A note on persistent private information*

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Abstract

We study the contracting problem in the persistent private information model of Williams [Wil11], in which an agent provides a report of a privately observed path to the principal, who in turn pays the agent in the least expensive way that induces truthful reporting. We first argue that, in the case of persistent information, the contract in [Wil11] does not induce truthful reporting if misreporting is allowed to grow sufficiently fast. The contract becomes incentive compatible (i.e., induces truthful reporting) if one imposes additional restrictions on misreporting, as shown also in Bloedel et al [BKS22] under different conditions. Under our restrictions, we show that the contract identified in [BKS22] is optimal among linear contracts. On the other hand, if additional restrictions are not imposed, we show that the contract optimal in a family of generalized linear contracts is deterministic.

Keywords: Principal-Agent problem, persistent information, truthful reporting, incentive-compatible contracts

JEL classification: D86, G00, G30, G35

1 Introduction

A classical problem in economics in general, and corporate finance and executive compensation in particular, is how to compensate an economic agent (manager) to report truthfully the information the agent has access to. The present paper is inspired by Williams [Wil11], the first paper to consider a continuous-time model of such a principal-agent problem, in which the principal's objective is to incentivize the agent to report truthfully the values of a process B observed by the agent. We also complement and build on the recent paper [BKS22] by Bloedel, Krishna, and Strulovici.[1] Work that considers

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[1] After we completed the first draft of this paper and made it publicly available, the existence of the paper [BKS22] was pointed out to us (by its authors). The paper, that was already under revision at that time, provides a critique and extensions of [Wil11]. Below we provide a comparison of our results to the results of that [BKS22].

similar problems in continuous-time models includes Prat and Jovanovic [PJ14] and He et al. [HWYG17]. In the model of those two papers, there is a drift component unobserved by both the principal and the agent, and they learn about it over time. All these papers consider the problem for the agent with CARA utility, and the risk-neutral principal. Mathematically, they encounter similar difficulties: (i) there is an additional state variable that depends on the cumulative agent's action, unobserved by the principal; (ii) there are technical issues in the usual weak formulation of the problem, due to the infinite horizon. These complicate the usual first-order approach, applied, for example, in Sannikov [San08], Cvitanić et al. [CPT17], [CPT18], and Cvitanić and Xing [CX18]. The first-order condition now depends additionally on the expected value of a functional of the future action. Still, [Wil11], [PJ14] and [HWYG17] provide the first-order necessary conditions for the optimal contract, and also conditions which are not necessary, but are sufficient, together with the first-order conditions, for a contract to be optimal.

In [Wil11] and [BKS22], as well as in the current paper, the principal minimizes the expected cost among the contracts that are incentive compatible (henceforth IC), under which the agent would report truthfully. In its main example (Section 6), [Wil11] provides an "optimal" contract, henceforth called contract W, and derives the main economic insights based on such a contract. We show that, if the agent is allowed to under-report with unbounded deviations, contract W is not incentive compatible in the most interesting case in which the persistency parameter λ (the rate of mean reversion) is strictly positive. Hence, it cannot be optimal, as it is outside of the set of admissible contracts.[2] We also show that the contract in [Wil11] becomes incentive compatible if the difference between the misreported process and the true process is constrained to remain sufficiently bounded.

In [Wil11], the agent's report y (realization of a process Y) for the realized path b of B has to be such that $y \leq b$ (i.e., $Y \leq B$). The paper justifies this by stating that, at least in some applications, one can imagine that the agent deposits the value y in an account, and cannot deposit more than the actual value b. However, in addition to $Y \leq B$, [Wil11] assumes much more, i.e., that

$$m_t := Y_t - B_t = \int_0^t \Delta_s ds \ , \quad \text{with} \quad \Delta_s \leq 0. \tag{1.1}$$

This assumption not only requires that $Y - B$ is non-positive, but also that it is differentiable, with the rate of change Δ non-positive. Our results do not require this assumption. In fact, we do not even apply the weak formulation (in which the process Δ represents a Girsanov change of measure), used in contract theory mainly for tractability – we are able to obtain our results in the strong formulation.[3]

Comparison to [BKS22]**.** The main differences between the present note and [BKS22] are: (a) we study in more detail the case of unbounded under-reporting; (b) we consider the family of linear contracts that is more general than the family of so-called self-insurance contracts (SIC) that [BKS22] consider; (c) [BKS22] also studies contracting in the case where the agent has a hidden savings account, which we do not consider. We provide a more detailed comparison next.

[2]We explain in the main text where the verification of incentive compatibility in [Wil11] goes wrong.

[3]This is, however, possible because, as [Wil11] and [BKS22], we do not find the optimal contract in the general family of contracts, but in a restricted family. Studying the former, a much harder problem, would require the weak formulation, and using it, [Wil11] does get some additional results in this direction.

Assumptions:

1. We consider separately the case with no assumptions on the growth of the misreporting process m . In the other case, in which we do impose restrictions on the growth of misreporting, our restrictions are different from the so-called no-Ponzi condition of [BKS22]. (Paper [Wil11] imposes no restrictions, at least not explicitly.)

Results (for the case $\lambda > 0$):

2. With no restrictions on the growth of misreporting, we show that contract W and other non-deterministic linear contracts are not IC, and we identify conditions under which linear contracts are IC. [BKS22] also shows that contract W is not IC if no additional assumptions are imposed, and that it is IC under their no-Ponzi condition, different from our conditions. The authors also show that, even under additional assumptions, contract W is not optimal.

3. [BKS22] finds the contract which is optimal among the self-insurance contracts, and shows that this contract is strictly better than contract W (which can also be implemented as a self-insurance contract). We show that the contract in [BKS22] is also optimal in the larger family of linear contracts, which includes the self-insurance contracts. When $\lambda > 0$, none of the three papers is able to find the optimal contract in the general family of contracts.

4. [BKS22] contains other results and interpretations. In particular, they show that contract W becomes optimal if the agent is allowed to secretly save at the interest rate equal to the discount rate. Moreover, in the so-called "permanent shocks" case, corresponding to the persistency parameter $\lambda = 0$, they find the optimal contract in the model with no savings account, which in this case is deterministic and, in fact, the same as in [Wil11].

