Geometric algebraic topology of embedding spaces

Part I

Geometric homology / cohomology

– homology represented by fundamental classes: maps from compact (oriented) n-∂ mflds (w/ corners)

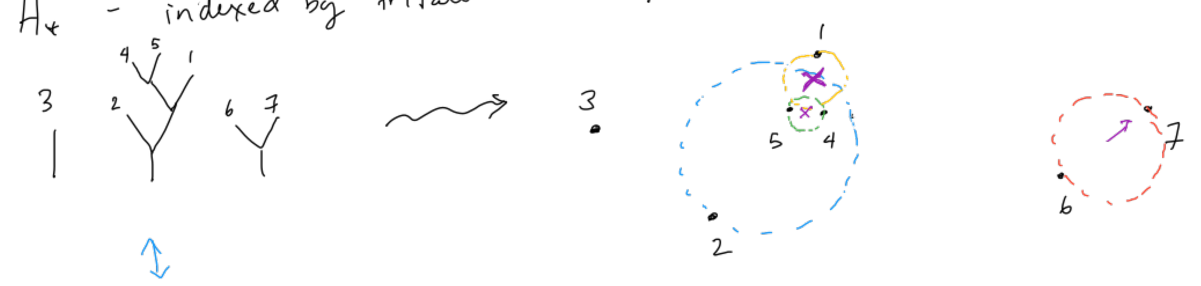
– cohomology represented by intersection / pullback w/ proper maps

(cf Lipyanskiy '14)

Full geometric presentation: H_*, H^* spanning sets of such, w/ relations & pairing.

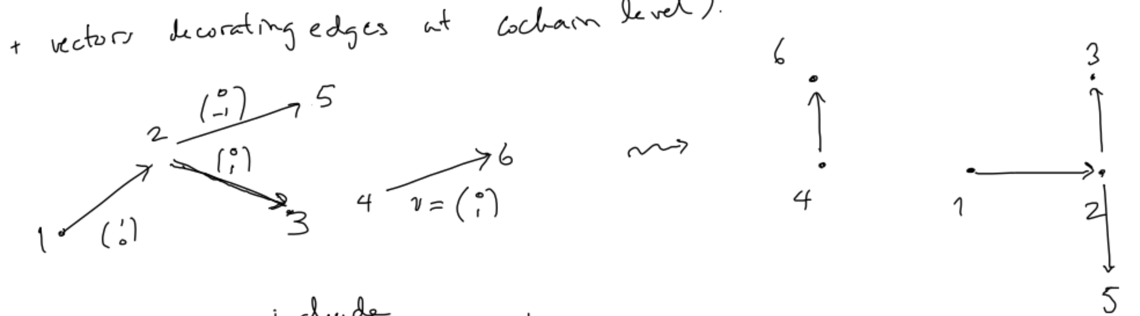
Ex $\mathrm{Conf}_n(\mathbb{R}^d)$ – H_* is the (graded) Poisson operad.

H_* – indexed by trivalent rooted forests



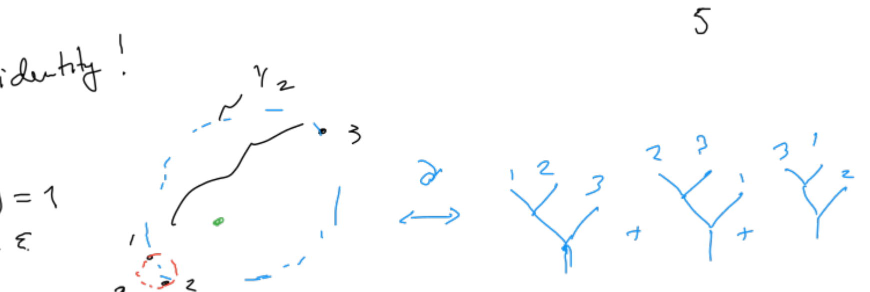
$x_3 \cdot [x_2,[[x_4,x_5],x_1]] \cdot [x_6,x_7]$ $\quad (S^1)^{\times 4} \hookrightarrow \mathrm{Conf}_7(\mathbb{R}^2)$ (d=2)

H^* – indexed by oriented graphs w/ no loops (+ vectors decorating edges at cochain level).



Relations in H_* – include Jacobi identity!

