$$\sqrt{n}\left(\hat{b}_j - \beta_j^0\right) \approx \mathcal{N}\left(0, \sigma^2 \underbrace{\frac{\|Z^{(j)}\|_2^2/n}{|(X^{(j)})^T Z^{(j)}/n|^2}}_{\text{known}}\right)$$

unknown

$$\boxed{\sqrt{n}\left(\widehat{bias} - bias\right) \xrightarrow{P} 0} \quad (*)$$

det. / known

$$\sqrt{n}\, C_j \left(\hat{b}_j - \beta_j^0\right) = W_j + \Delta_j \quad (j = 1, \ldots, p)$$

$$\underline{W} = (W_1, \ldots, W_p)^T \sim \mathcal{N}_p(0, \sigma^2 \Omega)$$

$$\max_{j=1,\ldots,p} |\Delta_j| \xrightarrow{P} 0 \qquad (\Omega)_{jk} = C_j C_k$$

$$\cancel{\sum_{j=1}^{p} |\Delta_j|^2 = \|\Delta\|_2^2 \xrightarrow{P} 0}$$

$\Longrightarrow$ one can not use test-statistic $\sum_{j=1}^{p} |\hat{b}_j|^2 \not\approx$ Gaussian χ^2

$$\sigma^{-1} \max_{j \in G} |W_j| \sim \, ?$$

max of dependent Gaussian variables

$$\text{simulate } \sigma^{-1} W_j \ (j \in G) \sim \underbrace{\mathcal{N}_G(0, \cancel{\sigma^2}\, \Omega_{GG})}_{\text{simulate } W^*_G}$$

$$\max_j |W_j^*| \approx \ldots$$

from simulation

$$(X^{(j)})^T Z^{(j)} / n \approx \|Z^{(j)}\|_2^2 / n$$

why?

$$X^{(j)} = \hat{X}^{(j)} + Z^{(j)}$$

$\boxed{(\hat{X}^{(j)})^T Z^{(j)} / n \text{ small}}$