Discussion:

The question arises which restrictions on the misreporting process are more natural. This is important, because the optimal (linear) contract is very different under different assumptions – in one case it depends on the reported path, and in the other it is deterministic. One could argue that in some applications the restriction that m has bounded growth is natural, in others that it is not. For example, it may be reasonable to assume that when the reported process becomes too different from the actual process, the principal will realize that the agent is not telling the truth. However, it may be less realistic to assume that there is an exact value of misreporting at which the principal will realize this, which corresponds to a sufficient condition we identify for a non-deterministic IC contract to be optimal to exist. In [BKS22] an asymptotic growth condition is imposed (the no-Ponzi condition), and the principal would only know at $t = \infty$ whether the condition has been violated, which is also not ideal.

Organization of the paper. Section 2 recalls the persistent private information reporting problem from [Wil11] and presents the technical assumptions needed to rigorously pose the problem. In Section 3 we present our counterexample to the contract claimed to be incentive compatible and optimal in [Wil11], and we elaborate on reasons why the argument presented there is erroneous under our assumptions. In Section 4, we show that, with additional restrictions on m, the optimal linear contract is deterministic. Moreover, we show that, under growth conditions on m, the optimal self-insurance contract is IC, and it is optimal among all linear contracts, not just among self-insurance contracts. Section

5 provides a discussion and conclusions.

Notation and Conventions. For a process X, we denote by $\mathbb{F}^X$ the raw filtration generated by X. With $\mathcal{C}(I; A)$ we denote the set of continuous functions $f : I \to A$, equipped with the topology of uniform convergence on compact sets.

2 The model and contracting problem

We recall first the framework of [Wil11]. We fix a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ supporting a Brownian motion W with its usual filtration $\mathbb{F}^W = \mathbb{F} = (\mathcal{F}_t : t \geq 0)$. The agent observes the realization of the process B, whose evolution is given by

$$dB_t = (\mu_0 - \lambda B_t)dt + \sigma dW_t, \quad B_0 = b_0, \tag{2.1}$$

where $\mu_0 \in \mathbb{R}, \lambda \geq 0, \sigma > 0$ and $b_0 \in \mathbb{R}$. We call λ the persistency parameter. If $\lambda > 0$, we say that information is persistent.

The principal knows the dynamics of the process B, that is, she knows μ_0, λ, σ and b_0 in (2.1), but does not observe the realization of B. To elicit B, she offers a contract to the agent, to receive a report on the observed realization. More formally, the agent chooses a *reporting strategy*

$$Y : [0, \infty) \times \Omega \to \mathbb{R}$$

which is an RCLL process adapted to $\mathbb{F}^B$. One easily verifies that up to completion and closure from the right, $\mathbb{F}^B = \mathbb{F}$. We shall thus assume that Y is actually $\mathbb{F}$-adapted.

Contracts

For a fixed reporting strategy Y, we denote by $y = (y_t)_{t \in [0,\infty)}$ its realization, which is the path reported to the principal by the agent as the realization $b = (b_t)_{t \in [0,\infty)}$ of B. We use $y_{s \leq t}$ as a shorthand for $(y_s)_{s \in [0,t]}$, the reported path up to time t. In exchange for the report, the principal pays the agent a salary $s = (s_t)_{t \in [0,\infty)}$ continuously in time. Those payments depend, in an adapted way, only on the path revealed by the agent. This means that the principal chooses a jointly Borel measurable function $s : [0, \infty) \times \mathcal{C}([0, \infty), \mathbb{R}) \to \mathbb{R}$ that satisfies

$$s_t(y) = s_t\left(y_{s \leq t}\right), \quad \text{for all } t \in [0, \infty).$$

We call $s = (s_t(y))_{t \in [0,\infty)}$ a *contract*.

Decision problems

As is standard in contract theory, the principal proposes a contract to the agent. Then, the agent may either accept or reject the contract. Once accepted, the agent continuously makes the report y_t to the principal and in turn receives the contractual compensation $s_t(y)$ from the principal. The principal, anticipating the agent's choices, will offer only contracts that maximize her objective, as we describe now.

For a given contract s, the agent aims to maximize

$$J_0(s(Y), B) := \mathbf{E}\left[\int_0^\infty e^{-\rho t} u(s_t(Y), B_t)\, dt\right] \tag{2.2}$$

over reporting strategies Y, where u is the flow utility function, and $\rho > 0$ is the discount factor. We call any strategy Y^s that maximizes (2.2), if it exists, an *optimal report* for s.

Definition 2.1. A contract s is said to satisfy the

- *Truth Revelation Constraint*, if $Y^s = B$ is an optimal reporting strategy;
- *Participation Constraint*, if
$$J_0 \left(s(B), B \right) \geq v_0,$$
for a given reservation utility v_0 of the agent.

We call a contract that satisfies the truth revelation constraint *incentive compatible* (IC).

As usual in contract theory, it is assumed that, if indifferent, the agent will report truthfully. Having formalized the agent's problem, we will consider the principal's problem of minimizing

$$\mathbf{E}\left[\int_0^\infty e^{-\rho t} s_t(Y^s) dt\right] \tag{2.3}$$

over contracts s that are incentive compatible and linear (as defined later). The problem of minimizing over a subset of more general contracts (possibly nonlinear and path-dependent) seems not to have a solution that can be nicely characterized, and is outside of the scope of this paper.

Denote the misreporting process by

$$m_t := Y_t - B_t. \tag{2.4}$$

Definition 2.2. We say that a misreporting strategy m is admissible if $Y = m + B$ is a reporting strategy, that is, if it is an RCLL $\mathbb{F}$-adapted process, and if

$$\int_0^t m_s ds < \infty \ \text{ for all } t \geq 0 \ , \mathbb{P} - a.s.$$

Remark 2.3. In [Wil11] the following condition is also imposed on reporting strategies:

$$\text{The agent can only under-report: } Y \leq B. \tag{2.5}$$

The justification provided in [Wil11] is that for some applications one can envision that the agent deposits the reported values y_t in a savings account observable by the principal, and is unable to deposit more than the actual values b_t. However, for many applications this may not be a valid restriction.[4] [Wil11] assumes in addition that the misreporting process is of the form

$$m_t = \int_0^t \Delta_s ds \ , \ \text{for some } \mathbb{F}\text{-adapted process } \ \Delta \leq 0. \tag{2.6}$$

Clearly, this requirement is stronger than (2.5).[5]

[4] In Section 5 below we discuss various possible restrictions on reporting strategies.

[5] The requirement that m_t is differentiable is justified in [Wil11] by saying that "the agent's report Y must be absolutely continuous with respect to his true state B". Indeed, in theory, it could be argued that, given the reported path, the principal could keep performing, continuously in time, (uncountably) many tests for the likelihood ratio $\frac{dP^B}{0.5(dP^B+dQ)}$, where P^B is the law of the true process B, for all possible alternative laws Q. For a path that comes from a law that is singular with respect to P^B, there would be a positive probability that the tests would reject P^B in favor of some singular Q in finite time. (We thank Martin Larsson for this observation.) In practice, this would, of course, be very hard to do.

