LECTURE 5

Recall: — Equivariant Morse chain complex —

$$C_k^{\tilde{\mathcal{U}}} := \mathbb{Z}\{ gH_i^k : g \in \pi, 1 \le i \le r_k \}$$

$$\delta_k^{\tilde{\mathcal{U}}}(gH_i^k) := \sum_{g' \in \pi, 1 \le j \le r_{k-1}} I(g\tilde{A}_i^k \pitchfork g'\tilde{B}_j^{k-1})\, g'H_j^{k-1}$$

def. Let N be an oriented smooth n-manifold and $f_i : M_i \longrightarrow N$ for $i = 1,2$ transverse maps with $\dim M_1 + \dim M_2 = \dim N$ and M_i simply connected, and w_{f_i} a path from the basepoint of N to a basepoint of f_i.

Let us write $\tilde{f}_i := f_i \cup w_{f_i}$.

We define the <u>equivariant intersection number</u>

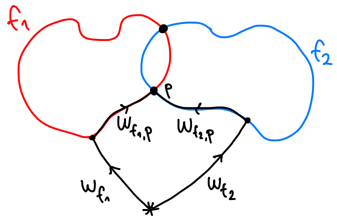
$$\tilde{I}(\tilde{f}_1 \pitchfork \tilde{f}_2) := \sum_{p \in f_1 \pitchfork f_2} \varepsilon_p g_p \in \mathbb{Z}[\pi_1 N]$$

for $\varepsilon_p \in \{+1,-1\}$ and $g_p \in \pi_1 N$ defined as follows:

- $\varepsilon_p := +1$ iff orientation of $Tf_1|_p \oplus Tf_2|_p$ agrees with that of $TN|_p$
- $g_p := [w_{f_1} \cdot w_{f_1,p} \cdot w_{f_2,g}^{-1} \cdot w_{f_2}^{-1}]$

where:

$w_{f_i,x}$ is a path in f_i from the basepoint of f_i to $p \in f_1 \pitchfork f_2$



NOTE. – M_i are simply connected

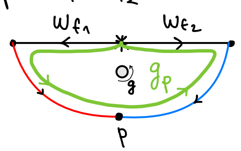
$\Rightarrow$ g_p does not depend on choices of $w_{f_i,x}$ but it <u>does</u> depend on choices of w_{f_i}.

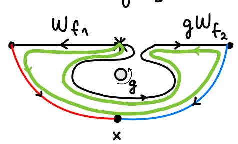
– $\tilde{f}_i$ is a based map, it is a choice of a lift of f_i to $\tilde{f}_i : M_i \longrightarrow \tilde{N}$.

(Recall: a map from simply connected lifts to the universal cover).

Lemma. $\tilde{I}(\tilde{f}_1 \pitchfork \tilde{f}_2) = \sum_{g \in \pi_1 N} I(\tilde{f}_1 \pitchfork g\tilde{f}_2) \cdot g$.

proof. $p \in f_1 \pitchfork f_2$ with $g_p = g$, $\varepsilon_p = \varepsilon$ $\iff$ $x \in \tilde{f}_1 \pitchfork g\tilde{f}_2$ with $g_x = 1$, $\varepsilon_x = \varepsilon$.





□

Cor. $\delta_k^{\tilde{\mathcal{U}}}(H_i^k) := \sum_{1 \le j \le r_{k-1}} \tilde{I}(\tilde{A}_i^k \pitchfork \tilde{B}_j^{k-1})\, H_j^{k-1}$ and $\delta_k^{\tilde{\mathcal{U}}}(gH_i^k) = g \cdot \delta_k^{\tilde{\mathcal{U}}}(H_i^k)$.

