A SHORT INTRODUCTION TO EMBEDDING CALCULUS AND CONFIGURATION CATEGORIES

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These are some rather terse notes written after a talk at the seminar "Building bridges" organised by Danica Kosanovic. I hope they are nevertheless more readable than the scribbles I made during the talk. More importantly, I'm including some references[1] for those who would like to learn more.

Throughout, $\mathrm{emb}(M,N)$ denotes the space of smooth embeddings from M to N with the weak topology (and so composition is continuous), and $\mathrm{imm}(M,N)$ denotes the space of smooth immersions, also with the weak topology.

1. Immersion theory

Theorem 1.1. *(Smale-Hirsch) Let M^m and N^n be smooth manifolds and $m < n$. Then the inclusion*

$$\mathrm{imm}(M,N) \hookrightarrow \mathrm{map}^{O(m)}(\mathrm{frame}_m(M), \mathrm{frame}_m(N))$$

is a weak homotopy equivalence[2]. Here frame_m stands for the tangent m-frame bundle, and $\mathrm{map}^{O(m)}(\dots,\dots)$ the space of $O(m)$-equivariant maps.

For a group G and G-spaces X and Y, a derived G-map $X \to Y$ is the data of: a map $f : X \to Y$; for each $g \in G$, a homotopy $\Delta^1 \to \mathrm{map}(X,Y)$ from $g \cdot f$ to $f \cdot g$; $(\cdots)$ for each n-tuple $g_1, \cdots, g_n$ of elements of G, an n-parameter homotopy

$$\Delta^n \to \mathrm{map}(X,Y)$$

restricting to the previously chosen homotopies on the boundary. I denote the space of such by $\mathrm{map}^{hG}(X,Y)$. Taking constant homotopies there is an inclusion

$$\mathrm{map}^G(X,Y) \hookrightarrow \mathrm{map}^{hG}(X,Y)$$

from the space of G-maps to the space of derived G-maps. (If X is a point, $\mathrm{map}^G(*,Y)$ is the space of G-fixed points of Y, whereas $\mathrm{map}^{hG}(*,Y)$ is the so-called space of homotopy fixed points.) In general, this inclusion is not a weak homotopy equivalence[3], but it is if the G-action on X is free.

The crucial property of $\mathrm{map}^{hG}(X,Y)$ is *homotopy invariance*: if $X \to X'$ is a G-map which is a weak homotopy equivalence, the induced map $\mathrm{map}^{hG}(X,Y) \to \mathrm{map}^{hG}(X',Y)$ is a weak homotopy equivalence (and similarly in the Y variable).

[1] The list is not exhaustive; it is meant as a possible entry point to the topic, and is invariably shaped by my perspective. Please let me know of any unforgivable omission.

[2] As Oscar Randal-Williams pointed out, if M is compact, both spaces are of the homotopy type of CW complexes, so this is an actual homotopy equivalence.

[3] E.g. for the circle with $\mathbb{Z}/2$-action given by reflection, the space of fixed points is discrete, but that's very much not the case for the homotopy fixed point space

Going back to immersion theory: since $O(m)$ acts freely on the m-frame bundle and the derivative-at-zero map $\mathrm{emb}(\mathbb{R}^m, M) \to \mathrm{frame}_m(M)$ is a weak equivalence of $O(m)$-spaces, we conclude that:

$$\mathrm{map}^{O(m)}(\mathrm{frame}_m(M), \mathrm{frame}_m(N)) \simeq \mathrm{map}^{hO(m)}(\mathrm{emb}(\mathbb{R}^m, M), \mathrm{emb}(\mathbb{R}^m, N))$$

In particular, Smale-Hirsch says that the homotopy type of the space of immersions only depends on the $O(m)$-homotopy type of the frame bundles. The reformulation in terms of derived mapping spaces is not necessarily helpful for direct computation here, but it has some advantages, e.g. it has a meaningful generalisation.