Neither of the above assumptions, that m_t is non-positive, and that it is differentiable, are needed in this paper, as we shall see. $\diamond$

3 A Counterexample to Incentive Compatibility

Before studying linear contracts, we show that the contract in the main example of [Wil11, Sec. 6] is not incentive compatible, if misreporting is not bounded from below.[6] In the example, the agent's expected utility is

$$\mathbf{E}\left[\int_0^\infty e^{-\rho t} u(s_t(Y) + B_t)dt\right],$$

where

$$u(x) = -e^{-\theta x},$$

for some parameter $\theta > 0$. We consider here the persistent case $\lambda > 0$. [7]

Consider the contract

$$S_t := s_t(Y) = \alpha - \frac{1}{\theta}\log(-q_t) - Y_t, \tag{3.1}$$

from [Wil11, p. 1271]. Here α is an appropriate constant, q_t is the promised utility (i.e., value) process of the agent under truthful reporting, given by

$$dq_t = -\sigma\theta\beta q_t dW_t^Y \ ,$$

with W_t^Y given by

$$W_t^Y := \frac{1}{\sigma}\left(Y_t - b_0 - \int_0^t [\mu_0 - \lambda Y_s]ds\right), \tag{3.2}$$

and β is a constant. Note that the functional s in (3.1) satisfies the measurability restrictions required in the definition of a contract from Section 2. Contract (3.1) may be written as

$$S_t + Y_t = \tilde{\alpha} + \frac{1}{2}\theta\beta^2\sigma^2 t + \beta\sigma W_t^Y, \tag{3.3}$$

where $\tilde{\alpha}$ is an appropriate constant.

It is claimed in [Wil11] that, with

$$\beta = \frac{\rho}{\rho + \lambda}\ , \tag{3.4}$$

this contract is optimal and thus, in particular, incentive compatible. We now show that this is not the case if misreporting is not bounded from below, and we find a sufficient condition on admissible misreporting strategies for the contract to be IC. [BKS22] show that, even under conditions that guarantee that this contract is IC, it is actually not optimal.

[6] [Wil11] does not explicitly impose a lower bound on misreporting.

[7] In the case $\lambda = 0$, the contract in [Wil11, Sec. 6] makes the agent indifferent, and is thus incentive compatible.

Proposition 3.1. *Any contract that satisfies (3.3) for an arbitrary constant $\beta \in (0,1)$ and an arbitrary constant $\tilde{\alpha}$ is not incentive compatible, if we allow deterministic misreporting that is not bounded from below.*

The proof will show that contract (3.1) actually does not admit an optimal reporting strategy. Instead, under this contract, the higher the misreporting, the higher the agent's utility.

Proof. We will provide a counterexample with misreporting processes m, even if they are required to satisfy properties (2.5) and (2.6). Just as in [Wil11, p. 1272], equation (3.2) can be written as

$$\begin{aligned}\sigma dW^Y &= dB_t - [\mu_0 - \Delta_t - \lambda(B_t + m_t)]dt \\ &= \sigma dW_t + [\Delta_t + \lambda m_t]dt.\end{aligned}$$

Contract (3.3) may be written as

$$S_t + B_t = \tilde{\alpha} + \frac{1}{2}\theta\beta^2\sigma^2 t + \beta\sigma W_t + \lambda\beta\int_0^t m_s ds - (1-\beta)m_t, \tag{3.5}$$

for some constant $\tilde{\alpha}$. Take, for $k > 0$,

$$\Delta_t = -e^{kt}. \tag{3.6}$$

Then, the sum of the last two terms in the expression for $S_t + B_t$ above is

$$\lambda\beta\int_0^t m_s ds - (1-\beta)m_t = \frac{1}{k}\lambda\beta\int_0^t (1-e^{ks})ds - \frac{1}{k}(1-\beta)(1-e^{kt})$$

$$= \frac{1}{k}\lambda\beta\left[t + \frac{1}{k}(1-e^{kt})\right] - \frac{1}{k}(1-\beta)(1-e^{kt}) = \frac{1}{k}\lambda\beta t + \frac{1}{k}(1-e^{kt})\left[\frac{1}{k}\lambda\beta - (1-\beta)\right].$$

A sufficient condition for this to be larger than zero is

$$k \geq \frac{\lambda\beta}{1-\beta}.$$

Thus, for any such k, the agent would be better off using the corresponding strictly negative Δ instead of zero. In fact, he would like to set k as large as possible. More precisely, plugging it back to the agent's expected utility $\mathbf{E}[\int_0^\infty e^{-\rho t}u(S_t + B_t)dt]$, we see that the agent can get as close as he wants to its maximum value, equal to zero, by increasing k. □

Since $\lambda > 0$, we have $\beta < 1$ in (3.4), and it follows, in particular, that the contract in [Wil11, Sec. 6] is not incentive compatible.

An inspection of the direct verification of incentive compatibility in the appendix of [Wil11] reveals the following omission. On [Wil11, p. 1272] there is an HJB equation for the agent's value given the contract (3.3), with $\beta = \rho/(\rho + \lambda)$ and with W^Y expressed in terms of q. The solution to that equation is provided as

$$V(q,m) = \frac{q\exp(\theta m)(\rho + \lambda)}{\rho + \lambda + \theta\lambda m}. \tag{3.7}$$

It is in fact immediately seen that this solution cannot be the value function of the agent for all values of m: since $q < 0$, this value is positive for $m < -\frac{\rho+\lambda}{\theta\lambda}$, hence it cannot be the value function corresponding to the negative CARA utility. An essential step neglected in [Wil11] is the verification that the solution of the HJB is in fact the agent's value function. For one thing, on infinite horizon the solution should satisfy the following transversality condition: for all admissible strategies,

$$\lim_{T\to\infty} \mathbf{E}_{q_t,m_t}\left[e^{-\rho(T-t)}V(q_T, m_T)\right] = 0.$$

It can be checked that the above function does not satisfy this unless m never reaches $-\frac{\rho+\lambda}{\theta\lambda}$.

