Short Note on Inf-Convolution Preserving the Fatou Property

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Abstract

We model agents' preferences by cash-invariant concave functionals defined on L^∞, and formulate the optimal risk allocation problem as their infimal-convolution. We study the case of agents whose choice functionals are law-invariant with respect to different probability measures and show how, in this case, the value function preserves a desirable dual representation (equivalent to the Fatou property).

1 Introduction

We consider the problem of the optimal allocation of risk among agents endowed with cash-invariant concave choice functionals. In this way well-known classes of functionals are included, as the coherent and convex risk measures introduced by Artzner et al. [2] and Föllmer and Schied [7]. We fix L^∞ as the space of admissible financial positions and formulate the optimization problem as the convolution of the agents' choice functionals. This problem has been studied by several authors, e.g. in Barrieu and El Karoui [3], Jouini et al. [10], Acciaio [1], Filipović and Svindland [6], where some answers are given about the existence and characterization of the optimal risk allocations. Here we analyze the case of agents whose choice functionals are law-invariant with respect to different probability measures, and prove a dual representation result for the value function of our optimization problem.

1.1 Set Up and Notations

We consider a measurable space $(\Omega, \mathcal{F})$ and two equivalent probability measures $\mathbb{P}_1, \mathbb{P}_2$ on it such that $(\Omega, \mathcal{F}, \mathbb{P}_i)$, $i = 1, 2$, are non-atomic standard probability spaces. Denote by $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P}_1) = L^\infty(\Omega, \mathcal{F}, \mathbb{P}_2)$ the space of all (equivalence classes of) essentially bounded random variables, and by $ba = (L^\infty)^*$ its dual space, which collects all bounded, finitely additive set functions μ on $(\Omega, \mathcal{F})$ with the property that $\mu(A) = 0$ whenever $\mathbb{P}_i(A) = 0$. The Yosida-Hewitt theorem (see [11]) implies, for each $\mu \in (L^\infty)^*$, the existence of a uniquely defined decomposition $\mu = \mu^a + \mu^p$, where μ^a is a σ-additive measure, absolutely continuous with respect to $\mathbb{P}_i$, and μ^p is a purely finitely additive measure.

We fix L^∞ as the space of admissible financial positions in a fixed future date, and on such space we define agents' choice functionals as follows.

Definition 1.1. *A proper functional $U : L^\infty \to [-\infty, +\infty)$ is called monetary utility functional (m.u.f.) if it is*

(i) concave: $U(\alpha X + (1-\alpha)Y) \geq \alpha U(X) + (1-\alpha)U(Y)$, $\forall\ X, Y \in L^\infty$, $\alpha \in (0,1)$,

(ii) cash-invariant: $U(X + c) = U(X) + c$, $\forall\ X \in L^\infty$, $c \in \mathbb{R}$,

(iii) monotone: $U(X) \geq U(Y)$, $\forall\ X, Y \in L^\infty$ s.t. $X \geq Y$,

i.e., if $-U$ is a convex risk measure in the sense of Föllmer and Schied [8].

By the Fenchel-Moreau theorem, each m.u.f. U admits dual representation

$$U(X) = \inf_{\mu \in (L^\infty)^*} \{V(\mu) + \langle \mu, X \rangle\}, \quad \forall X \in L^\infty, \tag{1}$$

through its convex conjugate function (or Fenchel-Legendre transform) $V : (L^\infty)^* \to [U(0), +\infty]$, defined as

$$V(\mu) := \sup_{X \in L^\infty} \{U(X) - \langle \mu, X \rangle\}, \quad \forall \mu \in (L^\infty)^*.$$

Note that, by cash-invariance and monotonicity, its domain satisfies

$$dom(V) \subseteq \mathcal{P} := \{\mu \in (L^\infty)^*_+ : \mu(\Omega) = 1\} \tag{2}$$

and therefore $(L^\infty)^*$ in (1) can be replaced by $\mathcal{P}$.

Let now recall two well-known properties, which are the basis for what follows.