Example 1.2. The space of immersions between discs $\mathrm{imm}(D^m, D^n) \simeq \Omega^m V_{m,n}$ where $V_{m,n}$ is the Stiefel manifold of linear injections $\mathbb{R}^m \to \mathbb{R}^n$.

2. Embedding calculus

The standard references are [26] and [17] (for the modern accounts closer to this section, see [7] and [4]). For $k \geq 0$, write $\underline{k} = \{1, \dots, k\}$ ($\underline{0}$ is the empty set) and $\mathbf{S}$ for the category of spaces and continuous maps. The idea, starting from immersion theory, is to replace:

$$\begin{array}{ccc} \text{the manifold } \mathbb{R}^m & \text{by} & \text{the manifolds } \mathbb{R}^m \times \underline{k}, \text{ for } k \geq 0 \\ \text{the group } O(m) & \text{by} & \text{a category } \mathsf{Disk} \\ \text{the space } \mathrm{emb}(\mathbb{R}^m, M) & \text{by} & \text{a functor } \mathrm{emb}(-, M) : \mathsf{Disk}^{\mathrm{op}} \to \mathbf{S} \end{array}$$

This category Disk has as objects the non-negative integers $i \geq 0$; a morphism $i \to j$ is a smooth embedding $\underline{i} \times \mathbb{R}^m \to \underline{j} \times \mathbb{R}^m$. (Note: the dimension m is fixed.) The morphism sets are naturally morphism *spaces*, and composition is continuous. The functor $\mathrm{emb}(-, M)$ is clear: the value on an object i is $\mathrm{emb}(\underline{i} \times \mathbb{R}^m, M)$ and morphisms act by composition.

Note also that $\mathrm{emb}(\underline{i} \times \mathbb{R}^m, M)$ is homotopy equivalent to the space of framed configurations of i points in M, i.e. the m-frame bundle of $\mathrm{emb}(\underline{i}, M)$. A point in the latter space is an injection $\underline{k} \hookrightarrow M$ together with an m-frame at each point.

Remark 2.1. In operadic language, Disk is the (PROP associated to) the framed little m-discs operad. A functor $\mathsf{Disk}^{\mathrm{op}} \to \mathbf{S}$ is a right module.

Varying the number of disks, we have subcategories

$$\mathsf{Disk}_{\leq 1} \subset \mathsf{Disk}_{\leq 2} \subset \mathsf{Disk}_{\leq 3} \subset \cdots \subset \mathsf{Disk}\,.$$

($\mathsf{Disk}_{\leq 1}$ is $O(m)$, viewed as a category with a single object, with the empty set thrown in as an object.). And by restriction $\mathrm{emb}(-, M)$ can be viewed as a functor on each $\mathsf{Disk}_{\leq k}$.

Definition 2.2. Let $k \geq 0$. The k-th approximation to the space of embeddings is the space

$$T_k\mathrm{emb}(M, N) := \mathrm{map}^h_{\mathsf{Disk}_{\leq k}}(\mathrm{emb}(-, M), \mathrm{emb}(-, N))$$

(k may be ∞, in which case $\mathsf{Disk}_{\leq k} = \mathsf{Disk}$.)

Explanations: Without the h, $\mathrm{map}_{\mathsf{Disk}_{\leq k}}(\mathrm{emb}(-, M), \mathrm{emb}(-, N))$ means *space of natural transformations* from the functor $\mathrm{emb}(-, M)$ to $\mathrm{emb}(-, N)$. It is a subspace of

$$\prod_{i\geq 0} \mathrm{map}(\mathrm{emb}(\underline{i}\times\mathbb{R}^m, M), \mathrm{emb}(\underline{i}\times\mathbb{R}^m, N))$$

where map denotes the space of maps (with respect to the compact-open topology). In analogy with what was said before, a point in $T_k\mathrm{emb}(M,N)$ can be concretely thought of as a coherent choice, for every n-tuple $(i_0,\dots,i_n)$, $n\geq 0$, of an n-parameter homotopy

