying Conditions 2.3 to 2.5, equation (3.2) can be written as

$$\begin{aligned}\sigma dW^Y &= dB_t - [\mu_0 - \Delta_t - \lambda(B_t + m_t)]dt \\ &= \sigma dW_t + [\Delta_t + \lambda m_t]dt.\end{aligned}$$

The second term measures the gap between the actual increment of the Brownian motion and the principal's calculated increment, if she assumes truth-telling. Contract (3.4) may be written as

$$S_t + B_t = \tilde{\alpha} + \frac{1}{2}\theta\beta^2\sigma^2 t + \beta\sigma W_t + \lambda\beta\int_0^t m_s ds - (1-\beta)m_t, \tag{3.5}$$

for some constant $\tilde{\alpha}$. Take, for $k > 0$,

$$\Delta_t = -e^{kt}. \tag{3.6}$$

Then, the sum of the last two terms in the expression for $S_t + B_t$ above is

$$\lambda\beta\int_0^t m_s ds - (1-\beta)m_t = \frac{1}{k}\lambda\beta\int_0^t (1-e^{ks})ds - \frac{1}{k}(1-\beta)(1-e^{kt})$$

$$= \frac{1}{k}\lambda\beta\left[t + \frac{1}{k}(1-e^{kt})\right] - \frac{1}{k}(1-\beta)(1-e^{kt}) = \frac{1}{k}\lambda\beta t + \frac{1}{k}(1-e^{kt})\left[\frac{1}{k}\lambda\beta - (1-\beta)\right].$$

A sufficient condition for this to be larger than zero is

$$k \geq \frac{\lambda\beta}{1-\beta}.$$

Thus, for any such k, the agent would be better off using the corresponding strictly negative Δ instead of zero. In fact, he would like to set k as large as possible. More precisely, plugging it back to the agent's expected utility $\mathbf{E}[\int_0^\infty e^{-\rho t}u(S_t + B_t)dt]$, we see that the agent can get as close as he wants to its maximum value, equal to zero, by increasing k. □

Since $\lambda > 0$, we have $\beta < 1$ in (3.3), and it follows, in particular, that the contract in [Wil11, Sec. 6] is not incentive compatible under the assumptions imposed in that paper, stated in Conditions 2.1 to 2.4.

An inspection of the direct verification of incentive compatibility in the appendix of [Wil11] reveals the following omission. On [Wil11, p. 1272] there is an HJB equation for the agent's value given the contract (3.4), with $\beta = \rho/(\rho + \lambda)$ and with W^Y expressed in terms of q. The solution to that equation is provided as

$$V(q, m) = \frac{q \exp(\theta m)(\rho + \lambda)}{\rho + \lambda + \theta\lambda m}. \tag{3.7}$$

It is in fact immediately seen that this solution cannot be the value function of the agent for all values of m: since $q < 0$, this value is positive for $m < -\frac{\rho+\lambda}{\theta\lambda}$, hence it cannot be the value function corresponding to the negative CARA utility. An essential step neglected in [Wil11] is the verification that the solution of the HJB is in fact the agent's value function. For one thing, on infinite horizon the solution should satisfy the following transversality condition: for all admissible strategies,

$$\lim_{T\to\infty} \mathbf{E}_{q_t,m_t}\left[e^{-\rho(T-t)}V(q_T, m_T)\right] = 0.$$

It can be checked that the above function does not satisfy this, unless m never reaches $-\frac{\rho+\lambda}{\theta\lambda}$.

[Wil11] provides another argument to claim that the contract is incentive compatible, by checking that the conditions of [Wil11, Theorem 4.1] are satisfied. However, the proof of that theorem goes through for infinite horizon only if $\mathbf{E}[e^{-\rho T}p_T m_T]$ converges to zero as T goes to infinity, where, as the paper shows, the adjoint process p satisfies $p_T = \theta q_T$, with q_T as above. Then, it is easy to verify that we do not have the desired convergence if m ever goes below the value $-\frac{\rho+\lambda}{\theta\lambda}$. This is because then $e^{-\rho T}q_T/q_t$ converges to infinity, and so does $\mathbf{E}[e^{-\rho T}p_T m_T]$.

Let us explain where the values $-\frac{\rho+\lambda}{\theta\lambda} = -\frac{\rho}{\theta\lambda\beta}$ and $\frac{\rho}{\rho+\lambda} = \beta$ are coming from. We have the following result.

Proposition 3.2. *Consider a contract of the form* (3.5)*. If* $\beta \geq 1$*, such a contract is incentive compatible. If* $\beta < \frac{\rho}{\rho+\lambda}$*, the contract is not incentive compatible. If* $\frac{\rho}{\rho+\lambda} \leq \beta < 1$*, the contract is not incentive compatible if unbounded misreporting is allowed (as already shown in the counterexample above), but it is incentive compatible if we admit only the strategies for which, for all* $t > 0$*, we have* $m_t \geq -M > -\frac{\rho}{\theta\lambda\beta}$ *a.s, for some constant* M*.*

Remark 3.3. Thus, to ensure that the contract from [Wil11] with $\beta = \frac{\rho}{\rho+\lambda}$ is incentive compatible, it is sufficient that the reporting gap $y - b$ be bounded from below by a large enough constant $-M > -\frac{\rho+\lambda}{\theta\lambda}$. However, it is not clear how to justify this specific bound on cumulative lies.