— Handle Slides Lemma —

The change of basis $\tilde{H}_j^{k+1} \xmapsto{B} \tilde{H}_j^{k+1} + g\tilde{H}_{j'}^{k+1}$ (or $\tilde{H}_j^{k+1} - g\tilde{H}_{j'}^{k+1}$) in $C_*^{\tilde{\mathcal{U}}}$ for some $1 \le j, j' \le r_{k+1}$, $g \in \pi$, can be realised geometrically on the handle decomposition. More precisely, the att. sphere of h_j^{k+1} can be isotoped so that the new handle decomposition induces B on $C_*^{\tilde{\mathcal{U}}}$.

proof.

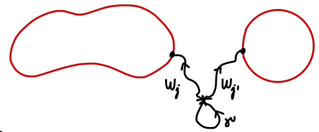
We can attach handles of the same index in any order, so consider $(W^{\le k} \cup h_j^{k+1}) \cup h_{j'}^{k+1}$.

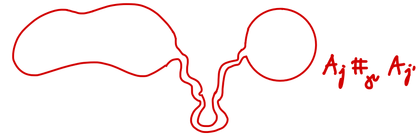
In $\partial_1(W^{\le k} \cup h_{j'}^{k+1})$ we have a push-off of $A_{j'}$, which bounds a disk Δ = push off of the core of $h_{j'}^{k+1}$.

Then we can form an ambient connected sum

$$A_j \#_\gamma A_{j'} = (A_j \setminus \nu(pt)) \cup \nu(w_j^{-1} \cdot \gamma \cdot w_{j'}) \cup (A_{j'} \setminus \nu(pt)) \text{ where } [\gamma] = g \in \pi$$

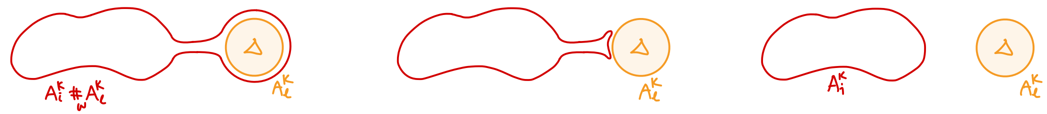
w_j = path from A_j to the basepoint





Since $A_\ell^k \setminus \nu(pt)$ is isotopic rel boundary to $\nu(pt)$, via Δ, we have isotopies

$$A_j \# A_{j'} \simeq A_i^k \cup \nu(w_j^{-1} \cdot \gamma \cdot w_{j'}) \cup \nu(pt) \simeq A_i^k.$$



On the other hand, we clearly have that a handle attached to $A_j \#_\gamma A_{j'}$ corresponds to $H_j^{k+1} + gH_{j'}^{k+1}$ (to get $-gH_j^{k+1}$ use oppositely oriented $A_{j'}$). □

§ Whitney Trick

Recall: Whitney used this trick in the proof of Whitney's embedding theorem in 1940s.
It is a regular homotopy that removes a pair of (self-) intersections.

Recall (L1): Every top. n-manifold admits a top. embedding into $\mathbb{R}^{n'}$, and $n' = 2n$ enough.

Thm [Whitney, 1936]
Every smooth manifold admits a smooth embedding into $\mathbb{R}^{n'}$, and $n' = 2n$ enough.

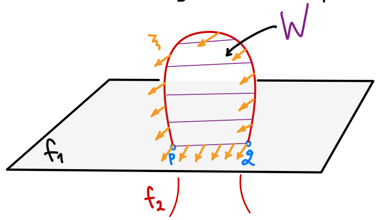
def. Fix smooth manifold M, N and a smooth map $f: M \times [0,1] \longrightarrow N$.
(So f is a smooth homotopy from f_0 to f_1, where $f_t := f(-,t): M \to N$, $t \in [0,1]$)
1° If f_t is an embedding for every t, then f is a smooth isotopy from f_0 to f_1
2° If f_t is an immersion for every t, then f is a regular homotopy from f_0 to f_1

Note: A Whitney move is a regular homotopy (in the local model, it goes from an immersion f_0 with 2 double points to an embedding).