$$\mathrm{mor}(i_0,\cdots,i_n)\times\Delta^n \to \mathrm{map}(\mathrm{emb}(i_0\times\mathbb{R}^m, M), \mathrm{emb}(i_n\times\mathbb{R}^m, N))$$

where $\mathrm{mor}(i_0) := \{i_0\}$ and, for $n\geq 1$, $\mathrm{mor}(i_0,\cdots,i_n)$ denotes the space of morphisms $i_0\to\cdots\to i_n$ in $\mathsf{Disk}_{\leq k}$, that is, the product $\mathrm{emb}(\underline{i}_0\times\mathbb{R}^m, \underline{i}_2\times\mathbb{R}^m)\times\cdots\times \mathrm{emb}(\underline{i}_{n-1}\times\mathbb{R}^m, \underline{i}_n\times\mathbb{R}^m)$.

Remark 2.3. As in immersion theory, the key point is *homotopy invariance*: it may not be completely apparent from the description above, but $T_k\mathrm{emb}(M,N)$ only depends on the weak homotopy type of the functors $\mathrm{emb}(-,M)$ and $\mathrm{emb}(-,N)$ on $\mathsf{Disk}_{\leq k}$. This is related to the recurring question: why *derived* mapping spaces? Ultimately, we are interested in the homotopy (and homology) of embedding spaces, so whatever spaces we find as approximations should be invariant under homotopy equivalences. The concrete description above does not really say what derived mapping spaces *are*, or how to compute them. This is explained by the theory of derived functors; but one can think of the relation between map and map^h as akin to the relation between Hom and Ext in homological algebra.

A side point: $T_k\mathrm{emb}(M,N)$ can be regarded as a homotopy limit over the category $\mathsf{Disk}_{\leq k}/M$. (We have a choice as to whether we remember the topology of objects and morphisms in $\mathsf{Disk}_{\leq k}/M$ or not. It turns that, for the purpose of computing that homotopy limit, it does not matter.)

Evaluation (restriction) provides inclusions

$$\mathrm{emb}(M,N)\hookrightarrow \mathrm{map}_{\mathsf{Disk}}(\mathrm{emb}(-,M),\mathrm{emb}(-,N))\hookrightarrow T_\infty$$

and hence a tower

$$\mathrm{emb}(M,N)\to T_\infty\to\cdots\to T_k\to\cdots\to T_1$$

Theorem 2.4 (Goodwillie-Klein [16]). *The k-th approximation*

$$ev_k : \mathrm{emb}(M,N)\to T_k\mathrm{emb}(M,N)$$

is $(3-n+(m+1))(n-m-2)$-connected, except if $m=1$ and $n=3$. (Kosanovic [20]) Holds also for 1-dimensional connected M and $n=3$.

Theorem 2.5 (Weiss [26]). *The k-th layer $L_k := \mathrm{hofiber}(T_k\to T_{k-1})$ is weakly equivalent to the space of partial sections over a certain fibration*

$$E\to C(k,M)$$

over the space $C(k,M)=\mathrm{emb}(\underline{k},M)/\Sigma_k$ of finite subsets of M of cardinality k, and with fiber given by the homotopy fiber of

$$\mathrm{emb}(\underline{k}\times\mathbb{R}^m, M)\to \operatorname*{holim}_{\underline{s}\subsetneq\underline{k}} \mathrm{emb}(\underline{s}\times\mathbb{R}^m, M)$$

"Partial" means sections are already prescribed in a neighbourhood of the fat diagonal in $M^{\times k}/\Sigma_k\subset C(k,M)$.