Proof. For $\beta \geq 1$, from (3.5), $S + B$ is obviously maximized when the agent uses $m \equiv 0$. For $\beta \leq 0$, the maximum is obtained at the lowest possible value of m, not at $m = 0$. The same is true if $\beta < 1$ and $\lambda = 0$. Thus, it only remains to consider the case $0 < \beta < 1$ and $\lambda > 0$.

Denote

$$X_t = x + \lambda\beta \int_0^t m_s ds + \frac{1}{2}\theta\beta^2\sigma^2 t + \beta\sigma W_t$$

The agent's value is equal to, using (3.5) and with $x = \tilde{\alpha}$,

$$-\mathbf{E}\left[\int_0^\infty e^{-\rho t} e^{-\theta(S_t+B_t)} dt\right] = -\mathbf{E}\left[\int_0^\infty e^{-\rho t} e^{-\theta[X_t-(1-\beta)m_t]} dt\right]$$

We can consider this as a stochastic control problem with control $m \leq 0$. Let us impose the constraint

$$m \geq -M,$$

for some $M > 0$. The Hamilton-Jacobi-Bellman equation is

$$\rho V(x) = \sup_{-M \leq m \leq 0} \left\{-\rho e^{-\theta[x+(\beta-1)m]} + (\lambda\beta m + \frac{1}{2}\theta\beta^2\sigma^2)V_x(x) + \frac{1}{2}\beta^2\sigma^2 V_{xx}(x)\right\}.$$

For incentive compatibility, we need the function corresponding to $m = 0$ to be the solution, that is,

$$V(x) = -\frac{1}{\rho}e^{-\theta x}.$$

Substituting, we get

$$1 = \sup_{-M \leq m \leq 0} \left\{-e^{-\theta(\beta-1)m} - \frac{\theta}{\rho}(\lambda\beta m + \frac{1}{2}\theta\beta^2\sigma^2) + \frac{1}{2\rho}\theta^2\beta^2\sigma^2\right\}.$$

The first derivative of the terms depending on m is proportional to $\rho(\beta-1)e^{-\theta(\beta-1)m} + \lambda\beta$. This is no less than zero at $m = 0$ if and only if $\beta \geq \frac{\rho}{\rho+\lambda}$, in which case the maximum is attained at zero. It remains to check the transversality condition $\lim_{T\to\infty} e^{-\rho T} E[V(X_T)] = 0$, for all admissible strategies. We have

$$e^{-\rho T} E[V(X_T)] = e^{-\rho T} E[e^{-\theta\left(z+\lambda\beta\int_0^T m_s ds + \frac{1}{2}\theta\beta^2\sigma^2 T + \beta\sigma W_T\right)}].$$

We see that this converges to zero if $m_t \geq -M > -\frac{\rho}{\theta\lambda\beta}$. □

Remark 3.4.

(i) Note that the above argument requires neither that m is differentiable, nor non-increasing.

(ii) It seems reasonable to conjecture that the above contracts with $1 > \beta \geq \frac{\rho}{\rho+\lambda}$ are incentive compatible also if m is restricted to be bounded from below by any constant $-M$, not only by those for which $-M > -\rho/(\theta\lambda\beta)$. This is because, if we start at $m_0 \leq -\rho/(\theta\lambda\beta)$ and not at $m_0 = 0$, and if m is bounded from below, then the agent's utility with $1 > \beta \geq \frac{\rho}{\rho+\lambda}$ can be shown to be minus infinity. However, for this argument, it is crucial that m is non-increasing, which is a strong assumption. The contracts with $\beta \geq 1$ are incentive compatible also when there are no restrictions on m being bounded or non-increasing.

4 Solving Examples in [Wil11]

4.1 Example I: Hidden Endowment

We revisit here the first example in [Wil11]. Consider the setup of the previous section and the contract process

$$\hat{S}_t = -\frac{1}{\theta}\log(-\theta V_0^A) - b_0 - \left(\mu_0 - \frac{1}{2}\theta\sigma^2\right)t + \lambda\int_0^t Y_s ds. \tag{4.1}$$

In this section, we show that this contract is optimal for the example of the previous section, for delivering to the agent the value V_0^A. In case $\lambda = 0$, it is the same contract as the one derived in [Wil11], except we present it in a simpler, completely deterministic form, while in [Wil11] it is presented in a form that depends on Y_t and V_t^A. We also provide a simpler argument, based on the approach of [CPT17] and [CPT18]. In case $\lambda > 0$, the contract is different from one in [Wil11]. As shown in the previous section, the contract provided in [Wil11] is not incentive compatible when $\lambda > 0$.

Agent's problem

For a given contract s, we write $S_t := s_t(Y)$ for the associated $\mathbb{F}$-adapted contract process.