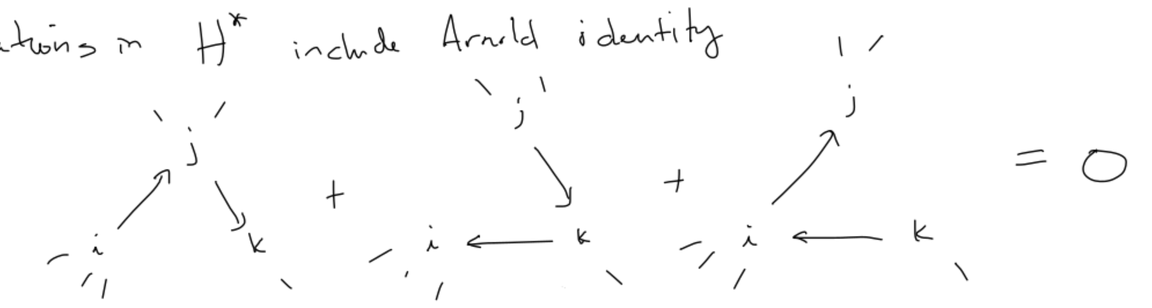
Consider (x_1,x_2,x_3) | $\Sigma x_i = 0$, $\Sigma\, d(x_i,x_j) = 1$, $d(x_i,x_j) \geq \varepsilon$



$[[x_1,x_2],x_3] + \dots = 0$

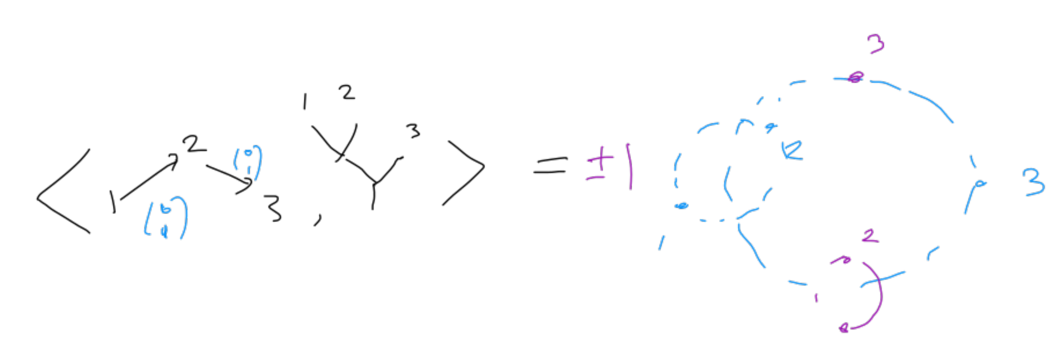
Jacobi / S_3 ≅ trefoil complement!

Relations in H^* include Arnold identity

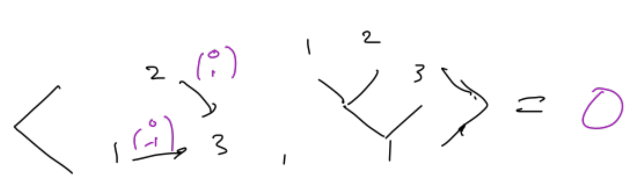


$= 0$

Pairing:



$= \pm 1$



$= 0$

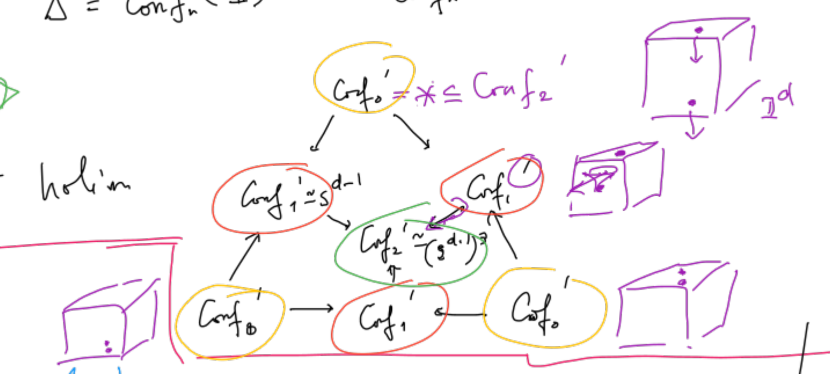
See arXiv: math/0610236 for expository treatment.

Back to spectral sequences.

