Embedding obstructions in 4-space from the Goodwillie-Weiss calculus and Whitney disks

Slava Krushkal
(joint work with Greg Arone)

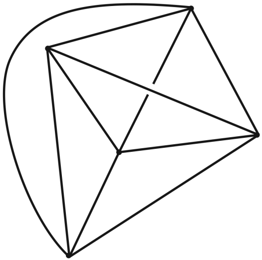
January 13, 2021

Outline:

- Background and history on embeddings of complexes into Euclidean spaces
- 2-complexes in $\mathbb{R}^4$
- New obstructions from intersections of Whitney disks
- From Whitney disks to maps of configuration spaces
- New obstructions from (mainly the bottom of) a (weaker version of) the Goodwillie-Weiss tower
- The two obstructions are equal!
- A cohomological obstruction
- Questions

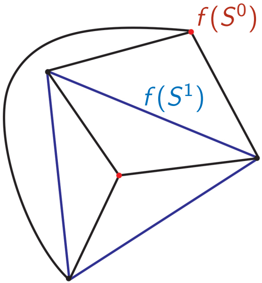
Warm-up: some basic topological combinatorics

A non-planar graph, K_5:



$K_5 \;=\;$ 1-skeleton of the 4-simplex

Note: K_5 minus an edge is planar.



Moreover, for *any* embedding

$$f : (K_5 \text{ minus edge}) \hookrightarrow \mathbb{R}^2,$$

$f(S^0)$ links $f(S^1)$ where $f(S^0)$ is the boundary of the missing edge.

Historical background:

By general position any n-complex embeds into $\mathbb{R}^d$, $d > 2n$.

A geometric description of the van Kampen obstruction (1929)

$$o(K) \in H^{2n}_{\mathbb{Z}/2}(K \times K \smallsetminus \Delta; \mathbb{Z})$$

to embeddability of an n-complex K into $\mathbb{R}^{2n}$.

Here Δ is the "simplicial diagonal" consisting of all products of simplices $\sigma_1 \times \sigma_2$ having a vertex in common.

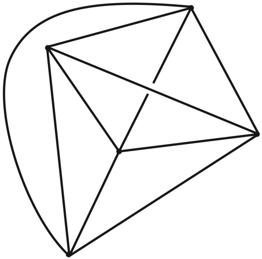
A geometric description of the van Kampen obstruction (1929)

$$o(K) \in H^{2n}_{\mathbb{Z}/2}(K \times K \smallsetminus \Delta; \mathbb{Z})$$

Consider any general position map $f\colon K \longrightarrow \mathbb{R}^{2n}$. Endow the n-cells of K with arbitrary orientations, and for any two n-cells σ_1, σ_2 without vertices in common, consider the algebraic intersection number $f(\sigma_1) \cdot f(\sigma_2) \in \mathbb{Z}$. This gives an equivariant cochain

$$o_f\colon C_{2n}(K \times K \smallsetminus \Delta) \longrightarrow \mathbb{Z}.$$

Since this is a top-dimensional cochain, it is a cocycle. Its cohomology class equals the van Kampen obstruction $o(K)$.



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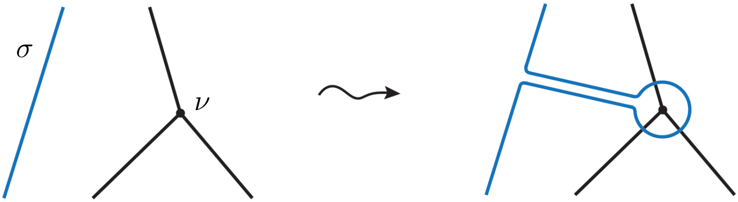
The fact that this cohomology class is independent of a choice of f may be seen geometrically. Any two general position maps $f_0, f_1 : K \longrightarrow \mathbb{R}^{2n}$ are connected by a 1-parameter family of maps f_t where at a non-generic time t_i an n-cell σ intersects an $(n-1)$-cell ν. Topologically the map $f_{t_i - \epsilon}$ and $f_{t_i + \epsilon}$ differ by tubing σ into a small n-sphere linking ν in $\mathbb{R}^{2n}$.



The effect of this elementary homotopy on the van Kampen cochain is precisely the addition of the elementary coboundary $\delta(\sigma \times \nu)$, where $\delta : C^{2n-1}(K \times K \setminus \Delta) \longrightarrow C^{2n}(K \times K \setminus \Delta)$.