We say that Δ is an *admissible strategy* of the agent if it is an $\mathbb{F}$-progressive processes satisfying Conditions 2.4 and 2.5. For each admissible strategy Δ of the agent, we write

$$m_t^\Delta := \int_0^t \Delta_s\, ds,$$

and define the *agent's payoff process*

$$J_t^A(S;\Delta) := \mathbf{E}\left[\int_t^\infty e^{-\rho(r-t)}u(S_r + Y_r - m_r^\Delta)\, dr \,\middle|\, \mathcal{F}_t\right]. \tag{4.2}$$

The *agent's value process* is then defined by

$$V_t^A(S;\Delta) := \operatorname{ess\,sup} J_t^A(S;\tilde{\Delta}), \tag{4.3}$$

where the supremum is taken over admissible strategies $\tilde{\Delta}$ satisfying $\tilde{\Delta}|_{[0,t]} = \Delta|_{[0,t]}$. The essential supremum here is well-defined, since it is taken with respect to the fixed measure $\mathbf{P}$, and Y denotes the unique strong solution $Y^\Delta = m^\Delta + B$ from (2.4) with B given by (2.1). We suppress the dependence on Δ in the notation and simply write Y.

We assume

$$u(x) = -\frac{\rho}{\theta}e^{-\theta x},$$

and note that the normalization by ρ/θ of the utility function used here is different from [Wil11] and from Section 3.

When $\lambda = 0$, this problem resembles well-known principal-agent models with hidden action Δ and cost process m^Δ. However, when $\lambda > 0$, m^Δ is an additional state variable because it appears in the dynamics of Y. Since this state variable is not observed, the standard methodology does not immediately apply.

Admissible contracts

We now define a family of contracts that will make the agent's problem tractable, and we motivate the definition below.

Definition 4.1. We call S an *admissible contract process*, if it can be represented as

$$S_t = -\frac{1}{\theta}\log(-\theta v_0) - Y_t - R_t + \int_0^t Z_s dY_s + \int_0^t \left(\frac{\rho}{\theta}[e^{\theta R_s} - 1] - \mu_0 Z_s + \frac{1}{2}\theta\sigma^2 Z_s^2 + \lambda Z_s Y_s\right) ds, \tag{4.4}$$

for some constant $v_0 < 0$, where Z and R satisfy:

(i) Z is $\mathbb{F}$-predictable and such that $Z \geq 1$ a.s., and $\mathbf{E}\Big[\int_0^t |Z_s|^2\, ds\Big] < \infty$ for all $t < \infty$;

(ii) R is an $\mathbb{F}$-predictable continuous semimartingale with $R_0 = 0$, and $\mathbf{E}\Big[\int_0^t e^{\theta R_s} ds\Big] < \infty$ for all $t < \infty$;

(iii) the following transversality condition holds:

$$\lim_{T\to\infty} \mathbf{E}\Big[e^{-\rho T}\int_0^T \left(\frac{\rho}{\theta}e^{\theta R_s} + \frac{1}{2}\theta\sigma^2 Z_s^2\right) ds\Big] = 0. \tag{4.5}$$

Condition (i) on Z implies in particular that process $\int_0^t Z_r dW_r$ is a martingale. Consequently, S itself is a semimartingale. From (4.11) below, transversality condition (4.5) is equivalent to $\lim_{T\to\infty} \mathbf{E}\Big[e^{-\rho T}(S_T + Y_T + R_T)\Big] = 0$, when $\Delta \equiv 0$. This means that the agent's certainty equivalent does not have an overly high average growth under truthful reporting. This can be interpreted as a kind of a participation constraint of the principal. It is needed to verify that the solution to the principal's HJB equation is equal to her value function.

We now informally justify the specific form (4.4) of the contract. It is motivated by the dynamic programming principle, or the martingale optimality principle, which states that, for any admissible Δ, the process

$$M_t^\Delta := e^{-\rho t}V_t^A(\Delta) - \frac{\rho}{\theta}\int_0^t e^{-\rho s}e^{-\theta(S_s + Y_s - m_s^\Delta)}ds \tag{4.6}$$

should be a local super-martingale for any admissible Δ, and a local martingale for any optimal control Δ^*. In particular, suppose we take R in Definition 4.1 to be the *residual certainty equivalent* of the agent's value process, defined implicitly by (suppressing the dependence of V^A, Z and R on Δ)

$$V_t^A = -\frac{1}{\theta}e^{-\theta(S_t + B_t + R_t)}. \tag{4.7}$$

Next, introduce the *reported certainty equivalent* C_t as

$$C_t := S_t + Y_t + R_t. \tag{4.8}$$

Assume we have the representation

$$C_t = c_0 + \int_0^t \sigma Z_s dY_s + H_t, \tag{4.9}$$

for a process H of locally finite variation with $H_0 = 0$, and where Z is predictable and a.s. locally square-integrable. [6] Next, observe that (4.7) may be written as $V_t^A = -\frac{1}{\theta}e^{-\theta(C_t - m_t^\Delta)}$, for which by Ito's rule we have the semimartingale expansion

$$-\frac{dV_t^A}{\theta V_t^A} = \sigma Z dY_t + dH_t - \Delta_t dt - \frac{\theta}{2}\sigma^2 Z^2 dt,$$

and thus

$$-\frac{dM_t^\Delta}{\theta e^{-\rho t}V_t^A} = -\frac{1}{e^{-\rho t}\theta V_t^A} d\left(e^{-\rho t}V_t^A\right) + \frac{\rho e^{-\theta\left(S_t + Y_t - m_t^\Delta\right)}}{\theta^2 V_t^A}dt$$

$$= \frac{\rho}{\theta}[1 - e^{\theta R_t}]dt + Z_t\left\{\sigma dW_t + [\Delta_t + \mu_0 - \lambda(Y_t - m_t^\Delta)]dt\right\} + dH_t - dm_t^\Delta - \frac{1}{2}\theta\sigma^2 Z_t^2 dt.$$