$\mathrm{Emb}(I, I^d) \hookrightarrow AM_n$ = "aligned" stratum preserving maps $\Delta^n = \mathrm{Conf}_n\langle I\rangle \to \mathrm{Conf}_n'\langle I^d\rangle$

$K \mapsto ev_n(K)$, $\mathrm{Conf}_n\langle K\rangle$

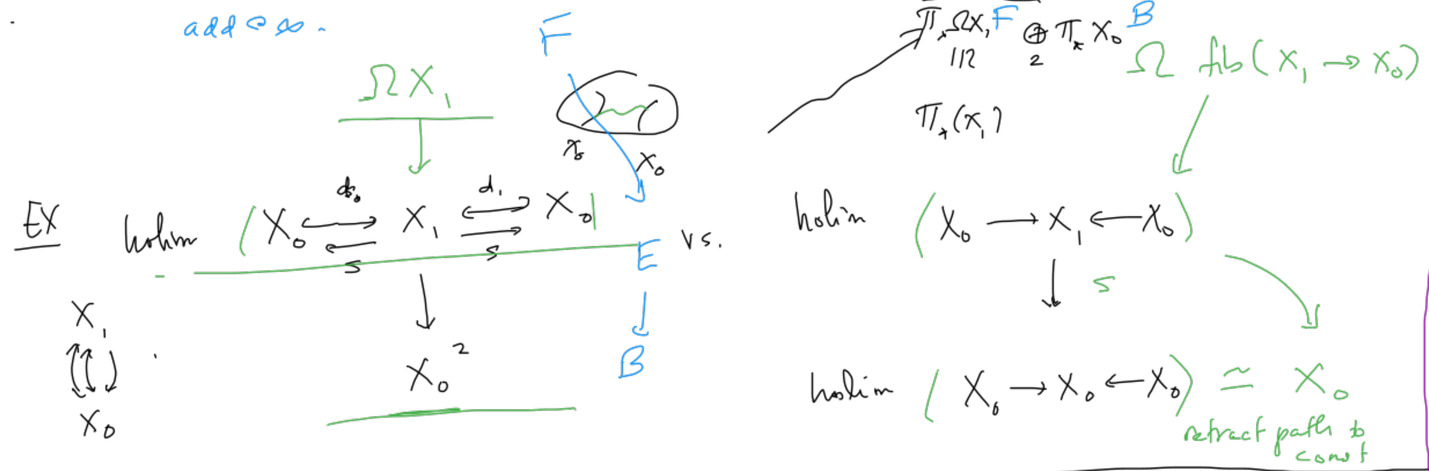
AM_2 = holim



splittings $\mathrm{Conf}_n'(\mathbb{R}^d) \rightleftarrows \mathrm{Conf}_{n-1}'(\mathbb{R}^d)$ (forget / add a pt.) ⇒ reduction.

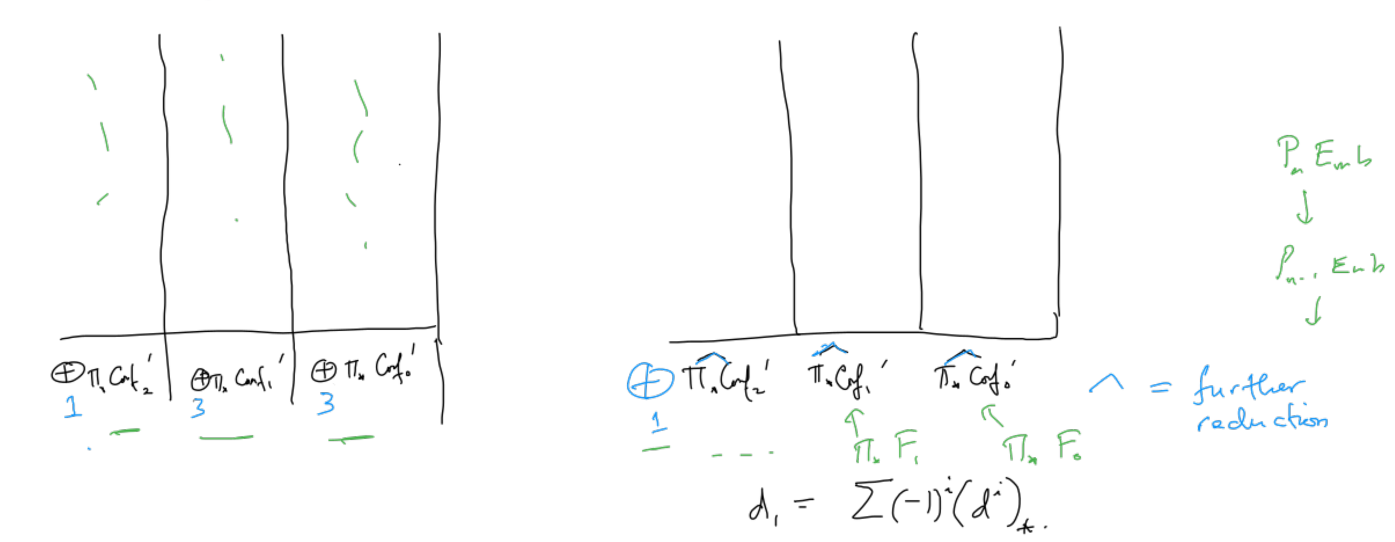
Ex holim $(X_0 \rightrightarrows X_1 \rightrightarrows X_0)$ vs. holim $(X_0 \to X_1 \leftarrow X_0)$

holim $(X_1 \to X_0 \leftarrow X_0) \simeq X_0$ (retract path to const)