If $f : K \hookrightarrow \mathbb{R}^{2n}$ then there is a $\mathbb{Z}/2$- equivariant map

$$f \times f : K \times K \setminus \Delta \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n} \setminus \Delta.$$

Algebraic characterization of the van Kampen obstruction:

$o(K) \in H^{2n}_{\mathbb{Z}/2}(K \times K \setminus \Delta; \mathbb{Z})$ is an obstruction to the existence of a $\mathbb{Z}/2$- equivariant map $K \times K \setminus \Delta \longrightarrow (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \setminus \Delta$.

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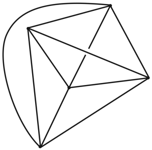
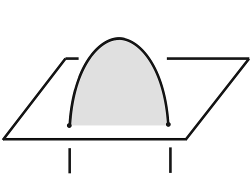
Example Let $K =$ the n-skeleton of the $2n+2$-simplex. Then $o(K) \neq 0$.



$K_5 \; = \;$ 1-skeleton of the 4-simplex

Theorem (van Kampen, Shapiro, Wu) Let $n > 2$. An n-complex K embeds into $\mathbb{R}^{2n}$ if and only if the van Kampen obstruction $o(K) \in H^{2n}_{\mathbb{Z}/2}(K \times K \setminus \Delta; \mathbb{Z})$ is trivial.

Idea of the proof: If the cohomology class $o(K)$ is trivial, there is a an immersion $f : K \longrightarrow \mathbb{R}^{2n}$ where non-adjacent simplices have zero algebraic intersection number. Use the Whitney trick (for intersections and self-intersections) to find an embedding.



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Building on work of Haefliger, Weber extended the embeddability result to the "metastable range" of dimensions:

Given an m-dimensional simplicial complex K and a $\mathbb{Z}/2$-equivariant map $f_2\colon\ \mathrm{C}(K,2) \longrightarrow \mathrm{C}(\mathbb{R}^d, 2)$ with

$$2d \geq 3(m+1),$$

there exists a PL embedding $f\colon\ K \longrightarrow \mathbb{R}^m$ such that the induced map f^2_Δ is $\mathbb{Z}/2$-equivariantly homotopic to f_2.

Theorem (van Kampen, Shapiro, Wu) Let $n > 2$. An n-complex K embeds into $\mathbb{R}^{2n}$ if and only if the van Kampen obstruction $o(K) \in H^{2n}_{\mathbb{Z}/2}(K \times K \setminus \Delta; \mathbb{Z})$ is trivial.

Idea of the proof: If the cohomology class $o(K)$ is trivial, there is a an immersion $f: K \longrightarrow \mathbb{R}^{2n}$ where non-adjacent simplices have zero algebraic intersection number. Use the Whitney trick to find an embedding.

Theorem (Freedman - K. - Teichner, 1994) There exist 2-complexes K which do not embed into $\mathbb{R}^4$ but the van Kampen obstruction $o(K)$ is trivial.

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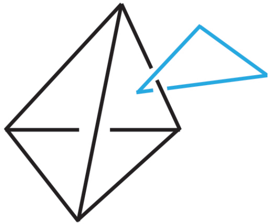
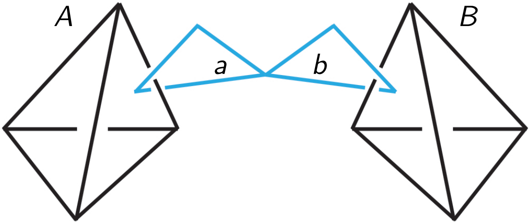
Idea of the proof: Let $K =$ the 2-skeleton of the 6-simplex. Then $o(K) \neq 0$, so K does not embed into $\mathbb{R}^4$.

Lemma Let K' be K with a single 2-cell removed. Then K' embeds into $\mathbb{R}^4$, and moreover for *any* embedding $f\colon K' \hookrightarrow \mathbb{R}^4$, the mod 2 linking number of $f(S^1)$, $f(S^2)$ is non-zero. Here S^1 is the boundary of the missing 2-cell.



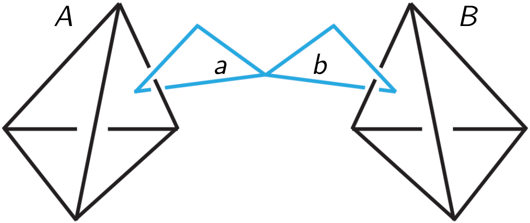
Sketch of the proof, continued



Take the wedge sum of two copies of K' and consider the 2-complex L is obtained by attaching a 2-cell along the commutator $[a, b]$.