Idea of proof. The basepoint T_{k-1} is a derived map $f : \mathsf{Disk}^{\mathrm{op}}_{\leq k-1} \to \mathbf{S}$. We want to describe the space of lifts to a derived map $F : \mathsf{Disk}^{\mathrm{op}}_{\leq k} \to \mathbf{S}$. So the first thing we must do is determine the image of F on the new object $\underline{k}$, i.e. a map

$$F(\underline{k}) : \mathrm{emb}(\underline{k} \times \mathbb{R}^m, M) \to \mathrm{emb}(\underline{k} \times \mathbb{R}^m, N)$$

which should be equivariant for the $\mathrm{Aut}(\underline{k}) = O(m)^{\times k} \rtimes \Sigma_k$ actions. But the space of G-equivariant maps $X \to Y$ between G-spaces, where the G-action on X is free, is identified with the space of sections of the fibration associated to $X \to X/G$ with fiber Y. So a map $F(\underline{k})$ is the same data as a section of the fibration associated to $\mathrm{emb}(\underline{k} \times \mathbb{R}^m, M) \to C(k, M)$ with fiber $\mathrm{emb}(\underline{k} \times \mathbb{R}^m, N)$.

Moreover, if F is to be an extension of f, the following should be verified:

(1) for every embedding $i : \underline{s} \times \mathbb{R}^m \hookrightarrow \underline{k} \times \mathbb{R}^m$ such that the induced map on path components is *injective* (but not bijective), the composition

$$\mathrm{emb}(\underline{k} \times \mathbb{R}^m, M) \xrightarrow{F(k)} \mathrm{emb}(\underline{k} \times \mathbb{R}^m, N) \xrightarrow{i^*} \mathrm{emb}(\underline{s} \times \mathbb{R}^m, N)$$

must factor through $f(\underline{s}) : \mathrm{emb}(\underline{s} \times \mathbb{R}^m, M) \to \mathrm{emb}(\underline{s} \times \mathbb{R}^m, N)$.

(2) for every embedding $i : \underline{k} \times \mathbb{R}^m \hookrightarrow \underline{\ell} \times \mathbb{R}^m$ such that the induced map on path-components is *surjective* (but not bijective), the composition

$$\mathrm{emb}(\underline{\ell} \times \mathbb{R}^m, M) \xrightarrow{i^*} \mathrm{emb}(\underline{k} \times \mathbb{R}^m, M) \xrightarrow{F(k)} \mathrm{emb}(\underline{k} \times \mathbb{R}^m, N)$$

must factor through $f(\ell) : \mathrm{emb}(\underline{\ell} \times \mathbb{R}^m, M) \to \mathrm{emb}(\underline{\ell} \times \mathbb{R}^m, N)$,

Reformulated in terms of section spaces, condition (1) says that the fiber over a point in $C(k, M)$ ought to be the homotopy fiber in the statement of the theorem. And (2) is the condition that sections are already given in a neighbourhood of the fat diagonal. (This is a Reedy-type argument.) □

Example 2.6. If M is the interval, and if we write ∂ for a neighbourhood of the fat diagonal, then the inclusion $\partial \subset C(k, M)$ is homotopy equivalent to the boundary inclusion $\partial\Delta^{k-2} \subset \Delta^{k-2}$ for $k > 2$. So, for $k > 2$, L_k is identified with the $(k-2)$-fold loop space Ω^{k-2} of the homotopy fiber in the statement of the theorem. This case has been studied extensively by Sinha [23], who first explained a relation to the little disks operad.

3. Configuration categories

Having thickened points to balls, the differential and configuration space data are intertwined. We would like to separate them.

Definition 3.1. Let M be a *topological* manifold. The configuration category of M – denoted $\mathsf{con}(M)$ – is the category whose

- objects are configurations of points in M, i.e. injections $\underline{k} \to M$, for $k \geq 0$.
- a morphism from $x : \underline{k} \to M$ to $y : \underline{\ell} \to M$ is a pair (f, H) where $f : \underline{k} \to \underline{\ell}$ is a map of finite sets and H is an exit path in M^k from x to fy: a path $H_t \in M^k$ such that if the i^{th} and j^{th} component of H_T agree for some T, then they agree for all $t \geq T$.