[Wil11] provides another argument to claim that the contract is incentive compatible, by checking that the conditions of [Wil11, Theorem 4.1] are satisfied. However, the proof of that theorem goes through for infinite horizon only if $\mathbf{E}[e^{-\rho T}p_T m_T]$ converges to zero as T goes to infinity, where, as the paper shows, the adjoint process p satisfies $p_T = \theta q_T$, with q_T as above. Then, it is easy to verify that we do not have the desired convergence if m ever goes below the value $-\frac{\rho+\lambda}{\theta\lambda}$. This is because then $e^{-\rho T}q_T/q_t$ converges to infinity, and so does $\mathbf{E}[e^{-\rho T}p_T m_T]$.

Remark 3.2. It will follow from the results of the following section that a contract of the form (3.5) with $\beta = \frac{\rho}{\rho+\lambda}$ becomes incentive compatible if we impose additional restrictions on admissible misreporting strategies. However, as shown in [BKS22], even under conditions under which that contract is IC, it is not optimal. ◇

4 Linear Contracts

We now restrict our attention to the family of linear contracts. We follow the tradition in the contracting literature of focusing on the contracts most observed in practice, i.e., linear contracts, when considering more general contracts is not tractable. For this, let β, C, k be constants, and $a(t)$ be a deterministic differentiable function with $a(0) = 0$. We consider the contracts s that satisfy

$$s_t(Y) + Y_t = C + a(t) + (k+1)(Y_t - Y_0) + \beta \int_0^t \lambda Y_s ds. \tag{4.1}$$

In what follows we will also use the function $\tilde{a}(t)$, given by

$$\tilde{a}(t) = a(t) + \beta\mu_0 t + [\lambda B_0 - \mu_0]\,\frac{1}{\lambda}(\beta - k - 1)(1 - e^{-\lambda t}). \tag{4.2}$$

Lemma 4.1. *Any contract s of the linear form in* (4.1) *satisfies*

$$s_t(B) + B_t$$
$$= C + a(t) + \beta\mu_0 t + [\lambda B_0 - \mu_0]\,\frac{1}{\lambda}(\beta - k - 1)(1 - e^{-\lambda t}) + \beta\sigma W_t - (\beta - k - 1)e^{-\lambda t}\int_0^t e^{\lambda s}\sigma dW_s.$$

Proof. We have

$$A_t := (k+1)(B_t - B_0) + \beta \int_0^t \lambda B_s ds$$

$$= (k+1)(B_t - B_0) - \beta(B_t - B_0 - \mu_0 t - \sigma W_t).$$

Using the fact that

$$B_t = e^{-\lambda t} B_0 + \mu_0 \int_0^t e^{-\lambda(t-s)} ds + \int_0^t e^{-\lambda(t-s)} \sigma dW_s, \tag{4.3}$$

we get

$$A_t = (k+1-\beta) \left[(e^{-\lambda t} - 1) B_0 + \mu_0 \frac{1}{\lambda}(1 - e^{-\lambda t}) + e^{-\lambda t} \int_0^t e^{\lambda s} \sigma dW_s \right] + \beta \mu_0 t + \beta \sigma W_t,$$

which concludes the proof. □

Using the function $\tilde{a}$ introduced in (4.2), we can write the utility of agent using a misreporting strategy m as

$$-e^{-\theta C} E \left[\int_0^\infty e^{-\rho t} e^{-\theta \left[\tilde{a}(t) + \beta \sigma W_t - (\beta - k - 1) e^{-\lambda t} \int_0^t e^{\lambda s} \sigma dW_s \right]} e^{-\theta \left[k m_t + \beta \int_0^t \lambda m_s ds \right]} dt \right]. \tag{4.4}$$

We consider the agent's problem as a stochastic control problem with the control process being the admissible misreporting strategy m. The agent's control problem is Markovian in the time variable t, and the following processes:

$$dW_t^w = dW_t \ , W_0^w = w;$$
$$dX_t^x = e^{\lambda t} \sigma dW_t \ , X_0^x = x;$$
$$dP_t^m = \lambda m_t dt \ , P_0^m = p.$$

We can write the agent's objective function as

$$J(t,p,w,x;m) = -e^{-\theta K} E_{t,p,w,x} \left[\int_t^\infty e^{-\rho v} e^{-\theta \left[\tilde{a}(v) + \beta \sigma W_v^w - (\beta - k - 1) e^{-\lambda v}) X_v \right]} e^{-\theta [k m_v + \beta P_v]} dv \right].$$

From this, we see that the HJB equation for the value function $V = V(t,p,w,x)$ is given by

$$0 = \sup_m \Big\{ -e^{-\rho t - \theta [C + \tilde{a}(t) + \beta \sigma w - (\beta - k - 1) e^{-\lambda t} x]} e^{-\theta(km + \beta p)} + \lambda m V_p \tag{4.5}$$
$$+ V_t + \frac{1}{2} V_{ww} + \frac{1}{2} \sigma^2 e^{2\lambda t} V_{xx} + \sigma e^{\lambda t} V_{xw} \Big\}.$$

In the remainder of the section, we assume that m is allowed to take values in an interval $(-\infty, N)$, with $0 \le N \le \infty$. Thus, zero misreporting is always allowed. We first consider the case with no additional restrictions on the admissible misreporting strategies, and then the case with additional restrictions. The proofs are postponed to Section 4.3.

4.1 No additional restrictions on misreporting

In this subsection, other than $m \leq N$, we impose no restrictions on the admissible processes m. When $N \in (0, \infty]$, we will show that the only IC contracts are the deterministic contracts; see Proposition 4.2. In the case $N = 0$, considered in [Wil11], even if non-deterministic IC contracts may exist, we will show that it is still the case that a deterministic contract is optimal; see Proposition 4.3.

Proposition 4.2. *Consider a linear contract of the form (4.1).*

(i) For the contract to be IC, it is necessary that either it is deterministic, that is,

$$\beta = k = 0 \ , \tag{4.6}$$

or it satisfies

$$\beta = k + 1. \tag{4.7}$$

Moreover, we have $\beta \geq 0$ and $k \geq 0$.

(ii) If $N > 0$, then the contract is IC iff $\beta = k = 0$.

We show next that, even if non-deterministic IC contracts may exist, a deterministic contract is still optimal in a more general family of linear contracts, regardless of whether $N > 0$ or $N = 0$.

Proposition 4.3. *Deterministic contract S^C given by*

$$S_t^C = C - e^{-\lambda t} b_0 - \frac{1}{\lambda}\mu_0(1 - e^{-\lambda t}) + \frac{\theta\sigma^2}{4\lambda}(1 - e^{-2\lambda t}), \tag{4.8}$$

where constant C is chosen so that the agent's participation constraint is satisfied as equality, is optimal among all IC contracts of the form (more general than the IC contracts of the form (4.1))

$$S_t = S_0 + \int_0^t k(s)dY_s + \int_0^t [a(s) + \gamma(s)S_s + \beta(s)\lambda Y_s]ds, \tag{4.9}$$

where a, k, γ, β are deterministic functions of time, and where $k \geq 0$ and $\beta \geq 0$.