Definition 1.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space and U a proper functional defined on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. We say that*

(i) U satisfies the Fatou property w.r. to $\mathbb{P}$, if for any bounded sequence $(X_n)_{n\in\mathbb{N}} \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ converging $\mathbb{P}$-a.s. to some X, then $U(X) \geq \limsup_n U(X_n)$;

(ii) U is $\mathbb{P}$-law-invariant, if $U(X) = U(Y)$ for any $X, Y \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ having the same law under $\mathbb{P}$.

In our framework, since we consider $\mathbb{P}_1, \mathbb{P}_2$ equivalent probability measures, we do not need to specify the measure we refer to when speaking of the Fatou property, fact that clearly is not true as far as the law-invariance is concerned. We recall that, for a concave upper semi-continuous (usc) function, the law-invariance is stronger than the Fatou property (which is the main result proved in [9]). On the other hand, a concave functional on L^∞ has the Fatou property if and only if it admits dual representation in the set of σ-additive measures absolutely continuous w.r. to the reference measure (see [4], [8], [1]), that in our case is $\mathcal{P}_\sigma := \{\mu : \mu \ \sigma\text{-additive measure on } (\Omega, \mathcal{F}),\ \mu \ll \mathbb{P}_i\}$. Therefore each m.u.f. U satisfying the Fatou property can be represented as

$$U(X) = \inf_{\mathbb{Q} \in \mathcal{Q}} \{V(\mathbb{Q}) + \mathbb{E}_{\mathbb{Q}}[X]\}, \quad \forall X \in L^\infty, \tag{3}$$

where $\mathcal{Q} := \{\mathbb{Q} : \mathbb{Q} \text{ probability measure on } (\Omega, \mathcal{F}),\ \mathbb{Q} \ll \mathbb{P}_i\} = \mathcal{P} \cap \mathcal{P}_\sigma$.

By means of the level sets of the dual function V of U:

$$C_k := \{\mu \in (L^\infty)^* : V(\mu) \leq k\}, \quad k \in \mathbb{R}, \tag{4}$$

the Fatou property can be restated as

$$\overline{C_k \cap \mathcal{Q}}^{w^*} = C_k, \quad \text{for all } k \in \mathbb{R}, \tag{5}$$

where the closure is taken in the weak*-topology. Indeed, if U satisfies the Fatou property, then it is weak*-usc and its acceptance set $\mathcal{A} := \{X \in L^\infty : U(X) \geq 0\}$ is weak*-closed. Then it is sufficient to apply the bipolar theorem twice, w.r. to the $(L^\infty, (L^\infty)^*)$- and to the (L^∞, L^1)-duality, to obtain (5). On the other hand, from (5) it follows that U admits dual representation over probabilities, i.e. (3) holds true. This characterization of the Fatou property will be used in Section 2 to prove our results.

1.2 Optimal Risk Allocation Problem

Consider two economic agents endowed with initial risky positions $\xi_1, \xi_2 \in L^\infty$ and preference relations represented by concave functionals U_1, U_2. The optimal sharing of the total risk $X := \xi_1 + \xi_2$ is formulated as the convolution of their choice functionals:

$$U(X) := U_1 \Box U_2(X) = \sup_{\substack{X_1, X_2 \in L^\infty, \\ X_1 + X_2 = X}} \left\{U_1(X_1) + U_2(X_2)\right\}. \tag{6}$$

Note that, when U_1 and U_2 are monetary utility functionals, the value function $U : L^\infty \to [-\infty, +\infty]$ inherits the properties of concavity, cash-invariance and monotonicity.

We denote by V_1, V_2 and V the dual conjugates of U_1, U_2 and U, and by C_k^1, C_k^2 and C_k their respective level sets, i.e., we define C_k as in (4) and similarly C_k^1 and C_k^2. From classical results of convex analysis we know that $V = V_1 + V_2$, thus clearly $C_k = \bigcup_{j=U_1(0)}^{k-U_2(0)} (C_j^1 \cap C_{k-j}^2)$ for $k \geq U_1(0) + U_2(0)$, and $C_k = \emptyset$ otherwise.

In general the Fatou property is not preserved by convolution (see, e.g., Delbaen [5]). However in the next section we prove that, for functionals law-invariant with respect to equivalent probability measures, the convolution functional has the Fatou property. This means that the value function in (6) admits dual representation in the space of σ-additive measures, and this constitutes a first step towards the study of that optimization problem.