Composition is given by concatenation, and the identity morphisms are the constant paths. There is an obvious projection functor $\mathsf{con}(M) \to \mathsf{Fin}$. The category $\mathsf{con}(M)$ is naturally a category internal to spaces (i.e. it has *spaces* of objects and morphisms). Finally, for each $k \geq 0$, there is a subcategory $\mathsf{con}(M; k)$ spanned by the objects consisting of $\ell \leq k$ points.

Evaluation gives maps of spaces

$$\mathrm{emb}(M,N) \hookrightarrow \mathrm{map}_{\mathsf{Fin}}(\mathsf{con}(M),\mathsf{con}(N)) \hookrightarrow \mathrm{map}^h_{\mathsf{Fin}}(\mathsf{con}(M),\mathsf{con}(N))$$

and the composition factors through $T_k\mathrm{emb}(M,N)$.

Theorem 3.2 ([8])**.** *Given smooth manifolds M and N and $k \geq 1$, there is homotopy pullback square:*

$$\begin{tikzcd} T_k\mathrm{emb}(M,N) \arrow[r] \arrow[d] & \mathrm{map}^h_{\mathsf{Fin}}(\mathsf{con}(M;k),\mathsf{con}(N)) \arrow[d] \\ \Gamma(p) \arrow[r] & \Gamma(p'_k) \end{tikzcd}$$

where the meaning of the lower row is explained below.

The space $\Gamma(p)$ is the space of sections over the fibration $p : E \to M$ where

$$E := \{(x,y,F) : x \in M, y \in N, F : T_xM \hookrightarrow T_yN\}\ .$$

So $\Gamma(p)$ is simply $T_1\mathrm{emb}(M,N)$, or $\mathrm{imm}(M,N)$, by Smale-Hirsch.

Moreover, a linear injective map $T_xM \to T_yN$ gives, by restriction, a (derived) map $\mathsf{con}(T_xM) \to \mathsf{con}(T_yN)$ over Fin. So, writing

$$E'_k := \{(x,y,F) : x \in M, y \in N, F : \mathsf{con}(T_xM;k) \to \mathsf{con}(T_yN;k)\}$$

and $p'_k : E'_k \to M$ for the projection, we have inclusions:

$$\Gamma(p) \to \Gamma(p'_k)\ .$$

for each k. This is the lower row in the theorem. It forgets the vector bundle structure, it only remembers its "configuration space" part.

Example 3.3. The case $k = 2$. It is not difficult to prove that

$$\mathrm{map}^h_{\mathsf{Fin}}(\mathsf{con}(\mathbb{R}^m;2),\mathsf{con}(\mathbb{R}^n;2)) \simeq \mathrm{map}^{h\Sigma_2}(S^{m-1},S^{n-1})$$

where Σ_2 acts via the antipodal map on the spheres. So, for $k = 2$, the lower row in the square of the theorem is essentially the forgetful map from the space of tangent bundle monomorphisms $TM \to TN$ to the space of Σ_2-equivariant maps of spherical tangent bundles $STM \to STN$, where Σ_2 acts antipodally on fibers. (The configuration category of M contains STM as the subspace of morphisms in $\mathsf{con}(M)$ whose underlying map of finite sets is $\underline{2} \to \underline{1}$.) The top right-hand corner is essentially the space of maps (with Σ_2-equivariance) from the diagram $M \leftarrow STM \hookrightarrow \mathrm{emb}(\underline{2},M)$ to the diagram $N \leftarrow STN \hookrightarrow \mathrm{emb}(\underline{2},N)$. c.f. Haefliger's metastable range approximation to embedding spaces.