4.2 Additional restrictions on misreporting, that allow non-deterministic IC contracts

In this section, we show that additional restrictions on admissible misreporting can be added to guarantee that (specific) linear contracts of the form (4.1) with $\beta = k+1$ are IC. We also show that the contract that is optimal among the linear IC contracts is superior to the deterministic contracts.

Denote

$$r = \frac{\lambda\beta}{1-\beta} \ . \tag{4.10}$$

It follows from (4.19) in the proof of Proposition 4.2 that, if $\beta = k + 1$ (which necessarily holds unless $\beta = k = 0$) process $\tilde{a}$ is uniquely determined and given by

$$\tilde{a}(t) = \frac{1}{\theta}r - \frac{\rho}{\theta}t + \frac{1}{2}\theta\sigma^2\beta^2 t. \tag{4.11}$$

Using this and expression (4.15) from the proof of Proposition 4.2, it is straightforward to verify that the agent's indirect utility function V^0 corresponding to $m = 0$, when offered the linear contract satisfying (4.7) and (4.11), is given by

$$V^0(t,p,w) = -\frac{1}{r}e^{-\theta\left(C+\beta p+\sigma\beta w+\frac{1}{2}\theta\sigma^2\beta^2 t\right)}. \tag{4.12}$$

We then obtain the following proposition.

Proposition 4.4. *Assume that the set of admissible strategies m is reduced to a subset for which*

(a) *V^0 satisfies the HJB equation (4.5) with $\beta = k+1$ and $\tilde{a}$ given by (4.11);*

(b) *the following transversality condition holds:*

$$\lim_{T\to\infty} e^{-\rho T}E[V^0(T, P_T^m, W_T^w)] = 0; \tag{4.13}$$

(c) *the following (consequence of) martingale property holds:*

$$E\left[\int_0^T e^{-\rho s}V_w^0(s, P_s^m, W_s^w)dW_s^w\right] = 0\ ,\ \forall\ T > 0\ . \tag{4.14}$$

Then, the linear contract of the form (4.1) satisfying (4.7) and (4.11) is IC, and V^0 is the value function of the agent.

Remark 4.5.

1. Linear contracts with $\beta = k+1$ were first introduced by [BKS22], from a different reasoning, as the contracts in which the principal offers to the agent a savings account with interest rate r given by (4.10), called self-insurance contracts, SIC's. They provide different sufficient conditions for the contract to be IC – the associated savings account process has to satisfy a transversality condition, called no-Ponzi condition. When the contracts are IC, [BKS22] describe how to find optimal $r = \hat{r}$. They also observe that the contract from [Wil11] corresponds to $r = \rho$ and show that $\hat{r} < \rho$, and hence contract [Wil11] is not optimal even among SIC's, thus also not optimal among the linear contracts.

2. Using Proposition 4.2, the principal's value for the optimal deterministic contract ($\beta = k = 0$) can be computed as

$$v^{P,0} = -\frac{1}{2}\theta\sigma^2\frac{1}{\rho(\rho+2\lambda)}.$$

The value corresponding to the contract in [Wil11], with $r = \rho$, and $\beta = k+1$, can be computed as

$$v^{P,\rho} = -\frac{1}{2}\theta\sigma^2\frac{1}{(\rho+\lambda)^2} > v^{P,0}.$$

Thus, that contract (and hence also the optimal linear contract) dominates the deterministic contracts, under the assumptions that guarantee that the contract is IC, as in this section.

3. A sufficient condition for (ii)-(a) and (ii)-(c) to hold is that admissible strategies satisfy $|m_t| \leq K$, a.s., for all $t > 0$, for some constant $K > 0$. Some sufficient conditions for (ii)-(b) to hold are given next, which basically say that m is sufficiently bounded

from below. More precisely, from (4.12) the transversality condition is, taking, wlog, $0 = w = p = t$,

$$0 = \lim_{T\to\infty} e^{-\rho T} E[V^0(T, P_T, W_T)]$$
$$= -\lim_{T\to\infty} \frac{1}{r} e^{-\rho T} E[e^{-\theta(K+\beta\int_0^T \lambda m_s ds)-\theta\sigma\beta W_T-\frac{1}{2}\theta^2\sigma^2\beta^2 T)}]$$

When $\beta > 0$ and m is allowed to be unbounded from below, we can find (deterministic) m's for which this limit will not equal zero. On the other hand, a sufficient condition for this to hold is, for some constant $c > 0$,

$$\rho + \theta\beta\frac{1}{T}\int_0^T \lambda m_s ds \geq c.$$

In particular, it holds if, for some $\epsilon > 0$,

$$m \geq -\frac{\rho}{\theta\lambda\beta} + \epsilon.$$

When $0 < \beta < 1$, as is the case for the self-insurance contracts, a sufficient condition is

$$m \geq -\frac{\rho}{\theta\lambda} + \epsilon.$$

◇

4.3 Proofs for Sections 4.1 and 4.2

Remark 4.6. It is possible to extend some of the proofs below to the case in which we allow β and k to be differentiable functions of time. In that case, one first shows that it is necessary for the contract to be IC that β and k be constant, after which the proofs below can be applied. ◇

Proof of Proposition 4.2. Fix an IC contract. The indirect utility function corresponding to $m = 0$ is, from (4.4),

$$V(t,p,w,x) \tag{4.15}$$
$$= -e^{-\theta C} E\left[\int_t^\infty e^{-\rho v-\theta\beta p-\theta\tilde{a}(v)} e^{-\theta\sigma\beta(w+W_v)} e^{\theta(\beta-k-1)e^{-\lambda v}(x+\int_t^v e^{\lambda s}\sigma dW_s)} dv\right]$$
$$= -e^{-\theta C}\int_t^\infty e^{-\rho v-\theta\beta p-\theta\left\{\tilde{a}(v)+\sigma\beta w-(\beta-k-1)xe^{-\lambda v}-\frac{1}{2}\theta\sigma^2\int_t^v\left[\beta-(\beta-k-1)e^{-\lambda(v-s)}\right]^2 ds\right\}} dv.$$

Since $m = 0$ is optimal among all admissible strategies m, it is optimal in the subset of strategies that also satisfy $-\epsilon \leq m \leq \epsilon$ for any given $0 < \epsilon < N$ when $N > 0$, or strategies that also satisfy $-\epsilon \leq m \leq 0$, when $N = 0$. Optimizing over this domain, one gets, from the usual stochastic control arguments using the dynamic programming principle when the set of control is bounded, that the corresponding HJB equation is a necessary condition for the value function (see, e.g., [Pha09]).