SS for $\pi_*(AM_n)$ goes from

(unreduced) (reduced)



$d_1 = \sum (-1)^i (d^i)_*$

^ = further reduction

Homology Spectral Sequence

Dold-Thom: $\pi_*(\mathbb{Z}X) \cong H_*(X)$

So can develop embedding calculus for the functor $\mathbb{Z}\mathrm{Emb}(U, N)$. ($U \mapsto$)

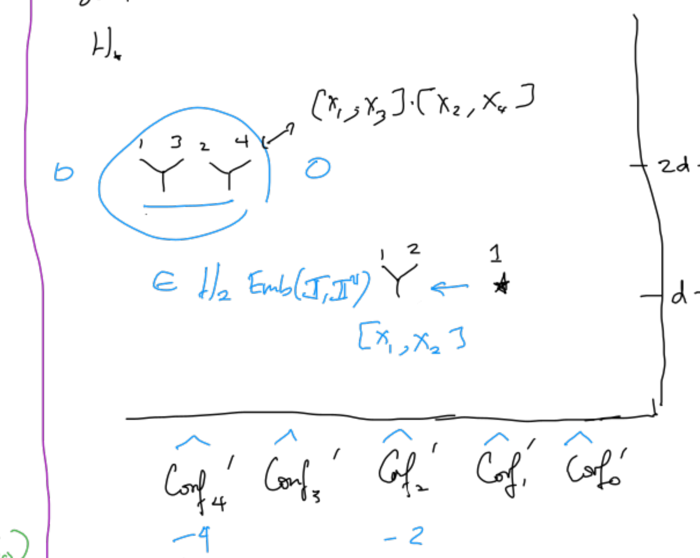
Excision / connectivity estimates follow in this case from standard ones.

In spectral sequences, replace $\pi_*\mathrm{Conf}_n'(\mathbb{R}^d)$ by $H_*\mathrm{Conf}_n'(\mathbb{R}^d)$! ($\pi_*(\mathbb{Z}\,\mathrm{Conf}_n'(\mathbb{R}^d)) \cong$)