Now, from the martingale optimality principle, we conclude that

$$dH_t = h_t dt,$$

with

$$h_t = -\sup_{\Delta_t}\left\{\frac{\rho}{\theta}(1 - e^{\theta R_t}) + Z_t\mu_0 - \frac{1}{2}\theta\sigma^2 Z_t^2 + \Delta_t(Z_t - 1) - \lambda Z_t(Y_t - m_t)\right\}. \tag{4.10}$$

For the incentive compatibility of S, we need $\Delta \equiv 0$ to be optimal. The form of h_t suggests we should then have $Z_t \geq 1$. Then, setting $\Delta \equiv 0$, and choosing a process $Z_t \geq 1$ in (4.9), gives a process h which together with (4.8) and (4.9) results in the contract process as in (4.4) of Definition 4.1.

Remark 4.2. The above is a heuristic justification for the representation (4.4) and for requiring $Z \geq 1$. We prove the necessity of $Z \geq 1$ in Proposition 4.9 below, using the weak formulation of the agent's control problem.

4.1.1 Principal's problem

It is convenient to have the principal's problem expressed as a maximization of

$$-\rho\mathbf{E}\left[\int_0^\infty e^{-\rho t}s_t(Y^s)dt\right]$$

over all contracts s that are incentive compatible.

The main result in this section is the following:

Proposition 4.3. *Assume Conditions 2.1 to 2.5 hold. The contract in (4.1) is optimal among all admissible contracts that are incentive compatible and that deliver V_0^A to the agent.*

The proof of the proposition is given below.

Corollary 4.4. *The conclusion of the proposition remains valid when only Condition 2.1 is assumed and*

[6] In the weak formulation, Y would be defined as a fixed Brownian motion, and the above representation would be obtained from the martingale representation theorem.

(i) for the case $\lambda = 0$: the agent can misreport by any $\mathbb{F}$-adapted process m such that $\int_0^t m_s\,ds < \infty$ for all $t > 0$;

(ii) for the case $\lambda > 0$: the agent can misreport by any $\mathbb{F}$-adapted process m such that $\int_0^t m_s\,ds \le 0$ for all $t > 0$.

Proof of Corollary 4.4. Using the contract from (4.1), we calculate

$$\hat{S}_t + B_t = -\frac{1}{\theta}\log(-\theta V_0^A) - b_0 - (\mu_0 - \frac{1}{2}\theta\sigma^2)t + \lambda\int_0^t B_s\,ds + B_t + \lambda\int_0^t m_s\,ds.$$

Clearly, when $\lambda = 0$ this is independent of the agent's action m_t, so that the agent is indifferent and reports truthfully. If $\lambda > 0$ and $\int_0^t m_s\,ds \le 0$, then setting $m_t = 0$ maximizes this expression. This proves that the contract from (4.1) is incentive compatible under the conditions of the corollary. The optimality follows by using the fact that, if the contract is optimal when the agent can only misreport by choosing m from a set $\mathcal{A}$, then the same contract is optimal for any set $\mathcal{B}$ containing $\mathcal{A}$ such that the contract is incentive compatible when the agent can misreport by choosing m from set $\mathcal{B}$. □

The corollary strengthens the results from [Wil11] by showing that in case $\lambda = 0$ the Conditions 2.3 to 2.5 are not necessary; and in case $\lambda > 0$ Conditions 2.4 and 2.5 are not necessary.

Remark 4.5. Let us comment on the case $\lambda = 0$. The principal's value for the contract (4.1) is

$$J^P(b_0, V_0^A) := -\rho\mathbf{E}\left[\int_0^\infty e^{-\rho t}\hat{S}_t dt\right] = \frac{1}{\theta}\log(-\theta V_0^A) + b_0 + \rho\int_0^\infty e^{-\rho t}\left(\mu_0 - \frac{1}{2}\theta\sigma^2\right)t\,dt,$$

which can be computed as

$$J^P(x, v) = \frac{1}{\theta}\log(-\theta v) + x + \frac{1}{\rho}\left(\mu_0 - \frac{1}{2}\theta\sigma^2\right).$$

The value function for the contract provided by [Wil11, p. 1253], which is incentive compatible in case $\lambda = 0$ (and this case only), is, in our notation, equal to $-\rho J(x, \theta v/\rho)$, where we have accounted for the different normalization. It is immediately verified that the two have the same value. The only difference is that we write the contract in a simpler form, depending only on the initial values $x = b_0$ and V_0^A of process Y and agent's value process V^A, while in [Wil11] the contract is written in a form that depends on Y_t and V_t^A.

Proof of Proposition 4.3. Considering only incentive compatible contracts, the principal assumes truthful reporting, so $\Delta = 0$, a.s. Then, from (4.4),

$$C_t = S_t + Y_t + R_t = c_0 + \int_0^t \sigma Z_s dW_s + \int_0^t \left(\frac{\rho}{\theta}\left[e^{\theta R_s} - 1\right] + \frac{1}{2}\theta\sigma^2 Z_s^2\right) ds \tag{4.11}$$

and

$$S_t = C_t - R_t - Y_t = C_t - R_t - e^{-\lambda t}b_0 - \int_0^t \sigma e^{\lambda(s-t)}dW - \int_0^t \mu_0 e^{\lambda(s-t)}ds.$$

We can interpret the principal's problem as optimally choosing Z and R, for a given value of V_0^A. Her expected value can then be written as

$$\begin{aligned}\sup_S -\rho \mathbf{E}\left[\int_0^\infty e^{-\rho t} S_t dt\right] \\ = \rho \int_0^\infty e^{-\rho t}\left\{b_0 e^{-\lambda t} + \int_0^t e^{\lambda(s-t)}\mu_0 ds\right\} dt + \sup_{Z\geq 1, R} \mathbf{E}\left[\rho \int_0^\infty e^{-\rho t}\left(R_t - C_t\right) dt\right],\end{aligned}$$

where the first supremum is over admissible contracts, and the second over processes Z and R corresponing to S, as in Definition 4.1.