Claims:

1. $o(L) = 0$
2. L does not embed into $\mathbb{R}^4$.



Claim: L does not embed into $\mathbb{R}^4$.

Use Stallings' theorem: Suppose $f\colon X \longrightarrow Y$ induces an isomorphism on H_1 and an epimorphism on H_2. Then for all k, f induces an isomorphism

$$\pi_1(X)/\pi_1^k(X) \;\cong\; \pi_1(Y)/\pi_1^k(Y)$$

where π^k is the k-th term of the lower central series, $\pi^2 = [\pi, \pi], \pi^k = [\pi^{k-1}, \pi]$.
Take $X = S^1 \vee S^1$, $Y = \mathbb{R}^4 \smallsetminus (S^2 \sqcup S^2)$. It follows that $\pi_1(\mathbb{R}^4 \smallsetminus (S^2 \sqcup S^2))$ is like the free group modulo any term of the l.c.s., so $[a, b] \neq 1$, a contradiction.

Theorem (Freedman - K. - Teichner, 1994) There exist 2-complexes K which do not embed into $\mathbb{R}^4$ but the van Kampen obstruction $o(K)$ is trivial.

Non-planarity of graphs is characterized by the Kuratowski theorem: a graph G is non-planar if and only if it contains K_5 or $K_{3,3}$ as a minor.

Question Is there a collection of “minors” characterizing none-embeddability of 2-complexes into $\mathbb{R}^4$?

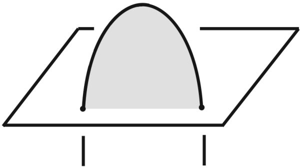
Question Is there a set of geometric moves relating any two maps of 2-complexes $f\colon\ K \longrightarrow \mathbb{R}^4$ with trivial van Kampen cochain?

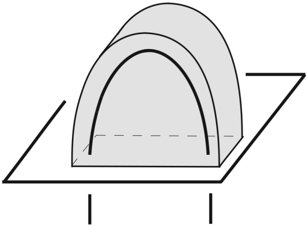
New work (joint with Greg Arone): a higher obstruction theory

In dimensions $n > 2$ the vanishing of the van Kampen class is necessary and sufficient for embeddability of an n-complex K into $\mathbb{R}^{2n}$.

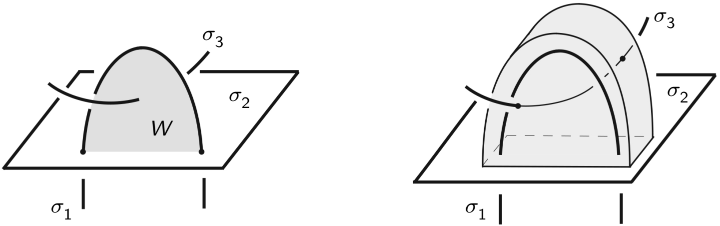
For $n = 2$ the obstruction is incomplete, due to the failure of the Whitney trick in 4 dimensions. The goal is to introduce higher obstructions, measuring the failure of the Whitney trick.

Whitney trick with embedded Whitney disk:

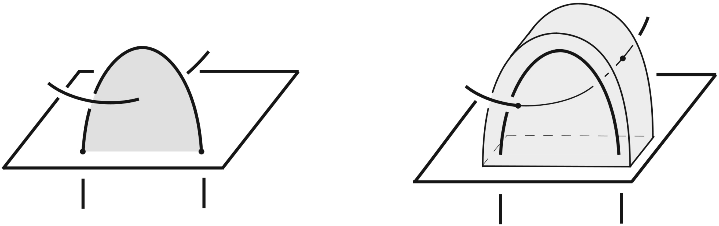




The generic case in dim 4: the Whitney disk W intersects another surface:



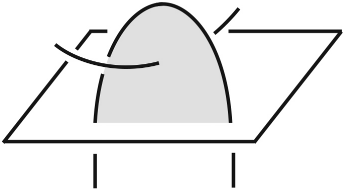
The Whitney move removes intersections $\sigma_1 \cap \sigma_2$ but creates new intersections $\sigma_2 \cap \sigma_3$

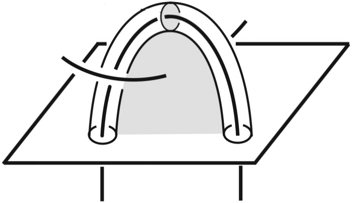


A neighborhood of the Whitney disk in $\mathbb{R}^4$ is a 4-ball D^4, and the the intersection of the three surfaces with ∂D^4 forms the Borromean rings:

Geometric higher obstructions for 2-complexes in $\mathbb{R}^4$:

Assume $o(K) = 0$. Consider an immersion $f\colon\ K \longrightarrow \mathbb{R}^4$ where all non-adjacent 2-cells have trivial algebraic intersection number. Pair up $\pm$ pairs of intersections, and choose Whitney disks W.