Geometric homology & cohomology of embedding spaces

d=4

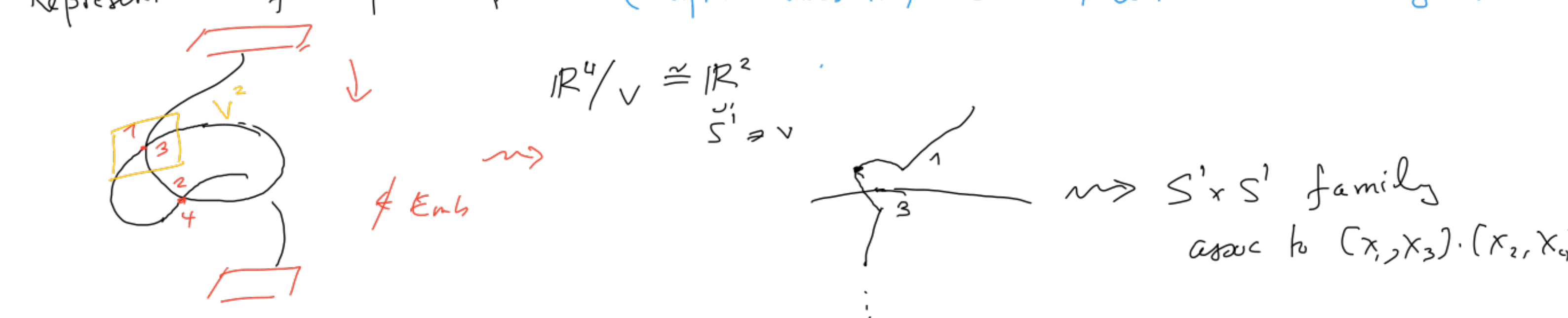
H_*



$[x_1,x_3][x_2,x_4]$ $\quad 2d-2 = 6$ $\quad d-1 = 3$

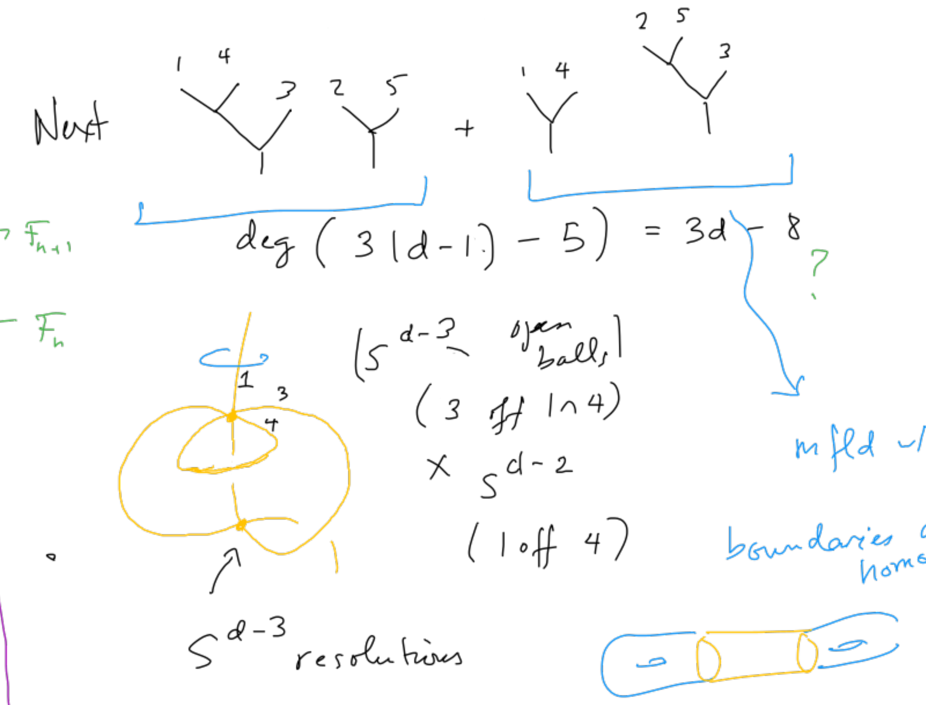
$\in H_2\,\mathrm{Emb}(I, I^4)$ $\quad [x_1,x_2]$

Representative of (after Vassiliev; Cattaneo, Cotta-Ramusino-Longoni).



$\mathbb{R}^4/v \cong \mathbb{R}^3$, S^1_v $\quad \notin \mathrm{Emb}$ $\quad \rightsquigarrow S^1 \times S^1$ family assoc to $[x_1,x_3]\cdot[x_2,x_4]$

Next



deg $(3(d-1) - 5) = 3d - 8$?

$(S^{d-2}$ (open balls) (3 of 1n4) $\times S^{d-2}$ (1 off 4)

S^{d-3} resolutions

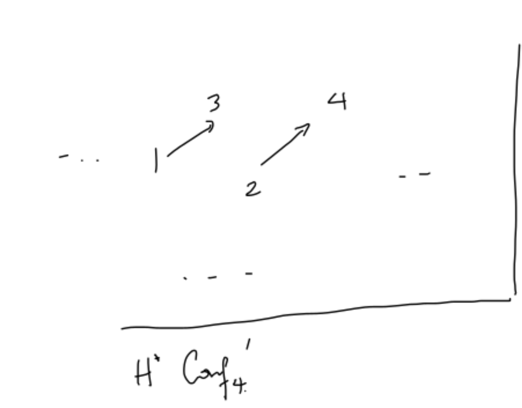
mfld w/ ∂ — boundaries are homologous

F_{n+1}, $P_n\mathrm{Emb} \leftarrow F_n$, $P_{n-1}\mathrm{Emb}$

Conjecture: Homology classes all produced by resolving singularities guided by d_1-differential in GW SS. In particular, SS collapses / $\mathbb{Z}$.

Compare w/ Vassiliev perspective

Cohomology:



$2(d-1)$

$H^*\mathrm{Conf}_4'$

"Subfold" of $f \in \mathrm{Emb}(I, I^d)$ s.t. $\exists\, t_1, \dots t_4$ s.t. (fixing $u, v \in S^{d-1}$) $\frac{f(t_3)-f(t_1)}{\|\ \|} = u$, $\frac{f(t_4)-f(t_2)}{\|\ \|} = v$

union with...

Conjecture: Chern-Simons integrals, as developed by Bott-Taubes, Cattaneo-Cotta-Ramusino-Longoni '02, span $H^*(\mathrm{Emb}(I, I^d); \mathbb{R})$.

$\int_{S^1}$ known to have same rank as H^* (abstractly), but not known to be ≅.

Conjecture: Collapse of H^*-SS / $\mathbb{Z}$, w/ classes produced through Koytcheff's cut-and-paste classes.