2 Representation Results

In the framework described in the previous section, we can formulate the following theorem that, together with Theorem 2.5, is the main result of the paper.

Theorem 2.1. *Let U_1 and U_2 be monetary utility functionals on L^∞, law-invariant w.r. to $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively, such that the value function $U_1 \Box U_2$ in (6) is proper. Then $U_1 \Box U_2$ has the Fatou property, i.e.*

$$U_1 \Box U_2(X) = \inf_{\mathbb{Q} \in \mathcal{Q}} \{V_1(\mathbb{Q}) + V_2(\mathbb{Q}) + \mathbb{E}_{\mathbb{Q}}[X]\}, \quad \forall X \in L^\infty. \tag{7}$$

The proof of Theorem 2.1 will be given at the end of this section. We first translate the notion of law-invariance to sets of elements in *ba*. Recall that a *measure preserving transformation* (mpt) τ of a given non-atomic standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is a bi-measurable bijection $\tau : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ which leaves $\mathbb{P}$ invariant, i.e. $\tau(\mathbb{P}) = \mathbb{P}$. Now, a set $D \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$ is called $\mathbb{P}$-*law-invariant* if, for any $\mu \in D$ and any measure preserving transformation τ, $\mu \circ \tau \in D$ as well. Note that, for a $\mathbb{P}$-law-invariant m.u.f., the level sets of its conjugate function are $\mathbb{P}$-law-invariant subsets of $L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$.

Lemma 2.2. *Let $D \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*_+$ be a convex, weak*-closed, $\mathbb{P}$-law-invariant set. Then, for any $\mu = \mu^a + \mu^p \in D$, the following inclusion holds true:*

$$\{\nu \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*_+ : \nu = \nu^a + \nu^p,\ \nu(\Omega) = \mu(\Omega) \text{ and } \nu^a \geq \mu^a\} \subseteq D.$$

Proof. Fix any $\mu = \mu^a + \mu^p \in D$. By the Yosida-Hewitt theorem (see [11]), μ^p purely finitely additive measure is characterized by some decreasing sequence of sets $(A_n)_{n\in\mathbb{N}}$ such that $A_n \downarrow \emptyset$ and $\mu^p(A_n) = \mu^p(\Omega) =: s$. W.l.g., we fix $\mathbb{P}(A_n) = 1/n$. We claim that, for any $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$, there exists a purely finitely additive measure β with $\beta(B) = \beta(\Omega) = s$ and s.t. $\mu^a + \beta \in D$. First consider a set B in $\mathcal{F}$ s.t. $\mathbb{P}(B) > 0$ and $B \cap A_n = \emptyset$ for n sufficiently large. Let B_k be a subset of B such that $\mathbb{P}(B_k) = 1/k$ and $B_k \cap A_k = \emptyset$, for some $k \in \mathbb{R}$. Fix a mpt $\tau_k : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ such that $\tau_k(A_k) = B_k, \tau_k(B_k) = A_k$ and $\tau_k = Id$ on $\Omega \setminus (A_k \cup B_k)$. Now, for each $n > k$, define $B_n := \tau_k^{-1}(A_n)$, so that $\mathbb{P}(B_n) = 1/n$ and $B_n \subseteq B_{n-1}$, and let $\tau_n : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega, \mathcal{F}, \mathbb{P})$ be a mpt such that $\tau_n\big|_{B_n} = \tau_k\big|_{B_n}$ (in particular, $\tau_n(B_n) = \tau_k(B_n) = A_n$), $\tau_n(A_n) = B_n$ and $\tau_n = Id$ on $\Omega \setminus (A_n \cup B_n)$. In this way, for any set $G \in \mathcal{F}$ we have

$$\begin{aligned} \mu^p \circ \tau_n(G) &= \mu^p(\tau_n(G) \cap A_n) = \mu^p(\tau_n(G \cap B_n)) = \mu^p(\tau_k(G \cap B_n)) \\ &= \mu^p(\tau_k(G) \cap A_n) = \mu^p \circ \tau_k(G), \end{aligned}$$

