GT action on the embedding calculus tower for knots

Geoffroy Horel (USPN, ENS)

BBS, November 25th 2020

Finite type theory

Definition

Fix a linear embedding $j : \mathbb{R} \to \mathbb{R}^3$. The space of long knots, denoted $Emb_c(\mathbb{R}, \mathbb{R}^3)$ is the space of embeddings from $\mathbb{R}$ to $\mathbb{R}^3$ that coïncide with j outside of a compact subset of $\mathbb{R}$.

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Finite type invariants for knots

Definition (Vassiliev, Gusarov, Stanford)

A map $\pi_0(Emb_c(\mathbb{R}, \mathbb{R}^3))) \to A$ with A an abelian group is an additive invariant of degree $\leq k$ if it is a monoid homomorphism and it is invariant under infection by pure braids lying in $\gamma_{k+1}(P_n)$.

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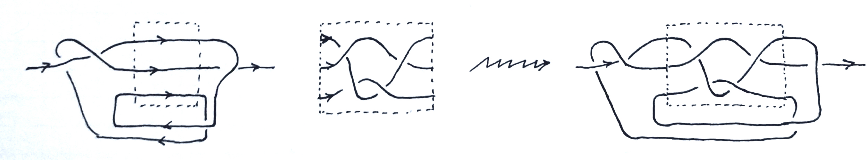

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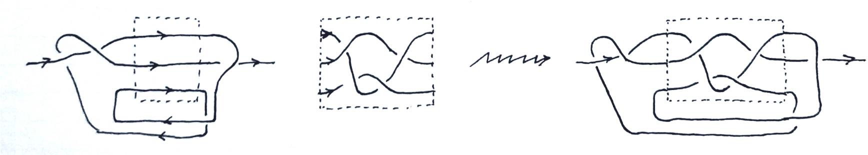

Conjecture (Goodwillie-Weiss,Budney-Conant-Koytcheff-Sinha)

The map $\mathrm{ev}_{k+1} : \pi_0(Emb_c(\mathbb{R}, \mathbb{R}^3)) \to \pi_0 T_{k+1} Emb_c(\mathbb{R}, \mathbb{R}^3)$ is the universal additive invariant of degree $\leq k$.

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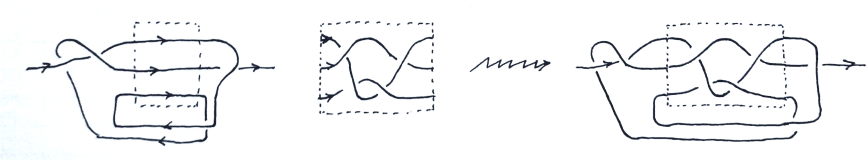

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True after tensoring with $\mathbb{Q}$ (Kontsevich integral). The map ev_{k+1} is a degree $\leq k$ invariant (Budney-Conant-Koytcheff-Sinha, Kosanović-Shi-Teichner)

Finite type invariants for knots

Theorem (Kosanović)

The map ev_{k+1} is the universal additive invariant of degree $\leq k$ if the spectral sequence for $T_{k+1}Emb_c(\mathbb{R}, \mathbb{R}^3)$ collapses at the E^2-page along the diagonal $t = s$.

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Theorem (Boavida de Brito, H.)

The map $\mathrm{ev}_{k+1} : \pi_0(Emb_c(\mathbb{R}, \mathbb{R}^3)) \to \pi_0 T_{k+1}Emb_c(\mathbb{R}, \mathbb{R}^3) \otimes \mathbb{Z}_{(p)}$ *is the universal p-local additive invariant of degree* $\leq k$ *if* $k \leq p+1$*. Furthermore, there is a non-canonical isomorphism*

$$\pi_0 T_{k+1}Emb_c(\mathbb{R}, \mathbb{R}^3) \otimes \mathbb{Z}_{(p)} \cong \oplus_{s \leq k} \mathcal{A}^I_s \otimes \mathbb{Z}_{(p)}$$

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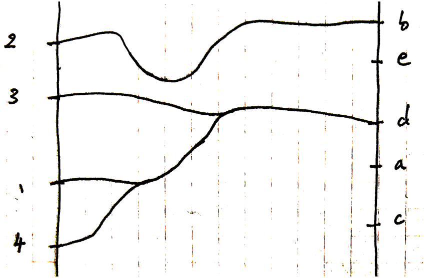

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Let M and N be two smooth manifolds. Then, there is a homotopy cartesian square

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with Γ the space of sections of a fiber bundle over M whose fiber over m is the space of pairs (n, α) with $n \in N$ and $\alpha : \mathrm{con}(T_m M) \to \mathrm{con}(T_n N)$ a map of configuration categories.