Remark 3.4. The right-hand column only depends on the configuration categories of M and N, and as such it does not depend on the smooth structures of M and N. Moreover (see below), the lower horizontal map is roughly $(2n-3m-4)$-connected, so in that range $T_k\mathrm{emb}(M,N)$ does not depend on the smooth structures of M and N. If M is 1-dimensional more is true: the lower horizontal map is an isomorphism on homotopy groups for $k = 2$ and injective for $k > 2$. Therefore, taking $k = \infty$ and using Goodwillie-Klein estimates, $\mathrm{emb}(M,N)$ does not depend on the smooth structure of N whenever $n \geq 4$. The relation between embedding calculus and smooth structures has been recently investigated by Arone-Szymik [3] and Knudsen-Kupers [19].

4. Long knots

In the case of long knots $D^m \to D^n$ which restrict to the standard inclusion near the boundary, the top-right hand corner in theorem 3.2 turns out to be contractible ("Alexander trick" ...), so we conclude:

Theorem 4.1. *There is a homotopy fiber sequence*

$$T_k\mathrm{emb}^\partial(D^m, D^n) \to \Omega^m V_{m,n} \to \Omega^m \mathrm{map}^h_{\mathsf{Fin}}(\mathsf{con}(\mathbb{R}^m; k), \mathsf{con}(\mathbb{R}^n; k))$$

where the basepoints are those corresponding to the standard inclusion $\mathbb{R}^m \to \mathbb{R}^n$.

Remark 4.2 (Operads!)**.** The space $\mathrm{map}^h_{\mathsf{Fin}}(\mathsf{con}(\mathbb{R}^m), \mathsf{con}(\mathbb{R}^n))$ turns out to be weakly equivalent to the space $\mathrm{map}^h(E_m, E_n)$ of derived operad maps between little disks operads [8]. Weiss also proved the corresponding truncated statement.

In particular, if $n - m \geq 2$, the space $\mathrm{emb}^\partial(D^m, D^n)$ is an m-fold loop space with delooping given by the homotopy fiber of the map

$$V_{m,n} \to \mathrm{map}^h_{\mathsf{Fin}}(\mathsf{con}(\mathbb{R}^m), \mathsf{con}(\mathbb{R}^n)) \ . \tag{4.1}$$

Writing $\overline{emb}^\partial(D^m, D^n)$ for the homotopy fiber of $\mathrm{emb}^\partial(D^m, D^n) \to \Omega^m V_{m,n}$, it follows that

$$\overline{emb}^\partial(D^m, D^n) \simeq \Omega^m \mathrm{map}^h_{\mathsf{Fin}}(\mathsf{con}(\mathbb{R}^m), \mathsf{con}(\mathbb{R}^n)) \ . \tag{4.2}$$

Remark 4.3. The equivalence (4.2) was first proved, when $m = 1$, by Dwyer-Hess [14] ($k = \infty$) and Turchin [24] (any k) by operadic methods. In [8], we proved the general case, theorem 4.1. A different proof has appeared in [12] and [13], generalising the operad-theoretic arguments of Turchin's first proof.

Here is a sample consequence:

Corollary 4.4 (Haefliger, Budney [9])**.** *For $n - m > 2$, $\mathrm{emb}^\partial(D^m, D^n)$ is $(2n - 3m - 4)$-connected.*

Proof. By Goodwillie-Klein, the map $\mathrm{emb}^\partial(D^m, D^n) \to T_2\mathrm{emb}^\partial(D^m, D^n)$ is $(2n - 3m - 4)$-connected. The right-hand map in theorem 4.1, with $k = 2$ is, by the discussion in example 3.3, the m-fold loop space Ω^m of the map

$$V_{m,n} \to \mathrm{map}^{\Sigma_2}(S^{m-1}, S^{n-1}) \ .$$

This map is $(2n - 2m - 3)$-connected (Haefliger-Hirsch: deduce it by induction on m. The map from the fiber of $V_{m,n} \to V_{m-1,n}$ to the fiber of $\mathrm{map}^{\Sigma_2}(S^{m-1}, S^{n-1}) \to \mathrm{map}^{\Sigma_2}(S^{m-2}, S^{n-1})$ is the unit map $S^{n-m} \to \Omega^m \Sigma^m S^{n-m}$, and so the Freudenthal suspension theorem applies). Therefore, $T_2\mathrm{emb}^\partial(D^m, D^n)$ is $(2n - 3m - 4)$-connected. ∎

