A LIGHT BULB THEOREM FOR DISKS

Danica Kosanović (Paris 13), joint with Peter Teichner (MPIM Bonn)

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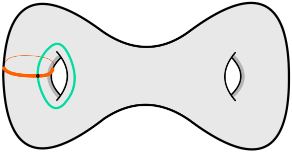
The main trick

Theorem (Space Level LBT)

For $k \leq d - 1$ let M be a compact smooth d-manifold with a pair of smoothly embedded spheres $s\colon S^{k-1} \hookrightarrow \partial M$ and $G\colon S^{d-k} \hookrightarrow \partial M$, such that G has trivial normal bundle and $G \pitchfork s = \{pt\}$.



- Note that a dual pair s, G does not exist in an arbitrary ∂M!

Space Level Light Bulb Theorem

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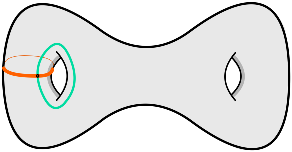
Then there is an explicit pair of homotopy equivalences

$$\mathrm{Emb}_\partial(D^k; M) \underset{\text{amb}}{\overset{\text{fol}^\varepsilon}{\rightleftarrows}} \mathrm{Emb}^\varepsilon_\partial(D^{k-1}; M \cup_G h^{d-k+1}).$$



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For $k \le d - 1$ let M be a compact smooth d-manifold with a pair of smoothly embedded spheres $\mathtt{s}\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$ and $\mathtt{G}\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$, such that $\mathtt{G}$ has trivial normal bundle and $\mathtt{G} \pitchfork \mathtt{s} = \{pt\}$.

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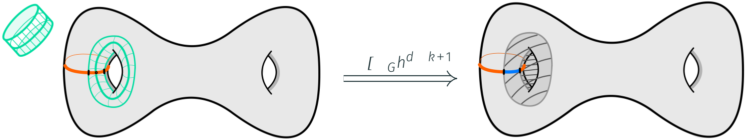
- $\mathrm{Emb}_\partial(\mathbb{D}^k; M)$ = space of neat embeddings $K\colon \mathbb{D}^k \hookrightarrow M$ with $K|_{\partial\mathbb{D}^k} = s$.
  Neat = transverse to ∂M and $K(X) \cap \partial M = K(\partial X)$.
- For $E = \mathrm{Emb}^\varepsilon_\partial(\mathbb{D}^{k-1}; M \cup_G h^{d-k+1})$ the boundary condition is $u_0 := \partial u_+$ and $\Omega E = \mathrm{Map}_*(\mathbb{S}^1; E)$ is the space of loops based at $\mathtt{u}_+ := \mathtt{s} \cap \mathtt{h}^{d-k+1}$.



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For $k \quad d \quad 1$ let M be a compact smooth d-manifold with a pair of smoothly embedded spheres $\mathsf{s}\colon \mathrm{S}^{k \quad 1} \,,!\; @M$ and $\mathsf{G}\colon \mathrm{S}^{d \quad k} \,,!\; @M$, such that G has trivial normal bundle and $\mathsf{G} \; \mathsf{t} \; \mathsf{s} = fptg$.

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- Supscript ε means each embedded disk is equipped with a "push-off"...
- Codimension increased by one! ($\Longrightarrow$ right hand side is easier)

Picture of Space Level LBT

Theorem (Space Level LBT)

For a d-manifold M and $s\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$, $G\colon \mathbb{S}^{d-k} \hookrightarrow \partial M$, such that G has trivial normal bundle and $G \pitchfork s = \{pt\}$, there is a pair of homotopy equivalences