$C_s(K, 3) :=$ "simplicial" configuration spaces with all products $\sigma_i \times \sigma_j \times \sigma_k$ of simplices removed where at least two of the simplices have a vertex in common.

Define the geometric obstruction measuring triple intersection numbers. This obstruction depends on the choice of Whitney disks W pairing the intersections of non-adjacent 2-cells of K. Let $\sigma_i, i = 1, 2, 3$ be three 2-cells of K.

Consider the 6-cochain:

$$w_3 \colon\ C_6(\mathrm{C_s}(K, 3)) \longrightarrow \mathbb{Z},$$

defined as follows. Let $\sigma_i, \sigma_j, \sigma_k$ be 2-cells of K which pairwise have no vertices in common, and define

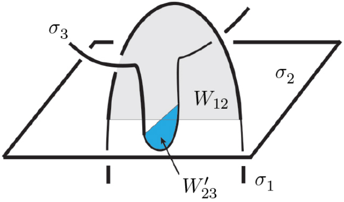
$$w_3(\sigma_i \times \sigma_j \times \sigma_k) \;=\; W_{ij} \cdot f(\sigma_k) + W_{jk} \cdot f(\sigma_i) + W_{ki} \cdot f(\sigma_j),$$

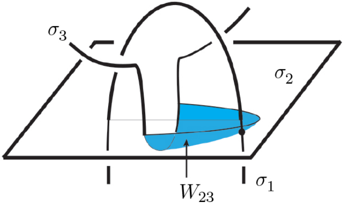
The resulting cohomology class is denoted

$$\mathcal{W}_3(K, f, W) \in H^6_{\Sigma_3}(\mathrm{C_s}(K, 3); \mathbb{Z}[(-1)]).$$

$$w_3(\sigma_i \times \sigma_j \times \sigma_k) = W_{ij} \cdot f(\sigma_k)$$

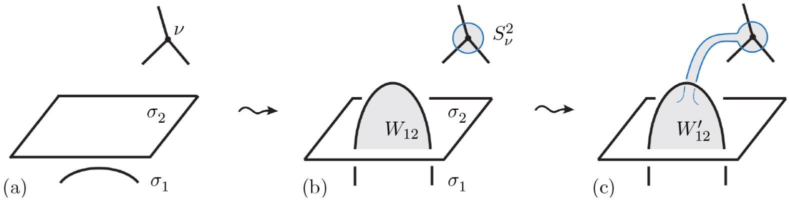
Why consider the sum?





A move that replaces an intersection $W_{12} \cap \sigma_3$ with $W_{23} \cap \sigma_1$

Operations on cochains similar to the van Kampen obstruction:



If the cohomology class is trivial, there exists a Whitney tower of height 2

Theorem *Let $f\colon K \longrightarrow \mathbb{R}^4$ be an immersion with double points paired up with Whitney disks W. Suppose the cohomology class*

$$\mathcal{W}_3(K, f, W) \in H^6(C_s(S, 3))$$

is trivial. Then there exists a map $\widetilde{f}\colon K \longrightarrow \mathbb{R}^4$ which admits a Whitney tower of order 2.

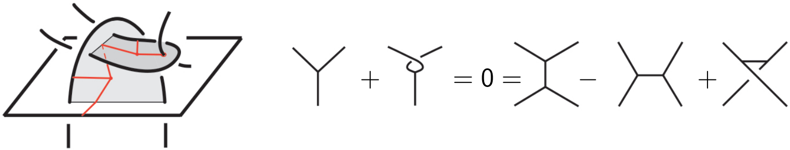
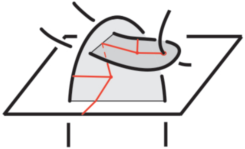
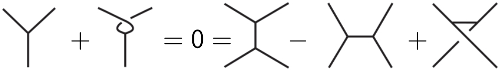


Figure: Left: a Whitney tower of order 2 and the associated tree. Right: the AS relation and the IHX relation

$\mathcal{T}_n :=$ the free abelian group generated by trivalent trees with n leaves, modulo the AS and IHX relations.