Note that the derivative with respect to m in the HJB equation is

$$D(t;m) = k\theta e^{-\theta[C+\tilde{a}(t)+\beta\sigma w-(\beta-k-1)e^{-\lambda t}x]} e^{-\theta(km+\beta p)} + \lambda V_p(t,p,w,x). \tag{4.16}$$

Denote

$$g(v,p,w,x) = -\rho v - \theta\Big\{\beta p + \tilde{a}(v) + \sigma\beta w - (\beta-k-1)e^{-\lambda v}x \tag{4.17}$$
$$-\frac{1}{2}\theta\sigma^2\int_0^v \Big[\beta - (\beta-k-1)e^{-\lambda(v-s)}\Big]^2 ds\Big\}.$$

For $m = 0$ to be optimal, we need to have $D(t;0) \geq 0$. That is, we have, for all (t,p,w,x),

$$ke^{g(t,p,w,x)} \geq -\lambda\beta\int_t^\infty e^{g(v,p,w,x)}dv.$$

This implies

$$k \geq -\lambda\int_t^\infty \beta e^{-\rho(v-t)}e^{-\theta\Big\{\tilde{a}(v)-\tilde{a}(t)+(\beta-k-1)[e^{-\lambda t}-e^{-\lambda v}]x-\frac{1}{2}\theta\sigma^2\int_t^v\big[\beta-(\beta-k-1)e^{-\lambda(v-s)}\big]^2ds\Big\}}dv.$$

Unless we have $\beta = 0$ or $\beta = k+1$, the right-hand side can be pushed to infinity, by sending x to minus or plus infinity. Notice also that when $\beta < 0$, we would need to have $k > 0$, which is not possible if $\beta = k+1$. Thus, $\beta \geq 0$.

We next show that necessarily $k \geq 0$. The agent's utility is, from (4.4),

$$J(m) := -e^{-\theta C}E\left[\int_0^\infty e^{-\rho t}e^{-\theta\left[\tilde{a}(t)+\beta\sigma W_t-(\beta-k-1)e^{-\lambda t}\int_0^t e^{\lambda s}\sigma dW_s\right]}e^{-\theta\left[km_t+\beta\int_0^t\lambda m_s ds\right]}dt\right].$$

Suppose $k < 0$, and introduce the (deterministic) strategy

$$m_t^A := \frac{1}{k}\int_0^t \left(1-\beta\lambda m_s^A\right)ds\ ,\quad m_0^A = 0.$$

Then, we have

$$km_t^A + \beta\int_0^t \lambda m_s^A ds = t\ .$$

Thus, strategy m^A is strictly better than $m \equiv 0$, and the contract is not IC. So, we need to have $k \geq 0$.

Suppose now $N > 0$ and $\beta = k+1$. Then, for $m \equiv 0$ to be optimal, we need to have $D(t;0) = 0$. This is equivalent to

$$(\beta-1)e^{-\rho t-\theta\tilde{a}(t)+\frac{1}{2}\theta^2\sigma^2\int_0^t\beta^2ds} = -\beta\lambda\int_t^\infty e^{-\rho v-\theta\tilde{a}(v)+\frac{1}{2}\theta^2\sigma^2\int_0^v\beta^2ds}dv. \tag{4.18}$$

Note that from this we conclude that β and $1-\beta$ have the same sign, so that $\beta \in (0,1)$. However, this implies $k < 0$, and the contract cannot be IC. Thus, we need to have $\beta = k = 0$.

We are now also in position to prove that necessarily (4.11) holds. Taking the derivative with respect to t in (4.18), we get

$$[\beta-1][-\rho-\theta\tilde{a}'(t)+\frac{1}{2}\theta^2\sigma^2\beta^2] = \beta\lambda.$$

From this, we get, since $\tilde{a}(0) = 0$,

$$\tilde{a}(t) = \frac{1}{\theta}\int_0^t\left\{-\frac{\lambda\beta}{\beta-1}-\rho+\frac{1}{2}\theta^2\sigma^2\beta^2\right\}ds. \tag{4.19}$$

$\square$

Proof of Proposition 4.3. For the contract satisfying (4.9), we have

$$e^{-\int_0^t \gamma(s)} S_t = S_0 + \int_0^t e^{-\int_0^s \gamma(u)du} k(s) dY_s + \int_0^t e^{-\int_0^s \gamma(u)du} [a(s) + \lambda\beta(s) Y_s] ds,$$

that is,

$$S_t = e^{\int_0^t \gamma(s)ds} S_0 + \int_0^t e^{\int_s^t \gamma(u)du} k(s) dB_s + \int_0^t e^{\int_s^t \gamma(u)du} k(s) dm_s + \int_0^t e^{\int_s^t \gamma(u)du} [a(s) + \lambda\beta(s)(B_s + m_s)] ds.$$

Assume the contract is IC (so that $m = 0$), and impose only the participation constraint. Then, the principal's problem is to minimize, with L being the Lagrange multiplier for that constraint,

$$\mathbf{E} \int_0^\infty e^{-\rho s} \left[-Lu(S_t + B_t) + S_t \right] dt$$

$$= \int_0^\infty e^{-\rho t} \Big[L\mathbf{E} e^{-\theta \left[B_t + e^{\int_0^t \gamma(s)ds} S_0 + \int_0^t e^{\int_s^t \gamma(u)du} k(s) dB_s + \int_0^t e^{\int_s^t \gamma(u)du} [a(s) + \lambda\beta(s) B_s] ds \right]}$$

$$+ e^{\int_0^t \gamma(s)ds} S_0 + \mathbf{E} \int_0^t e^{\int_s^t \gamma(u)du} k(s) dB_s + \mathbf{E} \int_0^t e^{\int_s^t \gamma(u)du} [a(s) + \lambda\beta(s) B_s] ds \Big].$$

Denote

$$f(t) = e^{\int_0^t \gamma(s)ds} S_0 + \int_0^t e^{\int_s^t \gamma(u)du} a(s) ds.$$

