

Note on rectangles with vertices on prescribed circles

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Inspired by [2], Ionascu and Stanica considered the following problem: The four subsequent vertices of a rectangle R in the plane are at distances r_1, r_2, r_3, r_4 from the origin. Given these data, what can be said about the area A of R ? In a recent paper [1] they proved

Theorem 1. (a) For given $r_i \geq 0$, rectangles R of the described kind exist iff

$$r_1^2 + r_3^2 = r_2^2 + r_4^2. \quad (1)$$

(b) The areas of these rectangles lie between the bounds

$$A_{\min} = |r_2 r_4 - r_1 r_3|, \quad A_{\max} = r_2 r_4 + r_1 r_3.$$

The authors call their problem “unusual because of its surprisingly simple answer in spite of our rather laborious solution” (which takes 10 pages). In this note we shall prove Theorem 1 in a simple way, making use of no more than the Pythagorean theorem and the formula for the derivative of a product.

After cyclic reordering, and neglecting special or degenerate cases, we may assume that $0 < r_1 < r_i$ ($2 \leq i \leq 4$). We may also assume that the sides of the rectangle are parallel to the axes and that the point $P_1 = (x, y)$ on the circle of radius r_1 is the lower lefthand vertex of R (Fig. 1). From P_1 we draw a horizontal to the right and obtain the vertex $P_2 = (\bar{x}, y)$ on the circle of radius r_2 ; in a similar way, going from P_1 vertically upwards, we obtain the vertex $P_4 = (x, \bar{y})$ on the circle of radius r_4 . So far we have

$$\begin{aligned} x^2 + y^2 &= r_1^2, \\ \bar{x}^2 + y^2 &= r_2^2, \quad \bar{x} > 0, \\ x^2 + \bar{y}^2 &= r_4^2, \quad \bar{y} > 0. \end{aligned} \quad (2)$$

The upper righthand vertex of R is $P_3 = (\bar{x}, \bar{y})$, and it lies on the circle of radius r_3 iff

$$r_3^2 = \bar{x}^2 + \bar{y}^2 = r_2^2 - y^2 + r_4^2 - x^2 = r_2^2 + r_4^2 - r_1^2,$$

i.e., iff (1) holds. Since the above construction is always possible, part (a) of the theorem follows.

Remark. The midpoint of R bisects both diagonals. It follows that (1) can be seen as an instance of the so-called Pappus-Fagnano-Legendre formula $a^2 + b^2 = (c^2 + z^2)/2$ where $z/2$ is the length of the median through C of a triangle with sides a, b, c .

For our extremal problem we have to consider admissible variations $dx, dy, d\bar{x}, d\bar{y}$ of the quantities x, y, \bar{x}, \bar{y} . To this end we differentiate the equations (2) and obtain

$$x dx + y dy = 0, \quad \bar{x} d\bar{x} + y dy = 0, \quad x dx + \bar{y} d\bar{y} = 0 \quad (3)$$

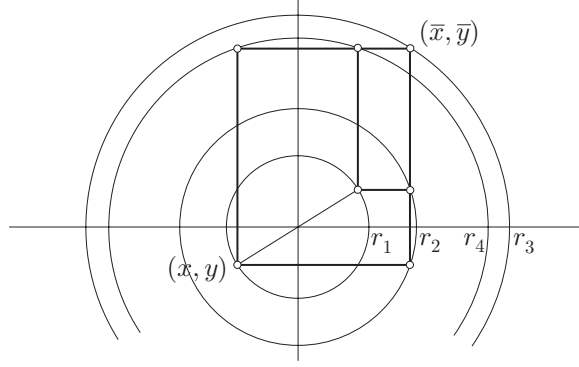


Fig. 1

from which we easily infer

$$d\bar{x} = \frac{x}{\bar{x}} dx, \quad d\bar{y} = \frac{y}{\bar{y}} dy. \quad (4)$$

The area of R is $A = (\bar{x} - x)(\bar{y} - y)$; it is maximal or minimal iff

$$dA = (d\bar{x} - dx)(\bar{y} - y) + (\bar{x} - x)(d\bar{y} - dy) = 0.$$

Using (4) we get

$$dA = \frac{x - \bar{x}}{\bar{x}}(\bar{y} - y) dx + (\bar{x} - x) \frac{y - \bar{y}}{\bar{y}} dy = -\frac{A}{\bar{x}\bar{y}}(\bar{y} dx + \bar{x} dy).$$

So the condition $dA = 0$ reduces to $\bar{y} dx + \bar{x} dy = 0$. Combined with the first equation (3) this shows that in the stationary situation we necessarily have

$$\frac{y}{x} = \frac{\bar{x}}{\bar{y}}. \quad (5)$$

In order to determine the maximal and minimal areas of the rectangle we argue as follows: From (5) and (2) we get

$$\frac{\bar{x}^2}{\bar{y}^2} = \frac{y^2}{x^2} = \frac{r_2^2 - \bar{x}^2}{r_4^2 - \bar{y}^2}, \quad \text{whence} \quad \bar{x}^2(r_4^2 - \bar{y}^2) = \bar{y}^2(r_2^2 - \bar{x}^2),$$

and after cancelling terms we see that

$$\frac{\bar{y}}{\bar{x}} = \frac{r_4}{r_2}, \quad \frac{y}{x} = \frac{r_2}{r_4},$$

the latter using (5) again.

We now know the ratios of these coordinates as well as their Pythagorean sums, whence they must be

$$x = \pm \frac{r_4}{\sqrt{r_2^2 + r_4^2}} r_1, \quad y = \pm \frac{r_2}{\sqrt{r_2^2 + r_4^2}} r_1, \quad \bar{x} = \frac{r_2}{\sqrt{r_2^2 + r_4^2}} r_3, \quad \bar{y} = \frac{r_4}{\sqrt{r_2^2 + r_4^2}} r_3.$$

Inserting these values into the formula for A we obtain, using (1):

$$A = \frac{(r_2 r_3 \mp r_4 r_1)(r_4 r_3 \mp r_2 r_1)}{r_2^2 + r_4^2} = \frac{r_1^2 + r_3^2}{r_2^2 + r_4^2} r_2 r_4 \mp r_1 r_3 = r_2 r_4 \mp r_1 r_3 .$$

This leads to

$$A_{\min} = r_2 r_4 - r_1 r_3 , \quad A_{\max} = r_2 r_4 + r_1 r_3 ,$$

as stated. It is easily seen that for the maximum the origin lies in the interior of R whereas for the minimum it is in the exterior.

References

- [1] Ionascu, Eugen J.; Stanica, Pantelimon: *Extremal values for the area of rectangles with vertices on concentric circles*. Elem. Math. 62 (2007), 30–39.
- [2] Zahlreich Problems Group: *Problem 11057*. Amer. Math. Monthly 111 (2004), 64. *Solution*: ibid. 113 (2006), 82.

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