for all $n > k$. On the other hand, since $\mathbb{P}(A_n \cup B_n) \to 0$ for $n \to \infty$ and μ^a is absolutely continuous w.r. to $\mathbb{P}$, by definition of τ_n we have $w^*\text{-}\lim_{n\to\infty} (\mu^a \circ \tau_n) = \mu^a$. Therefore, from D law-invariant and weak*-closed, the sequence $(\mu \circ \tau_n)_{n\geq k} \subseteq D$ has weak*-limit $\mu^a + \beta \in D$, where $\beta = \mu^p \circ \tau_k$ is a purely finitely additive measure with $\beta(B_n) = \beta(\Omega) = s$ $\forall n \geq k$. Now, for any other set in $\mathcal{F}$ that does not intersect B_n for n sufficiently large, we can follow the same procedure and so on, thus obtaining what we claimed at the beginning of the proof. We call $\mathcal{E}$ the collection of all convex combinations of measures $\mu^a + \beta$ obtained in this way. Clearly $\mathcal{E} \subseteq D$ being D convex.

Let now fix any measure $\eta \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*_+$ with $\eta(\Omega) = s$. We claim that $(\mu^a + \eta) \in D$. Call $\mathcal{H}$ the family of all finite partitions H of Ω, i.e. $H = \{H_1, \ldots, H_n\} \subseteq \mathcal{F}$ s.t. $\Omega = \bigcup_{i=1}^n H_i$ and $H_i \cap H_j = \emptyset$ $\forall i \neq j$. For each $H = \{H_1, \ldots, H_n\} \in \mathcal{H}$, we define a purely finitely additive function η_H s.t. $\mu^a + \eta_H \in \mathcal{E}$ with $\eta_H(H_i) = \eta(H_i)\,\forall\, i = 1, \ldots, n$. Consider the net $(\eta_H)_{H\in\mathcal{H}}$ where the order relation given on $\mathcal{H}$ is the refinement of partitions. In this way we obtain $\eta = w^*\text{-}\lim_{H\in\mathcal{H}} \eta_H$. Indeed, by definition of limit of a net, we have to show that for any neighborhood N of η there exists an index $H \in \mathcal{H}$ such that, for all refinements $K \in \mathcal{H}$ of H, η_K is in N. Consider the following basis of neighborhoods of η in the weak*-topology:

$$N(\eta, X_1, \ldots, X_m, \epsilon) = \{\xi \in (L^\infty)^* : |\langle \xi - \eta, X_i \rangle| < \epsilon,\ \forall\, i = 1, \ldots, m\},$$

for some $X_1, \ldots, X_m \in L^\infty$ and $\epsilon > 0$. Fix $X_1, \ldots, X_m \in L^\infty$ and $\epsilon > 0$ and consider simple random variables $Y_i = \sum_{j\in J_i} y_{ij} 1_{F_{ij}}$, with $|J_i| < \infty$, such that $\|Y_i - X_i\|_{L^\infty} \leq 1/2s\epsilon$, for $i =$

$1, \dots, m$. Call H the partition of Ω generated by $\{F_{ij}, j \in J_i, i = 1, \dots, m\}$. Then for any refinement K of H we have

$$\begin{aligned} |\langle \eta_K - \eta, X_i \rangle| &\leq |\langle \eta_K - \eta, Y_i \rangle| + |\langle \eta_K - \eta, X_i - Y_i \rangle| = |\langle \eta_K - \eta, X_i - Y_i \rangle| \\ &\leq 2s \frac{1}{2s\epsilon} = \epsilon, \end{aligned}$$

for all $i = 1, \dots, m$, as wanted. Therefore, $\mu^a + \eta = \mu^a + w^*\text{-}\lim_{H \in \mathcal{H}} \eta_H = w^*\text{-}\lim_{H \in \mathcal{H}} (\mu^a + \eta_H) \in D$, being D weak*-closed and $(\mu^a + \eta_H) \in \mathcal{E} \subseteq D$, and this concludes the proof. □

Corollary 2.3. *Let D be as in Lemma 2.2. If there is some measure $\mu \in D$ s.t. $\mu = \mu^p$, then*

$$\{\nu \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*_+ : \nu(\Omega) = \mu(\Omega)\} \subseteq D.$$

Now we are able to prove our main theorem.