A tower of obstructions defined using intersections of Whitney towers, studied by (Conant-)Schneiderman-Teichner.





$$\tau_n(W) := \sum_p \epsilon(p)\, t_p \ \in \mathcal{T}_n,$$

where the sum is taken over all unpaired (order n) intersections points p, and $\epsilon(p)$ is the sign of the intersection. Consider the Σ_n-equivariant $2n$-cochain:

$$w_n\colon\ C_{2n}(\mathrm{C_s}(K,n)) \longrightarrow \mathcal{T}_{n-2},$$

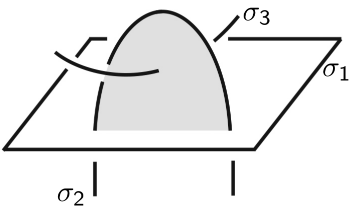
The resulting cohomology class is denoted

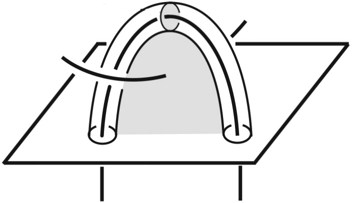
$$\mathcal{W}_{\mathrm{n}}(K,W) \in H^{2n}_{\Sigma_n}(\mathrm{C_s}(K,n);\mathcal{T}_{n-2}).$$

From Whitney disks to $Z/2Z$-equivariant maps of 2-point configuration spaces

Lemma. *Let K be a 2-complex and $f\colon\ K \longrightarrow \mathbb{R}^4$ a general position map such that all intersections of non-adjacent 2-cells are paired up with split Whitney disks W. This data determines a Σ_2-equivariant map $F_{f,W}\colon\ \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^d,2)$.*

Claim: the definition of $F := F_{f,W}$ on the 4-cells $\sigma_i \times \sigma_j$ on the next slide can be extended to give a continuous Σ_2-equivariant map $\mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^d,2)$. (This uses the fact that the original map f and the result of the Whitney move are isotopic.)





$\mathbb{Z}/2\mathbb{Z}$-equivariant map $F\colon\ \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\left(\mathbb{R}^d,2\right)$. Since $f(\sigma_3)$ is disjoint from both $f(\sigma_1)$ and $f(\sigma_2)$, define

$$F|_{\sigma_1\times\sigma_3\,\cup\,\sigma_2\times\sigma_3} := f\times f|_{\sigma_1\times\sigma_2\,\cup\,\sigma_2\times\sigma_3}.$$

Now consider the map $f'\colon\ K \longrightarrow \mathbb{R}^4$ which coincides with f everywhere except for a small disk neighborhood of the Whitney arc in σ_1. In this neighborhood f is defined to be the result of the Whitney move along the Whitney disk W_{12}, making $f'(\sigma_1)$ disjoint from $f'(\sigma_2)$. (As a result of this move, $f'(\sigma_1)$ intersects $f'(3)$.) Define

$$F|_{\sigma_1\times\sigma_2} := f'\times f'|_{\sigma_1\times\sigma_2}.$$

Higher obstructions from the Goodwillie-Weiss calculus

Denote $\mathrm{C}(K,n)$: the ordered n-point configuration space.

If $f : K \hookrightarrow \mathbb{R}^4$ then for each $n \geq 2$ there is a Σ_n- equivariant map

$$\mathrm{C}(K,n) \longrightarrow \mathrm{C}(\mathbb{R}^4,n)$$

Recall: the van Kampen obstruction $o(K) \in H^{2n}_{\Sigma_2}(\mathrm{C}(K,2);\mathbb{Z})$ is an obstruction to the existence of such a map for $n = 2$.

Suppose it is trivial. In this case, does there exist a map for $n = 3$?

If $f : K \hookrightarrow \mathbb{R}^4$ then there is an induced map of of cubical diagrams for K and for $\mathbb{R}^4$:

$$
\begin{tikzcd}
 & \mathrm{C}(X,2) \arrow[rr, "p^1"] \arrow[dd, "p^2"] & & X \arrow[dd] \\
\mathrm{C}(X,3) \arrow[ur, "p^2"] \arrow[rr, "p^1"] \arrow[dd, "p^3"] & & \mathrm{C}(X,2) \arrow[ur, "p^1"] \arrow[dd, "p^2"] & \\
 & X \arrow[rr] & & \{*\} \\
\mathrm{C}(X,2) \arrow[ur, "p^2"] \arrow[rr, "p^1"] & & X \arrow[ur] &
\end{tikzcd}
\tag{1}
$$

When $X = \mathbb{R}^4$, the entries marked X are contractible.