The first term is a constant depending only on b_0, so consider the optimization problem

$$F(C_t) = \sup_{Z\geq 1, R} \mathbf{E}\left[\rho \int_t^\infty e^{-\rho s}\left(R_s - C_s\right) ds\right].$$

This is a standard control problem with state C and controls Z and R. Since $Z \geq 1$ and $\sigma > 0$, the HJB equation is well-posed and regular and given by

$$\rho F = \sup_{z\geq 1, r}\left\{\rho(r-c) + \left(\frac{\rho}{\theta}[e^{\theta r} - 1] + \frac{1}{2}\theta\sigma^2 z^2\right)\partial_c F + \frac{1}{2}\sigma^2 z^2 \partial^2_{cc} F\right\}.$$

To find a solution F of the HJB equation, we first assume that it satisfies $\partial_c F < 0$ and $\partial^2_{cc} F = 0$, and verify that these conditions are indeed valid for the solution we derive. Observe that in this case clearly $z = 1$ must be chosen in the supremum. Moreover, $\partial_c F < 0$ implies that the term inside the supremum is a convex function of r, so that the first order condition

$$\partial_c F = \exp(-\theta r)$$

uniquely identifies the optimal value of r. Thus, the HJB equation becomes

$$\rho F = -\frac{\rho}{\theta}\log(-\partial_c F) - \rho c + \left(-\rho/\theta + \frac{1}{2}\theta\sigma^2\right)\partial_c F - \rho/\theta,$$

with solution

$$F(c) = -\frac{1}{2\rho}\theta\sigma^2 - c.$$

Clearly, this function verifies the assumed conditions. Consequently, the candidates for the optimal strategy are $R \equiv 0$ and $Z \equiv 1$, and these processes satisfy conditions (i) and (ii) in Definition 4.1. Substituting in (4.4), we get

$$S_t = -\frac{1}{\theta}\log(-\theta V_0^A) - b_0 - [\mu_0 - \frac{1}{2}\theta\sigma^2]t + \lambda \int_0^t Y_s\, ds,$$

obtaining the contract in (4.1), as claimed.

It remains to verify that $F(c)$ is indeed the value function. For that, we need that the stochastic integral in the expression $\int_0^t F'(C_t) dC_t$, that is $-\int_0^t \sigma Z_t dW_t$, is a true martingale, for all admissible Z. This is true by the assumption of square-integrability of Z in Definition 4.1 (i). Finally, we need the transversality condition $\lim_{T\to} \mathbf{E}[e^{-\rho T} F(C_T)] = 0$. This corresponds precisely to the transversality condition in Definition 4.1 (iii). Thus, our solution F to the HJB equation is the principal's value function, and $R \equiv 0, Z \equiv 1$ are optimal. □

4.2 Example II: Private Taste Shock

Here, we revisit the second example in [Wil11]. For some $\theta > 0$, we let

$$u(s,x) = -\rho e^{-x}\frac{s^{-\theta}}{\theta}.$$

For this example, [Wil11] only considers the case $\lambda = 0$, and so do we. Notice that we can write

$$u(s,x) = -\frac{\rho}{\theta}e^{-x-\theta\log(s)} = -\frac{\rho}{\theta}e^{-\theta(x/\theta+\log(s))}.$$

Thus, as far as the agent's problem goes, this is the same as the example before, except we work with

$$B'_t = B_t/\theta \qquad \text{and} \qquad S'_t = \log(S_t).$$

Therefore, we say that contract process S is admissible, if S' is admissible in the sense of Definition 4.1, with Y replaced by Y/θ. In particular,

$$\log(S_t)+\frac{1}{\theta}Y_t+R_t = c_0+\frac{1}{\theta}\int_0^t Z_s dY_s+\int_0^t\left(\frac{\rho}{\theta}[e^{\theta R_s}-1]-\frac{1}{\theta}\mu_0 Z_s+\frac{1}{2}\sigma^2 Z_s^2+\frac{1}{\theta}\lambda Z_s Y_s\right)ds, \tag{4.12}$$

for some constant c_0.

The principal's objective is the same as before. As in the first example, the result is basically the same as in [Wil11], only obtained almost directly from the previous example.

Proposition 4.6. *Suppose Condition 2.1 holds and assume $\lambda = 0$. Suppose, moreover, that the agent may misreport by any adapted process $m_t \le 0$ a.s.. Then, the contract S from (4.16) with $Z \equiv 1$, $R \equiv 0$ and c_0 such that $s_0 = (-\theta e^{b_0}V_0^A)^{-\frac{1}{\theta}}$ is optimal for the principal among all admissible contracts that deliver V_0^A to the agent.*

Proof. As before, we prove it first under Condition 2.4, and then we note that the agent is indifferent with respect to the contract, thus Condition 2.4 is not needed.