We can write the sum of expectations inside the integral as

$$L e^{-\theta[f(t) + G_t(\beta,k,\gamma)]} + f(t) + \mu_0 \int_0^t e^{\int_s^t \gamma(u)du} k(s) ds + \int_0^t e^{\int_s^t \gamma(u)du} [\lambda(\beta(s) - k(s)) \mathbf{E}[B(s)]] ds,$$

where G_t is defined by

$$e^{-\theta G_t(\beta,k,\gamma)} = \mathbf{E} \left[e^{-\theta \left[B_t + \int_0^t e^{\int_s^t \gamma(u)du} k(s)[\mu_0 ds + \sigma dW_s] + \int_0^t e^{\int_s^t \gamma(u)du} \lambda(\beta(s) - k(s)) B_s ds \right]} \right].$$

Fixing everything else, the necessary condition for optimal $f(t)$ gives

$$-\theta \left[f(t) + G_t(\beta, k, \gamma) \right] = -\log(L\theta).$$

It is easily checked that this is also a sufficient condition. In the case of the deterministic contracts, that is, with $k \equiv \beta \equiv 0$, using the fact that

$$B_t = e^{-\lambda t} b_0 + \mu_0 \int_0^t e^{-\lambda(t-s)} ds + \int_0^t e^{-\lambda(t-s)} \sigma dW_s \tag{4.20}$$

to compute $\mathbf{E}[e^{-\theta B_t}]$, it is a matter of simple computation to check that such f gives rise to the contract of the form S^C, where C is chosen so that the participation constraint is satisfied. (Note also that γ can be chosen to be zero.) Thus, for that C, S^C is optimal among deterministic contracts. Let us continue to prove that it is also optimal among all the IC contracts of the form (4.9) with $\beta \geq 0, k \geq 0$.

Using the optimal $f(t)$, minimizing what is inside of the integral means minimizing

$$\frac{1}{\theta}[1+\log(L\theta)]-G_t(\beta,k,\gamma)+\mu_0\int_0^t e^{\int_s^t\gamma(u)du}k(s)ds+\int_0^t e^{\int_s^t\gamma(u)du}[\lambda(\beta(s)-k(s))\mathbf{E}[B(s)]]ds.$$

Now, we want to compute $G_t(\beta,k,\gamma)$. We can write

$$e^{-\theta G_t(\beta,k,\gamma)}=\mathbf{E}\left[e^{-\theta N_t}\right], \tag{4.21}$$

where N_t is a normally distributed random variable, given by

$$N_t := B_t+\int_0^t e^{\int_s^t\gamma(u)du}k(s)dB_s+\int_0^t e^{\int_s^t\gamma(u)du}\lambda\beta(s)B_sds$$

$$=B_0+\int_0^t\left\{e^{\int_s^t\gamma(u)du}k(s)+1\right\}dB_s+\int_0^t e^{\int_s^t\gamma(u)du}\lambda\beta(s)B_sds$$

$$=B_0+\int_0^t\left\{e^{\int_s^t\gamma(u)du}k(s)+1\right\}[\mu_0ds+\sigma dW_s]$$

$$+\lambda\int_0^t\left\{e^{\int_s^t\gamma(u)du}[\beta(s)-k(s)]-1\right\}\left[e^{-\lambda s}b_0+\int_0^s\mu_0e^{-\lambda(s-u)}du+\int_0^s e^{-\lambda(s-u)}\sigma dW_u\right]ds.$$

Note that, for deterministic A, C, D, we have

$$Var\left[\int_0^t A(s;t)dW_s+\int_0^t\{C(s;t)\int_0^s D(u)dW_u\}ds\right]$$

$$=Var\left[\int_0^t A(s;t)dW_s+\int_0^t D(s)dW_s\int_0^t C(s;t)ds-\int_0^t\{D(s)\int_0^s C(u;t)du\}dW_s\right]$$

$$=\int_0^t\left[A(s;t)+D(s)\int_0^t C(u;t)du-D(s)\int_0^s C(u;t)du\right]^2ds$$

$$=\int_0^t\left[A(s;t)+D(s)\int_s^t C(u;t)du\right]^2ds.$$

Thus, taking $D(u)=e^{\lambda u}$, N_t has normal distribution with mean

$$M_t=B_0+\mu_0\int_0^t\left\{e^{\int_s^t\gamma(u)du}k(s)+1\right\}ds$$

$$+\lambda\int_0^t\left\{e^{\int_s^t\gamma(u)du}[\beta(s)-k(s)]-1\right\}\left[e^{-\lambda s}b_0+\int_0^s\mu_0e^{-\lambda(s-u)}du\right]ds$$

and variance

$$\Sigma_t^2 := \sigma^2\int_0^t\left(e^{\int_s^t\gamma(u)du}k(s)+1+\lambda e^{\lambda s}\int_s^t e^{-\lambda u}\left\{e^{\int_u^t\gamma(v)dv}[\beta(u)-k(u)]-1\right\}du\right)^2ds.$$

Then, from (4.21), we have

$$G_t(\beta,k,\gamma)=M_t-\frac{1}{2}\theta\Sigma^2.$$

Thus, we consider minimizing, given a fixed value of t,

$$-\lambda \int_0^t \left\{ e^{\int_s^t \gamma(u)du}[\beta(s) - k(s)] - 1 \right\} \left[e^{-\lambda s} b_0 + \int_0^s \mu_0 e^{-\lambda(s-u)} du \right] ds$$
$$+ \int_0^t e^{\int_s^t \gamma(u)du} [\lambda(\beta(s) - k(s)) \left[e^{-\lambda s} b_0 + \mu_0 \int_0^s [e^{-\lambda(s-u)} du \right]]ds$$
$$+\frac{1}{2}\theta\Sigma_t^2$$
$$= \lambda \int_0^t \left[e^{-\lambda s} b_0 + \mu_0 \int_0^s [e^{-\lambda(s-u)} du \right] ds + \frac{1}{2}\theta\Sigma_t^2.$$

This is the same as minimizing Σ_t^2/σ^2, which is equal to

$$\int_0^t \left(e^{-\lambda(t-s)} + e^{\int_s^t \gamma(v)dv} k(s) + \lambda \int_s^t e^{-\lambda(u-s)} e^{\int_u^t \gamma(v)dv} [\beta(u) - k(u)] du \right)^2 ds$$
$$= \int_0^t e^{2\lambda s} \left(e^{-\lambda t} + e^{-\lambda s} e^{\int_s^t \gamma(v)dv} k(s) + \lambda \int_s^t e^{-\lambda u} e^{\int_u^t \gamma(v)dv} [\beta(u) - k(u)] du \right)^2 ds.$$

Denote, by omitting dependence on t,

$$g(s) = -\int_s^t e^{-\lambda u} e^{\int_u^t \gamma(v)dv} k(u) du$$
$$h(s) = -\int_s^t e^{-\lambda u} e^{\int_u^t \gamma(v)dv} \beta(u) du.$$