Question 4.5. By regarding $\mathbb{R}^n$ as $\mathbb{R}^m \times \mathbb{R}^{n-m}$ and using the known corresponding statement for operads $\mathsf{con}(\mathbb{R}^n) \simeq \mathsf{con}(\mathbb{R}^m) \boxtimes \mathsf{con}(\mathbb{R}^{n-m})$ (the so-called additivity theorem), it follows that the space of linear isometries $O(n - m)$ acts on the map (4.1) in a basepoint preserving manner. This means that $T_\infty\mathrm{emb}^\partial(D^m, D^n)$ is in fact an E_m-algebra in $O(n - m)$-spaces. Is the evaluation map $\mathrm{emb}^\partial(D^m, D^n) \to T_\infty\mathrm{emb}^\partial(D^m, D^n)$ equivariant? (What should the action be on the source? There's an obvious one, spinning.)

4.1. **The first non-trivial homotopy group, $m = 1$.** Specifying to $m = 1$, we have that $\mathrm{emb}^\partial(D^1, D^n)$ is $(2n-7)$-connected. It is also known that

$$\pi_{2n-6} T_3 \mathrm{emb}^\partial(D^1, D^n) \simeq \mathbb{Z}$$

which is $\pi_{2n-6}\mathrm{emb}^\partial(D^1, D^n)$ for $n \geq 4$ by Goodwillie-Klein. (For a sketch, see below.) When $n = 3$, there is a nice geometric interpretation of $\pi_0 \mathrm{emb}^\partial(D^1, D^3) \to \pi_0 T_3 \mathrm{emb}^\partial(D^1, D^3) \cong \mathbb{Z}$ in [11] and its relation to the finite type invariant of degree 2.

About π_{2n-6}: By investigating the layers, one can identify $T_3\mathrm{emb}^\partial(D^1, D^n)$ with Ω^2 of the space of maps (rel ∂)

$$\Delta^1 \simeq C(3, \mathbb{R}^1) \to F_3$$

where

$$F_3 = \mathrm{hofiber}(\mathrm{emb}(\underline{3}, \mathbb{R}^n) \to \prod_{\underline{2} \subset \underline{3}} \mathrm{emb}(\underline{2}, \mathbb{R}^n)) \simeq \mathrm{hofiber}(S^{n-1} \vee S^{n-1} \to S^{n-1} \times S^{n-1})$$

(c.f. Theorem 2.5). Therefore: $T_3 \simeq \Omega^3 \mathrm{hofiber}(S^{n-1} \vee S^{n-1} \to S^{n-1} \times S^{n-1})$. The generator of $\pi_{2n-6} T_3 = \pi_{2n-3} \mathrm{hofiber}(S^{n-1} \vee S^{n-1} \to S^{n-1} \times S^{n-1}) \simeq \mathbb{Z}$ is the Whitehead product of the two inclusions of S^{n-1} in the wedge $S^{n-1} \vee S^{n-1}$.

Remark 4.6. This is really the beginning of a story. The rational homotopy and homology of the space of long knots has been studied extensively using embedding calculus and its relation to the little disks operads and their formality, [2], [1], [21], [4], [5], culminating in [15], see also references in these papers and forthcoming talks in this seminar by Arone and Turchin. For the connection to finite type invariants, starting points are [25], [10], [20]. For non-rational results about the tower, see Geoffroy's talk and [6].

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