Considering only the contracts for which the agent chooses $\Delta \equiv 0$, the principal computes

$$\log(S_t) = \log(s_0)+\frac{1}{\theta}\int_0^t (Z_s-1)dY_s-\frac{1}{\theta}\int_0^t[\mu_0 Z_s-\frac{1}{2}\sigma^2 Z_s^2]ds+\int_0^t\frac{\rho}{\theta}[e^{\theta R_s}-1]ds.$$

The last term is again minimized by $R \equiv 0$. Substituting for that value and for dY (with $\Delta \equiv 0$), the principal's problem becomes a familiar stochastic control problem for which it is well-known that the optimal Z is deterministic and therefore obtained by solving

$$\inf_{z\ge 1}\left[\frac{1}{2}(z-1)^2(\sigma/\theta)^2-\mu_0/\theta+\frac{1}{2}\sigma^2 z^2/\theta\right].$$

This is minimized for $z = 1$. It is straightforward to check that the agent is indifferent given the contract, and that, with $x = b_0$, the agent's value is $-\frac{1}{\theta}e^{-x}s_0^{-\theta} = V_0^A$.

□

It is easy to compute that the principal's value is

$$s_0 + \frac{1}{\rho\theta}[\mu_0 - \frac{1}{2}\sigma^2] = (-\theta e^x V_0^A)^{-\frac{1}{\theta}} + \frac{1}{\rho\theta}[\mu_0 - \frac{1}{2}\sigma^2].$$

This is the same as the optimal value in [Wil11], accounting for the assumption in that paper (for this example) that $\mu_0 = \sigma^2/2$, that we have a factor ρ in principal's objective, and that, in [Wil11] notation, θ is replaced by $\theta - 1$.

4.3 Arguing necessity of $Z \geq 1$

In this subsection we want to prove that, for the contract represented by (4.4) to be incentive compatible, we necessarily need to have $Z \geq 1$. In order to do that, we adopt the weak formulation, standard in contract theory.

We take $(\Omega, \mathcal{F}, \mathbf{P})$ to be $\Omega = \mathcal{C}([0,\infty);\mathbb{R})$ with Borel σ-algebra $\mathcal{F}$ and probability measure $\mathbf{P}^0$ defined as the unique law of the strong solution of

$$dX_t = (\mu_0 - \lambda X_t)dt + \sigma dW_t, \quad X_0 = x_0, \tag{4.13}$$

for a Brownian motion W. We denote the canonical process by $Y : [0,\infty) \times \Omega \to \mathbb{R}$,

$$(t,\omega) \mapsto Y_t(\omega) := \omega_t,$$

and the canonical filtration, completed and closed from the right, by $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$. These conditions imply in particular that the process W_t^0 satisfying $dW_t^0 := \sigma^{-1} dY_t - \sigma^{-1}(\mu_0 - \lambda Y_t)\,dt$ is a Brownian motion under $\mathbf{P}^0$.

Denote

$$L_t^\Delta := e^{\frac{1}{\sigma}\int_0^t \left(\Delta_s + \lambda m_s^\Delta\right) dW_s^0 - \frac{1}{2\sigma^2}\int_0^t \left(\Delta_s + \lambda m_s^\Delta\right)^2 ds}. \tag{4.14}$$

We define admissible strategies Δ analogously as before, but we add the constraint that, for all $T > 0$,

$$\int_0^T (m_s^\Delta)^2\,ds < \infty\ , \qquad \mathbf{P}\text{-almost surely.}$$

In the standard approach with weak formulation, the choice of a reporting strategy is then modeled in terms of a change of measure. That is, the agent reports the realized path y of the process Y_t under the law induced by a given misreporting strategy. The Girsanov change of measure is applied on interval $[0,T]$ using $L^\Delta(T)$ as the Radon-Nikodym derivative. However, instead of changing the measure, we change the definition of the agent's objective (as in Hu and Schweizer 2011, [HS11]), and assume that the agent maximizes[7]

$$J^A(S;\Delta) := \lim_{T\to\infty} \mathbf{E}^0 \left[\int_0^T e^{-\rho s} L_s^\Delta u(S_s + Y_s - m_s^\Delta)\,ds \right]. \tag{4.15}$$

Note that with exponential utility function $u(x)$ this limit always exists.

We now define admissible contracts S in weak formulation.

[7]Under technical conditions, this is equivalent to changing the measure.

Definition 4.7. We call S an *admissible contract process*, if it can be represented as

$$S_t = -\frac{1}{\theta}\log(-\theta v_0) - Y_t - R_t + \int_0^t Z_s dY_s + \int_0^t \left(\frac{\rho}{\theta}[e^{\theta R_s} - 1] - \mu_0 Z_s + \frac{1}{2}\theta\sigma^2 Z_s^2 + \lambda Z_s Y_s\right) ds, \tag{4.16}$$

for some constant $v_0 < 0$, such that:

(i) Z is $\mathbb{F}$-predictable, and $\int_0^t |Z_s|^2\, ds < \infty$ for all $t > 0$;

(ii) R is $\mathbb{F}$-predictable continuous semimartingale with $R_0 = 0$, and $\int_0^t e^{\theta R_s} ds < \infty$ for all $t > 0$.