Then, we minimize

$$\int_0^t e^{2\lambda s} F_s(h, g, g') ds,$$

where

$$F_s(h, g, g') = \left(e^{-\lambda t} + g'(s) + \lambda[g(s) - h(s)] \right)^2,$$

under the constraints

$$g' \geq 0, \ h' \geq 0, \ g(t) = h(t) = 0.$$

Let's compare the choice $(g, h) = (0, 0)$ to any other admissible choice. It is easily verified that $F_s(x, y, z)$ is a convex function of (x, y, z). Thus, using integration by parts in the last equality below and that $g(t) = 0$, we have

$$\int_0^t e^{2\lambda s} \left[F_s(0, 0, 0) - F_s(g, g', h) \right] ds$$
$$\leq -\int_0^t e^{2\lambda s} \left[g(s)\partial_x F_s(0, 0, 0) + g'(s)\partial_y F_s(0, 0, 0) + h(s)\partial_z F_s(0, 0, 0) \right] ds$$
$$= -\int_0^t e^{2\lambda s} 2e^{-\lambda t} \left[\lambda(g(s) - h(s)) + g'(s) \right] ds$$
$$= -\int_0^t e^{2\lambda s} 2e^{-\lambda t} \left[\lambda(g(s) - h(s)) - 2\lambda g(s) \right] ds + 2e^{-\lambda t}[g(0) - g(t)e^{2\lambda t}]$$
$$\leq 0 \ .$$

Therefore, $(g, h) = (0, 0)$ is optimal. This corresponds to the deterministic contract S^C.

□

Proof of Proposition 4.4: To show that V^0 is the value function, we perform the usual verification argument from the standard stochastic control theory to verify that $m = 0$ is optimal.

Take any admissible strategy m. Since we assume that the set of admissible strategies m is such that the corresponding value function V of (4.12) satisfies the HJB equation, we have, by Ito's rule,

$$e^{-\rho T}V^0(T, P_T^m, W_T^w) \leq e^{-\rho t}V^0(0, p, w) + \int_0^T e^{-\rho s - \theta[C+\tilde{a}(s)+\beta\sigma W_s^w]}e^{-\theta(km_s+\beta P_s^m)}ds$$

$$+ \int_0^T e^{-\rho s}V_w^0(s, P_s^m, W_s^w)dW_s^w.$$

Taking expectation, and using assumption (ii) -(c), we get

$$-E\left[\int_0^T e^{-\rho s-\theta[C+\tilde{a}(s)+\beta\sigma W_s^w]}e^{-\theta(km_s+\beta P_s^m)}ds\right] \leq V^0(0,p,w) - e^{-\rho T}E[V^0(T, P_T^m, W_T^w)] \ .$$

Taking the limit as $T \to \infty$ and using assumption (ii)-(b) concludes the proof. □

5 Discussion and Conclusions

Discussion: ways the agent can misreport. We now discuss, somewhat informally, what kind of reports by the agent could be considered credible by the principal. In [Wil11] it is assumed that the only way the agent can lie is with a reporting process Y that equals the true state process plus a non-positive differentiable drift. In general, we can ask the following question:

For a reported path $y = (y_t)_{t\in[0,\infty)}$ to be accepted by the principal as credible, what are the properties it has to satisfy?

The answer is not obvious. For example, since the diffusion process (2.1) has continuous paths almost surely, we may want to impose a restriction that the reported path y be continuous, almost surely. However, the qualifier 'almost sure' is relevant in this context, for – depending on the space on which the diffusion is defined – a discontinuous path could actually happen, and some principals might be willing to accept even such reports. Nevertheless, it is reasonable that the principal insists on the reported path y having properties that are known to hold for almost all paths of B. This can be rephrased as saying that the principal fixes a set of paths deemed as unlikely to happen and thus deemed as lies if reported by the agent:

The principal fixes a Borel measurable set $N \subset \mathcal{C}([0,\infty),\mathbb{R})$, with the property that $\mathbf{P}[B \in N] = 0$. Then, if the agent reports $y \in N$, the principal considers it a lie.

The set N could, for instance, include all differentiable paths, for Brownian motion is almost surely nowhere differentiable. It could also include paths that do not have a quadratic variation process equal to $\sigma^2 t$, because quadratic variation can be calculated pathwise; see e.g. [Kar83].

On the other hand, no continuous $\mathbb{L}^2$ drift can be recognized as a lie in the above sense. Indeed, for every $t < \infty$, the law of B in (2.1) is equivalent to the law of $\sigma W_\cdot$ on $\mathcal{C}([0,t],\mathbb{R})$. This follows from Cameron-Martin-Girsanov theory, see e.g. [KS, Ch. 3.5]. Moreover, if

$A \subset \mathcal{C}([0,t],\mathbb{R})$ is measurable, and such that $\mathbf{P}[\sigma W. \in A] > 0$, then also $\mathbf{P}[\sigma W. \in A_\mu] > 0$ for any absolutely continuous $\mu \in \mathbb{L}^2([0,t])$, where $A_\mu := \{s \mapsto w_s + \mu_s \ : \ w \in A\}$ is the set of path translated by μ. Thus, if $N_t \subset \mathcal{C}([0,t],\mathbb{R})$ is a null set, then set $A = N^c$ has a full measure, and so does A_μ, so that reporting an additional drift μ to any report that is not considered a sure lie, is not a sure lie. The same conclusion holds also for B.

We leave for further research these issues, and, in particular, the following question: What is the minimal set of restrictions that are reasonable to assume on the reported process, while still having interesting examples, that is, examples in which the optimal contract is not such that the agent is indifferent with respect to how much to misreport?

Conclusions. In this paper, we show that the optimal contract provided in [Wil11] is not incentive compatible in the case of persistent shocks, if the growth on under-reporting is not restricted. It becomes incentive compatible if the difference between the reported process and the true process is sufficiently bounded, but, as shown in [BKS22], it is not optimal. In the case there is no limit on how much the agent can under-report, we show that deterministic contracts are the only IC contracts in an extended family of linear contracts. If additional restrictions on admissible contracts are imposed, we show that the contract that [BKS22] identify as optimal among self-insurance contracts is also optimal among the more general class of linear contracts. It is still an open question what the optimal contract is if we allow general contracts, and/or under additional restrictions on misreporting, for example, if it is restricted to take values in a bounded interval. We also leave for future research the question of what types of misreporting would be credible in continuous-time models driven by Brownian motion, while still resulting in non-trivial optimal contracts.

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