Thm [Whitney]
A smooth map $f: M \to N$ is homotopic to an embedding if $n \geq 2m+1$ and to an immersion if $n = 2m$. If $f, g: M \hookrightarrow N$ are two homotopic embeddings and $n \geq 2m+2$, then they are also isotopic.

Let $f_i: M_i \hookrightarrow N$ be two embeddings with $\overset{=:k}{\dim M_1} + \dim M_2 = \overset{=:n}{\dim N}$ and $f_1 \pitchfork f_2$.
We assume points $p, q \in f_1 \pitchfork f_2$ are such that:
there are arcs $\gamma_i \subseteq f_i(M_i)$ from p to q and an embedded disc $W: D^2 \hookrightarrow N$,
with $\partial W = \gamma_1 \gamma_2^{-1}$ and $\operatorname{int} W$ disjoint from $f_i(M_i)$, $i = 1, 2$.

I. Now define a subbundle ξ of $\nu_{W \subseteq N}|_{\partial W}$ of rank $k-1$.
1° along γ_1 define $\xi|_{\gamma_1} := \nu_{\gamma_1 \subseteq f_1}$
2° along γ_2 define ξ as the rank $k-1$ subbundle of $\nu_{f_2 \subseteq N}|_{\gamma_2}$ (this is rank k) that is normal to W (so $\xi|_{\gamma_2}$ is the complement of $\nu_{\gamma_2 \subseteq f_2} \subseteq \nu_{f_2 \subseteq N}|_{\gamma_2}$)



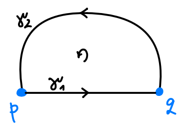
Thus, ξ is tangent to f_1 and normal to f_2.

The normal bundle $\nu_{W \subseteq N}$ is trivial (as $D^2 \simeq *$) and we would like to extend ξ to a subbundle of $\nu_{W \subseteq N}$ over the whole disc W.
This corresponds to extending the map $\partial D^2 \xrightarrow{\xi} Gr_{k-1} \mathbb{R}^{n-2}$ over D^2.
This is possible iff $[\xi] = 1 \in \pi_1 Gr_{k-1} \mathbb{R}^{n-2} \cong \mathbb{Z}/2$, unless $k-1 = 1$, $n-2 = 2$ ☆
whose nontrivial element is given by taking a plane to itself with opposite orientation
For ☆ use that oriented Grassm. $Gr_i^+ \mathbb{R}^j \cong SO(j) / SO(i) \times SO(j-i)$ double covers $Gr_{k-1} \mathbb{R}^{n-2}$
and is simply connected as $\pi_1 SO(i) \twoheadrightarrow \pi_1 SO(j)$, if $j \geq i > 1$,
and $\pi_1 SO(j-i) \twoheadrightarrow \pi_1 SO(j)$ if $i = 1$, $j-i > 1$
whereas $\pi_1 Gr_1(\mathbb{R}^2) \cong \pi_1 \mathbb{R}P^1 \cong \pi_1 SO(2) \cong \mathbb{Z}$

We will have $[\xi] = 1$ iff ξ is an orientable bundle. (For $k=2, n=4$ consider proj. to $\mathbb{Z}/2$)

Lemma. ξ is orientable iff the intersection points have opposite signs, $\varepsilon_q = -\varepsilon_p$
i.e. if $df_1|_x \oplus df_2|_x \cong TN|_x$ is orient. pres at precisely one $x = p, q$.

proof.



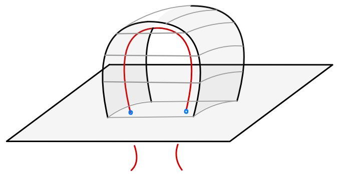
By definition:
$$\xi|_{\gamma_1} \oplus T\gamma_1 \cong Tf_1|_{\gamma_1} \quad \text{and} \quad \xi|_{\gamma_2} \oplus \nu_{\gamma_2 \subseteq W} \cong \nu_{f_2 \subseteq W}|_{\gamma_2}$$
Pick an orientation for W. This orients ∂W so also $T\gamma_1$ and $\nu_{\gamma_2 \subseteq W}$.
Note that $T\gamma_1|_x \cong \nu_{\gamma_2 \subseteq W}|_x$ is orient. pres. for precisely one of $x = p, q$. Say p.
On the other hand, ξ is orientable iff induced orientations of $\xi|_{\gamma_1}$ and $\xi|_{\gamma_2}$ either agree or disagree at both p, q.
They agree at p iff $\varepsilon_p = +1$, and they agree at q iff $\varepsilon_q = -1$. □

Case $k=2, n=4$.