(iii) Given $n > 0$, introduce the set

$$A(n) := \left\{(t,\omega)\ :\ Z_t(\omega) \leq 1 - \frac{1}{n}\right\}.$$

and the strategies[8]

$$\tilde{\Delta}_t^n := \mathbb{1}_{A(n)} \frac{1}{Z_t - 1}\left(e^{-nt} - \mathbb{1}_{\{Z_t \geq 0\}} \tilde{m}_t^n \lambda Z_t\right),$$

$$\Delta_t^0 \equiv 0\ .$$

Also, define

$$\bar{V}_t^A(\Delta) := -\frac{1}{\theta} e^{-\theta(S_t + Y_t - m_t^\Delta + R_t)}.$$

We assume that

(a) for all $T > 0$ and $n > 0$, and for $\Delta = \tilde{\Delta}^n, \Delta^0$,

$$\mathbf{E}^0\left[\int_0^T e^{-\rho t} L_t^\Delta \bar{V}_t^A(\Delta)(\sigma Z_t - \Delta_t/\sigma - \lambda m_t^\Delta/\sigma) dW_t^0\right] = 0$$

and

(b) the following transversality condition holds for Δ^0:

$$\lim_{T\to\infty} \mathbf{E}^0\left[e^{-\rho T}\bar{V}_T^A(\Delta^0)\right] = 0\ . \tag{4.17}$$

Remark 4.8.

(i) Note that in the weak formulation nothing in the representation (4.16) of admissible S depends on Δ – everything is a functional of the canonical process Y. Thus, Z and R do not change their values $Z(\omega), R(\omega)$ when changing Δ.

(ii) Condition (a) is satisfied if the stochastic integral under the expectation is a martingale rather than just a local martingale. It is satisfied if, for example, Z and R are bounded almost everywhere on $[0,T]$, for all $T > 0$.

[8] Note that $\tilde{\Delta}_t^n + \lambda m_t^{\tilde{\Delta}^n}$ is bounded on the finite interval $[0,T]$, so that $\tilde{\Delta}^n$ is admissible.

Proposition 4.9. *Consider an admissible contract process S with the representation (4.16). Suppose that $\Delta^0 \equiv 0$ is optimal for the agent. Then, it is necessary that $Z \geq 1$ on a set of full measure.*

Proof. Define

$$\bar{M}_t^{\Delta} := e^{-\rho t} L_t^{\Delta} \bar{V}_t^A(\Delta) - \frac{\rho}{\theta} \int_0^t e^{-\rho s} L_s^{\Delta} e^{-\theta(S_s + Y_s - m_s^{\Delta})} ds. \tag{4.18}$$

We get, by Ito's rule,

$$-\frac{d\bar{V}_t^A}{\theta \bar{V}_t^A} = \sigma Z_t dW_t^0 + \left(\frac{\rho}{\theta}[e^{\theta R_t} - 1] - \Delta_t \right) dt,$$

$$\frac{dL_t^{\Delta}}{L_t^{\Delta}} = \frac{1}{\sigma}(\Delta_t + \lambda m_t^{\Delta}) dW_t^0,$$

and thus

$$\begin{aligned} -\frac{d\bar{M}_t^{\Delta}}{\theta e^{-\rho t} L_t^{\Delta} \bar{V}_t^A(\Delta)} &= -\frac{1}{e^{-\rho t} L_t^{\Delta} \theta \bar{V}_t^A} d\left(e^{-\rho t} L_t^{\Delta} \bar{V}_t^A \right) + \frac{\rho e^{-\theta\left(S_t + Y_t - m_t^{\Delta}\right)}}{\theta^2 \bar{V}_t^A} dt \\ &= (\sigma Z_t - \frac{\Delta_t}{\sigma} - \frac{\lambda m_t^{\Delta}}{\sigma}) dW_t^0 + \left\{ \Delta_t (Z_t - 1) + \lambda Z_t m_t^{\Delta} \right\} dt. \end{aligned} \tag{4.19}$$

Then, we have

$$\begin{aligned} F(T, \Delta) &:= -\frac{\rho}{\theta} \int_0^T e^{-\rho s} L_s^{\Delta} e^{-\theta(S_s + Y_s - m_s^{\Delta})} ds = \bar{M}_T^{\Delta} - e^{-\rho T} L_T^{\Delta} \bar{V}_T^A(\Delta) \\ &= \bar{M}_0^{\Delta} - e^{-\rho T} L_T^{\Delta} \bar{V}_T^A(\Delta) - \int_0^T \theta e^{-\rho t} L_t^{\Delta} \bar{V}_t^A(\Delta) \left\{ \Delta_t (Z_t - 1) + \lambda Z_t m_t^{\Delta} \right\} dt \\ &\quad - \int_0^T \theta e^{-\rho t} L_t^{\Delta} \bar{V}_t^A(\Delta) (\sigma Z_t - \Delta_t/\sigma - \lambda m_t^{\Delta}/\sigma) dW_t^0. \end{aligned}$$