Proof. [of Theorem 2.1] We use the characterization of the Fatou property given in (5). Fix $k \in \mathbb{R}$. Fix $k \in \mathbb{R}$. The inclusion '$\subseteq$' is immediate consequence of the weak*-lower semi-continuity of the dual function V. Let now consider $\mu = \mu^a + \mu^p \in C_k$, i.e. $\mu \in C^1_j \cap C^2_{k-j}$ for some $j \in \mathbb{R}$. If $\mu = \mu^a$ there is nothing to prove. On the other hand, if $\mu = \mu^p$, then $C^1_j = C^2_{k-j} = \mathcal{P}$ from (2) and Corollary 2.3, and $C_k = \mathcal{P} = \overline{\mathcal{Q}}^{w^*}$. As last, consider the case $\mu = \mu^a + \mu^p \in C_k$ with $\mu^p(\Omega) \in (0, 1)$. Since $\mathcal{P}_\sigma$ is weak*-dense in $(L^\infty)^*$, there exists a net $(\eta_j)_j$ in $\mathcal{P}_\sigma$ converging weak* to μ^p, where of course η_j can be chosen positive and such that $\eta_j(\Omega) = \mu^p(\Omega)$. Therefore, applying Lemma 2.2 to C^1_j and C^2_{k-j}, we get $(\mu^a + \eta_j) \in C^1_j \cap C^2_{k-j} \cap \mathcal{Q} \subseteq C_k \cap \mathcal{Q}$, with $w^*\text{-}\lim_j(\mu^a + \eta_j) = \mu$, which concludes the proof. □

By applying, to the negative and positive parts of signed measures, arguments similar to those used in the proof of Lemma 2.2, we obtain the following lemma.

Lemma 2.4. *Let $D \subseteq L^\infty(\Omega, \mathcal{F}, \mathbb{P})^*$ be a convex, weak*-closed, $\mathbb{P}$-law-invariant set. Then, for any $\mu = \mu^a + \mu^p \in D$, the following inclusion holds true:*

$$\begin{aligned} \{\nu \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* : \nu = \mu^a + \eta, \ \text{with } \eta \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})^* \text{ such that} \\ \eta^+(\Omega) = (\mu^p)^+(\Omega) \text{ and } \eta^-(\Omega) = (\mu^p)^-(\Omega)\} \subseteq D. \end{aligned}$$

This allows us to extend the result of Theorem 2.1 to the case of possibly non-monotone choice functionals. Note that for an usc proper functional $U : L^\infty \to [-\infty, +\infty)$ satisfying properties (i)-(ii) in Definition 1.1, the dual representation in (1) still holds. In this case the Fatou property is equivalent to

$$U(X) = \inf_{\mathbb{Q}\in\mathcal{Q}_s} \{V(\mathbb{Q}) + \mathbb{E}_{\mathbb{Q}}[X]\}, \quad \forall X \in L^\infty,$$

where $\mathcal{Q}_s := \{\mathbb{Q} : \mathbb{Q}\,\sigma\text{-additive measure on } (\Omega, \mathcal{F}),\, \mathbb{Q} \ll \mathbb{P}_i,\, \mathbb{Q}(\Omega) = 1\}$, and it can be restated as $\overline{C_k \cap \mathcal{Q}_s}^{w^*} = C_k$ for all $k \in \mathbb{R}$, similarly to the monotone case.

Theorem 2.5. *Let U_1 and U_2 be usc proper functionals on L^∞ satisfying properties (i)-(ii) in Definition 1.1, law-invariant w.r. to $\mathbb{P}_1$ and $\mathbb{P}_2$ respectively, and such that the value function $U_1 \Box U_2$ in (6) is proper and usc. Then $U_1 \Box U_2$ has the Fatou property, i.e.*

$$U_1 \Box U_2(X) = \inf_{\mathbb{Q}\in\mathcal{Q}_s} \{V_1(\mathbb{Q}) + V_2(\mathbb{Q}) + \mathbb{E}_{\mathbb{Q}}[X]\}, \quad \forall X \in L^\infty. \tag{8}$$

Theorems 2.1 and 2.5 ensure the representation of the convolution functional in terms of σ-additive measures, and then allow to deal with the more tractable spaces $\mathcal{Q}$, $\mathcal{Q}_s$ instead of the whole dual space $(L^\infty)^*$.

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