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There is a map $Imm(M,N) \to \Gamma'$ with Γ' the space of section of a fiber bundle over M whose fiber over m is the space of pairs $(n.\beta)$ with β an injective linear map $T_mM \to T_nN$. This is often an equivalence (Smale-Hirsch).

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We write $T_k = T_k Emb_c(\mathbb{R}, \mathbb{R}^d)$. We denote by L_k the homotopy fiber of the map $T_k \to T_{k-1}$. We have a fiber sequence

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Theorem

There is a weak equivalence for $2 \leq k \leq \infty$

$$L_k \simeq \Omega^2 \mathrm{hofib}[Map(\mathrm{con}(\mathbb{R}, k), \mathrm{con}(\mathbb{R}^d, k))$$

$$\to Map(\mathrm{con}(\mathbb{R}, k-1), \mathrm{con}(\mathbb{R}^d, k-1))]$$

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We have $\pi_{t-s}(L_s) = \bigcap_{i=0}^{s-1} \ker(\pi_t(s^i))$ *with*

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This can be computed completely in terms of homotopy groups of spheres using the fiber sequence

$$\bigvee_{s-1} S^{d-1} \to Emb(\underline{s}, \mathbb{R}^d) \to Emb(\underline{s-1}, \mathbb{R}^d)$$

Main theorem

Theorem (Boavida-H.)

Let p be a prime. Let $E^r_{-s,t}$ be the Goodwillie-Weiss spectral sequence for $T_\infty Emb(\mathbb{R}, \mathbb{R}^d)$. In the spectral sequence $E^r_{-s,t} \otimes \mathbb{Z}_{(p)}$, in the range $t < 2p - 2 + (s-1)(d-2)$, the only possibly non-zero differential are the d^r with $r - 1$ a multiple of $(p-1)(d-2)$.

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Corollary

- *For $n\leq(p-1)(d-2)+3$ and $i\leq 2p-6+2(d-2)$:*

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- *For $d > 4$ (resp. $d = 4$) and $i < 2p + 2d - 4$ (resp. $i < 2p$) :*

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Main theorem, sketch of proof

Definition

Let X be a simply connected finite type CW-complex. There exists a unique space up to homotopy L_pX called the p-completion of X with a map $X \to L_pX$ such that

- *The map $X \to L_pX$ induces an isomorphism in $H_*(-, \mathbb{F}_p)$*
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Theorem (Boavida, H.)

There is a non-trivial action of $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the tower $\{T_n \otimes \mathbb{Z}_p\}_{n \in \mathbb{N}}$. This action is what forces some of the differentials to be zero.

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- *In the range $t < 2p - 2 + (s-1)(d-2)$, we have $E^1_{-s,t} \otimes \mathbb{Z}_p = 0$ unless $t = n(d-2)+1$.*
- *The Γ-action on $E^1_{-s,n(d-2)+1} \otimes \mathbb{Z}_p$ is cyclotomic of weight n.*

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Construction (Étale homotopy type)

Let X be an algebraic varitey defined over the rational numbers. Then the algebraic p-completion of the homotopy groups of $X(\mathbb{C})_{top}$ have an action of Γ. In fact (in good cases) the homotopy type $L_p X(\mathbb{C})_{top}$ has an action of Γ

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In particular, there is a Galois action on the p-completion of the pure braid groups. This extends to a Galois action on the p-completion of $\mathrm{con}(\mathbb{R}^2)$ (Drinfel'd).
This can be extended to the p-completion of $\mathrm{con}(\mathbb{R}^d)$ via the following theorem.

Theorem (Boavida de Brito-Weiss)

Let M and N be two manifold. There is a functorial way to construct $\mathrm{con}(M \times N)$ from $\mathrm{con}(M)$ and $\mathrm{con}(N)$.