For a fixed $n > 0$, consider the strategy $\tilde{\Delta} := \tilde{\Delta}^n$ from Definition 4.7, and denote $\tilde{L} := L^{\tilde{\Delta}}$, $\tilde{m}_t := \int_0^t \tilde{\Delta}_s \, ds$. By assumption (a) in Definition 4.7, we have, since $\bar{M}_0^{\Delta^0} = \bar{M}_0^{\tilde{\Delta}}$,

$$\begin{aligned} &\mathbf{E}^0[F(T, \Delta^0) - F(T, \tilde{\Delta})] \\ &\leq \theta \mathbf{E}^0 \left[\int_0^T e^{-(\rho+n)s} \tilde{L}_s \bar{V}_s^A(\tilde{\Delta}) \mathbb{1}_{\{(s,\omega) \in A(n)\}} ds \right] + e^{-\rho T} \left(\mathbf{E}^0 \left[\tilde{L}_T \bar{V}_T^A(\tilde{\Delta}) - \bar{V}_T^A(\Delta^0) \right] \right) \\ &\leq \theta \mathbf{E}^0 \left[\int_0^T e^{-(\rho+n)s} \tilde{L}_s \bar{V}_s^A(\tilde{\Delta}) \mathbb{1}_{\{(s,\omega) \in A(n)\}} ds \right] - e^{-\rho T} \mathbf{E}^0 \left[\bar{V}_T^A(\Delta^0) \right]. \end{aligned}$$

From the transversality condition (b) in Definition 4.7, we have that the last term converges to zero as $T \to \infty$. If $\Delta^0 \equiv 0$ is optimal, the above expression is non-negative. We get

$$0 \leq \lim_{T \to \infty} \mathbf{E}^0 \left[\theta \int_0^T e^{-(\rho+n)s} \tilde{L}_s \bar{V}_s^A(\tilde{\Delta}) \mathbb{1}_{\{(s,\omega) \in A(n)\}} ds \right].$$

Using the monotone convergence theorem, we get that the $dt \otimes d\mathbf{P}^0$ measure of $A(n)$ must be zero. It follows that $A(n)^c$ is a full measure set, and so is $\bigcap_{n \in \mathbb{N}} A(n)^c$, from which we conclude $Z \geq 1$ a.e.. □

5 Ways the agent can misreport

In this section we discuss, somewhat informally, what kind of reports by the agent could be considered credible by the principal. In [Wil11] and in this paper, we assume that the only way the agent can lie is with a reporting process Y that equals the true state process plus a non-positive differentiable drift. In general, we can ask the following question:

For a reported path $y = (y_t)_{t\in[0,\infty)}$ to be accepted by the principal as credible, what are the properties it has to satisfy?

The answer is not obvious. For example, since the diffusion process (2.1) has continuous paths almost surely, we imposed Condition 2.1. However, the qualifier 'almost sure' is relevant in this context, for – depending on the space on which the diffusion is defined – a discontinuous path could actually happen, and some principals might be willing to accept even such reports. Nevertheless, it is reasonable that the principal insists on the reported path y having properties that are known to hold for almost all paths of B. This can be rephrased as saying that the principal fixes a set of paths deemed as unlikely to happen and thus deemed as lies if reported by the agent:

The principal fixes a Borel measurable set $N \subset \mathcal{C}([0,\infty), \mathbb{R})$, with the property that $\mathbf{P}[B \in N] = 0$. Then, if the agents reports $y \in N$, the principal considers it a lie.

The set N could, for instance, include all differentiable paths, for Brownian motion is almost surely nowhere differentiable. It could also include paths which do not have quadratic variation process equal to $\sigma^2 t$, because quadratic variation can be calculated pathwise; see e.g. [Kar83].

On the other hand, no continuous $\mathbb{L}^2$ drift can be recognized as a lie in the above sense. Indeed, for every $t < \infty$, the law of B in (2.1) is equivalent to the law of $\sigma W_.$ on $\mathcal{C}([0,t], \mathbb{R})$. This follows from Cameron-Martin-Girsanov theory, see e.g. [KS, Ch. 3.5]. Moreover, if $A \subset \mathcal{C}([0,t], \mathbb{R})$ is measurable, and such that $\mathbf{P}[\sigma W_. \in A] > 0$, then also $\mathbf{P}[\sigma W_. \in A_\mu] > 0$ for any absolutely continuous $\mu \in \mathbb{L}^2([0,t])$, where $A_\mu := \{s \mapsto w_s + \mu_s \ : \ w \in A\}$ is the set of path translated by μ. Thus, if $N_t \subset \mathcal{C}([0,t], \mathbb{R})$ is a null set, then set $A = N^c$ has full measure, and so does A_μ, so that reporting an additional drift μ to any report that is not considered a sure lie, is not a sure lie. The same conclusion holds also for B.

We leave for further research studying these issues, and, in particular, the following question: What is the minimal set of restrictions that are reasonable to assume on the reported process, while still having interesting examples, that is, examples in which the optimal contract is not such that the agent is indifferent with respect to how much to misreport?

6 Conclusions

In this paper, we show that the optimal contract provided in [Wil11] is not incentive compatible in the case of persistent shocks, under the assumptions of that paper. It becomes incentive compatible if the difference between the reported process and the true process is sufficiently bounded. In case of permanent shocks, the contract in [Wil11] is, as stated in that paper, incentive compatible and optimal. For that case, we find, using a simpler approach, the same optimal contract, and we write it in a somewhat simpler form. We show that a contract of the similar form is optimal in the case of persistent shocks. We leave for future research the question on what types of misreporting would be credible in continuous-time models driven by Brownian motion, while still resulting in non-trivial optimal contracts.

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