Since van Kampen's obstruction vanishes, there exists

$$f_2\colon\ \mathrm{C}\,(K,2) \longrightarrow \mathrm{C}\,(\mathbb{R}^4,2)$$

The new obstruction is the homotopy lifting obstruction:

$$\begin{tikzcd}
 & & \mathrm{C}\,(\mathbb{R}^d,3) \arrow[d, "p_{\mathbb{R}^4}"] \\
\mathrm{C}\,(K,3) \arrow[r, "p_K"] \arrow[urr, dashed] & \mathrm{C}\,(K,2)^{\times 3} \arrow[r, "(f_2)^3"] & \mathrm{C}\,(\mathbb{R}^4,2)^{\times 3}
\end{tikzcd} \tag{2}$$

It is an element

$$\mathcal{O}_3(K) \in H^6_{\Sigma_3}(\mathrm{C}\,(K,3);\mathbb{Z}[-1])$$

The two obstructions are equal!

Theorem. *Given a 2-complex K with trivial van Kampen's obstruction, let W be a collection of split Whitney disks for double points of a map $f\colon\ K \longrightarrow \mathbb{R}^4$. Let $F_{f,W}\colon\ \mathrm{C_s}(K,2) \longrightarrow \mathrm{C}\,(R^4,2)$ be the Σ_2-equivariant map determined by f, W. Then*

$$\mathcal{W}_3(K,f,W) = i^*\,\mathcal{O}_3(K,F_{f,W}) \in H^6_{\Sigma_3}(\mathrm{C_s}(K,3);\mathbb{Z}[(-1)]),$$

where $i\colon\ \mathrm{C_s}(K,3) \longrightarrow \mathrm{C}\,(K,3)$ is the inclusion map.

(An implicit assumption: the 2-complex has been subdivided so that $\mathrm{C_s}(K,2)$ is (equivariantly) homotopy equivalent to $\mathrm{C}\,(K,2)$.)

Idea of the proof: lift a map of the 5-skeleton and relate the two obstruction cochains.

Outline of the connection with the Arnold class:

A well-known fact:

Lemma

$$p_{\mathbb{R}^d}\colon \; \mathrm{C}\left(\mathbb{R}^d, 3\right) \to \mathrm{C}\left(\mathbb{R}^d, 2\right)^3$$

is surjective in cohomology, and its kernel in cohomology is the ideal generated by the Arnold class.

Here the *Arnold class* is the cohomological element:

$$u_{12} \otimes u_{23} \otimes 1 + 1 \otimes u_{23} \otimes u_{31} - u_{12} \otimes 1 \otimes u_{31} \in$$

$$H^6(\mathrm{C}\left(\mathbb{R}^4, \{1,2\}\right) \times \mathrm{C}\left(\mathbb{R}^4, \{2,3\}\right) \times \mathrm{C}\left(\mathbb{R}^4, \{3,1\}\right)).$$

where u_{ij} is a (certain preferred) generator of $\mathrm{C}\left(\mathbb{R}^4, \{i,j\}\right)$

Outline of the connection with the Arnold class:

By definition, our obstruction $\mathcal{O}_3(K)$ is an element in the Σ_3-equivariant cohomology group $H^6_{\Sigma_3}(\mathrm{C}(K,3);\mathbb{Z}^{\pm})$. There is a natural homomorphism

$$H^6_{\Sigma_3}(\mathrm{C}(K,3);\mathbb{Z}^{\pm}) \to H^6(\mathrm{C}(K,3);\mathbb{Z}^{\pm})^{\Sigma_3} \subset H^6(\mathrm{C}(K,3))$$

Lemma

The image of $\mathcal{O}_3(K)$ in $H^6(\mathrm{C}(K,3))$ under this homomorphism is (the image of) the Arnold class under the map

$$p_k \circ f_2^3\colon\ \mathrm{C}(K,3) \to \mathrm{C}(K,2)^3 \to \mathrm{C}(\mathbb{R}^4,2)^3.$$

Work in progress with Greg Arone, Danica Kosanović, Rob Schneiderman and Peter Teichner:

• A "non-repeating" tower T_n for link maps of 2-spheres into a 4-manifold (any π_1)

• A Whitney tower of height n gives a point in T_n

• The Whitney tower obstruction of Schneiderman-Teichner equals the obstruction to lifting to T_{n+1}