Recap

Oracle inequality for the Lasso

Theorem 6.1 in Bühlmann and van de Geer (2011)

assume: compatibility condition holds with compatibility constant ϕ_0^2 ($\geq L > 0$)

Then, on $\mathcal{T}$ and for $\lambda \geq 2\lambda_0$:

$$\|X(\hat{\beta} - \beta^0\|_2^2/n + \lambda\|\hat{\beta} - \beta^0\|_1 \leq 4\lambda^2 s_0/\phi_0^2$$

recall: $\mathcal{T} = \{2 \max_{j=1,\ldots,p} |\varepsilon^T X^{(j)}/n| \leq \lambda_0\}$

$\mathbb{P}[\mathcal{T}]$ large if $\lambda_0 \asymp \sqrt{\log(p)/n}$

and errors have Gaussian or sub-Gaussian distribution implications:

$$\|X(\hat{\beta} - \beta^0\|_2^2/n = O_P(s_0 \log(p)/n) \quad \text{(fast rate)}$$
$$\|\hat{\beta} - \beta^0\|_1 = O_P(s_o\sqrt{\log(p)/n})$$

these are the (minimax) optimal rates:

no other method can do better

Variable Screening

assume compatibility condition and (e.g.) Gaussian errors
in addition, require beta-min condition:

$$\min_{j \in S_0} |\beta_j^0| \gg s_0 \sqrt{\log(p)/n}$$

$$\Longrightarrow \ \mathbb{P}[\hat{S} \supseteq S_0] \to 1 \ (p \geq n \to \infty)$$

with high probab: Lasso selects a superset of the active set S_0
$\leadsto$ Lasso does not miss an important active variable!

in practice: $\lambda = \lambda_{CV} \leadsto$ leads "typically" to a too large model

LASSO: Least Absolute Shrinkage and Screening Operator

originally (Tibshirani, 1996):
LASSO: Least Absolute Shrinkage and Selection Operator

Theory versus Practice

theory:

$$\mathbb{P}[\hat{S} \supseteq S_0] \to 1$$

if the following hold:

- compatibility condition for the (fixed) design X
- beta-min condition
- i.i.d. Gaussian errors (can be relaxed)

in addition: $|\hat{S}| \leq \min(n, p)$
hence: huge dimensionality reduction if $p \gg n$

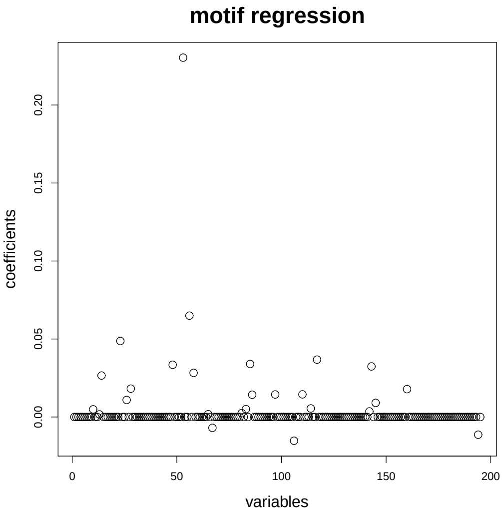
in practice: $\mathbb{P}[\hat{S} \supseteq S_0]$ may not be so large...
even if one chooses λ very small which results in a typically larger set $\hat{S}$...

possible reasons to explain with theory:

- compatibility constant ϕ_0^2 might be very small (due to highly correlated columns in X or near linear dependence among a few columns of X)
  $\leadsto \|\hat{\beta} - \beta^0\|_1 \leq 4\lambda s_0/\phi_0^2$
  $\leadsto$ for variable screening: need a stronger beta-min condition!
- errors are non-Gaussian (heavy tailed)

it is "empirically evident" though: $\mathbb{P}[\hat{S} \supseteq S_{\mathrm{substantial}(C)}]$ large
where $S_{\mathrm{substantial}(C)} = \{j;\ |\beta_j^0| \geq \underbrace{C}_{\text{large}}\}$

The Lasso workhorse



$$p = 195, n = 143, |\hat{S}(\lambda_{CV})| = 26$$