In this case it does not suffice that $\varepsilon_q = -\varepsilon_p$: this only ensures that ξ is orientable, but it might not extend to W. Namely, $[\xi] \in \mathbb{Z} \cong \pi_1 SO(2)$ is the relative Euler number of W, obtained by restricting a framing $\nu_{W \subseteq N} \cong \mathbb{D}^2 \times \mathbb{R}^2$ to ∂W.
In practice, one has to change W to correct the framing (might not always be possible).

II.

Now assume we found a framed Whitney disc (so W together with an extension of ξ to W).
We can then perform the Whitney move:



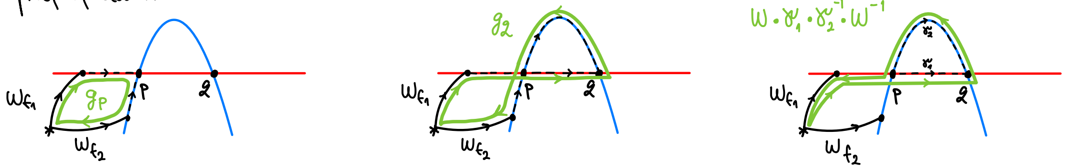
1° extend W slightly
2° pick a tubular neighbourhood $\mathbb{D}^2 \times \mathbb{D}^{n-2}$ together with the sub-nbhd corresponding to $\xi \cong \mathbb{D}^2 \times \mathbb{D}^{k-1}$
3° the boundary of this sub-nbhd is a sphere $S^k \hookrightarrow N$ which intersects f_1 in a nbhd of γ_1.
4° we push this strip $\gamma_1 \times \mathbb{D}^{k-2}$ across the sub-nbhd to become the union of:
a nbhd of $\gamma_2^{\uparrow}$ $(\cong \mathbb{D}^1 \times \mathbb{D}^{k-2})$ with the sphere bundle over $\mathbb{D}^2$ $(\cong \mathbb{D}^2 \times S^{k-2})$.

This is an isotopy of f_1 that removes 2 of its intersection points with f_2.
Viewed together, this is a regular homotopy of the immersion $M_1 \sqcup M_2 \xrightarrow{f_1 \cup f_2} N$.

Lemma $\tilde{I}(\tilde{f}_1 \pitchfork \tilde{f}_2)$ is an invariant of based maps $\tilde{f}_i$ under Whitney moves.

proof. We check that $\tilde{I}(f_1 \pitchfork f_2)$ remains unchanged after a Whitney move.
To perform the Whitney move we need points p, q and paths γ_1, γ_2 from p to q such that $\gamma_1 \cdot \gamma_2^{-1}$ is nullhomotopic in N, and $\varepsilon_q = -\varepsilon_p$.
Claim. $\gamma_1 \cdot \gamma_2^{-1}$ is nullhomotopic in N iff $g_q = g_p$.

proof of Claim.



We have $g_q = \underbrace{w_{f_1} \cdot (w_{f_1,p} \cdot \gamma_1)}_{w} \cdot (w_{f_2,p} \cdot \gamma_2)^{-1} \cdot w_{f_2}^{-1} = w \cdot \gamma_1 \cdot \gamma_2^{-1} \cdot w^{-1} \cdot \underbrace{(w \cdot w_{f_2,p}^{-1} \cdot w_{f_2}^{-1})}_{= g_p}$. □