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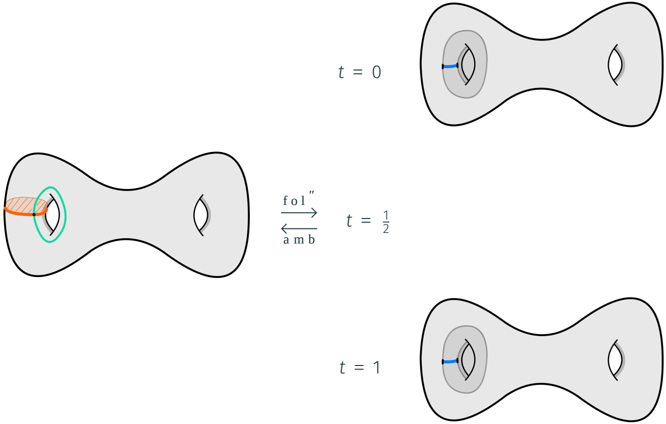

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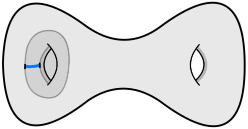
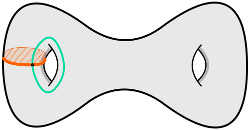
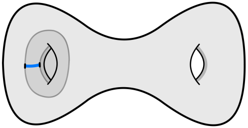




$\underset{\text{amb}}{\overset{\text{fol}^\varepsilon}{\rightleftarrows}}$ $t = \frac{1}{2}$

ε Only a schematic: for any $t \in [0;1]$, the time t arc is isotopic to u_+.

$t = 1$



$k = d$: Recovers a theorem (and proof) of Cerf '68:

$$\mathrm{Diff}^{+}_{\partial}(D^d) = \mathrm{Emb}_{\partial}(D^d; D^d) \simeq \Omega\,\mathrm{Emb}_{\partial}(D^{d-1}; D^d).$$

In particular, $\pi_0\,\mathrm{Diff}^{+}_{\partial}(D^4) = \pi_1(\mathrm{Emb}_{\partial}(D^3; D^4); \iota)$.

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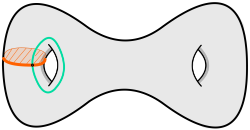
$k = 3; d = 4:$ $\pi_0\mathrm{Emb}_\partial(\mathrm{D}^3;\mathrm{S}^1 \times \mathrm{D}^3) = \pi_1\mathrm{Emb}_\partial(\mathrm{D}^2;\mathrm{D}^4)$, cf. Budney–Gabai.



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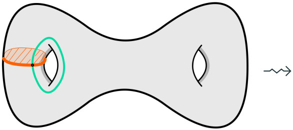
$K\colon \mathrm{D}^k \ ,!\ M$, with $@K = s$ $\qquad$ $J\colon ⅁^k \ ,!\ X := M\ [\ {}_G h^d \ {}^{k+1}$, with $@J = u \ \ [\ u_+$

□

Picture Proof of Space Level LBT



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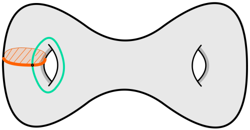

$\rightsquigarrow$ $=$

$K\colon \mathbb{D}^k \hookrightarrow M$, with $\partial K = s$ $\qquad$ $J\colon \mathbb{D}^k \hookrightarrow X := M \cup_G h^{d-k+1}$, with $\partial J = u_- \cup u_+$

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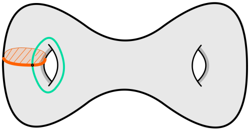
$$\mathrm{Emb}_{\partial\varepsilon}(\mathbb{D}^k; X) \hookrightarrow \mathrm{Emb}_{\mathrm{D}_-^{\varepsilon}}(\mathbb{D}^k; X) \xrightarrow{K \mapsto K|_{\mathrm{D}_+^{\varepsilon}}} \mathrm{Emb}^{\varepsilon}_{\partial\varepsilon}(\mathrm{D}^{k-1}; X)$$

$\square$



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The total space is contractible (shrink the half-disk to its $u^"$ -collar), so:

$$\mathrm{Emb}^{"}_{@"}(\mathrm{D}^{k \ \ 1}; X) \underset{\mathrm{fol}^{"}_{a}}{\overset{\mathrm{amb}_{a}}{\rightleftarrows}} \mathrm{Emb}_{@"}(ɑ^k; X)$$

where:

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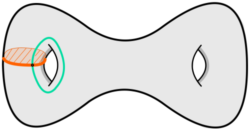
Picture Proof of Space Level LBT



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The total space is contractible (shrink the half-disk to its u_--collar), so:

$$\mathrm{Emb}_{\partial}(\mathrm{D}^{k-1}; X) \underset{\mathrm{fol}}{\overset{\mathrm{amb}}{\rightleftarrows}} \mathrm{Emb}_{\partial}(ɑ^k; X)$$

where: amb is the connecting map (use the family ambient isotopy theorem to extend loops),
fol(K) is the loop of augmented $(k-1)$-disks foliating the sphere $\cup K$. □

LBT for 2-disks in 4-manifolds

The 4D setting

Let M be an oriented compact smooth 4-manifold together with

- a knot $s\colon \mathrm{S}^1 \ ,\!\ @M$,
- an embedded sphere $G\colon \mathrm{S}^2 \ ,\!\ @M$,

so that s and G intersect transversely and positively in a single point.

Let M be an oriented compact smooth 4-manifold together with

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Notation. Let

- $m \quad = s(\quad i) \in M$ be the basepoint and denote $\quad = \ _1(M; m \quad)$,
- $\mathbb{Z}[\quad]$ be the group ring, and $\mathbb{Z}[\quad r 1] := \{r = \sum \quad _i g_i : g_i \neq 1\}$ its subgroup,

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Notation. Let

- $m = s(-i) \in M$ be the basepoint and denote $\pi = \pi_1(M; m)$,
- $\mathbb{Z}[\pi]$ be the group ring, and $\mathbb{Z}[\pi \setminus 1] := \{r = \sum n_i g_i : g_i \neq 1\}$ its subgroup,
- $\sigma\colon \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$ be the usual involution $r = \sum n_i g_i \mapsto \sigma(r) = \overline{r} = \sum n_i g_i^{-1}$,
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- $\mathbb{Z}[\;\setminus 1]$ be the subgroup of $\mathbb{Z}[\;\setminus 1]$ of the invariants: those r with $r = \bar{r}$,
- $: {}_2M \quad {}_2M \to \mathbb{Z}[\;]$ be the equivariant intersection form of M.

Let M be an oriented compact smooth 4-manifold together with

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Notation. Let

- $m_- = s(-i) \in M$ be the basepoint and denote $\pi = \pi_1(M; m_-)$,
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- $\lambda\colon \pi_2 M \times \pi_2 M \to \mathbb{Z}[\pi]$ be the equivariant intersection form of M.

We study the set of isotopy classes $\mathrm{Emb}_\partial[\mathbb{D}^2; M] := \pi_0 \mathrm{Emb}_\partial(\mathbb{D}^2; M)$ of neat smooth embeddings $K\colon \mathbb{D}^2 \hookrightarrow M$ which on $\partial \mathbb{D}^2$ agree with s.

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By Space Level LBT we have $\mathrm{Emb}_\partial[\mathbb{D}^2; M] := \ _1 \mathrm{Emb}^{\ "}_\partial(\mathbb{D}^1; M \cup_G h^3)$ and we can compute the latter group.

LBT for 2-disks

Theorem A. There is an exact sequence of sets

$$\mathbb{Z}[\quad r\,1] \quad \mathrm{dax}(\quad {}_3M) \underset{\mathrm{Dax}}{\overset{+\ \mathrm{fm}(\ \ )^G}{\rightleftarrows}} \mathrm{Emb}_{@}[\mathrm{D}^2; M] \xrightarrow{j} \mathrm{Map}_{@}[\mathrm{D}^2; M] \xrightarrow{2} \mathbb{Z}[\quad r\,1] \;{}_{hr} \quad {}_{\bar{r}i}$$

In detail:

LBT for 2-disks

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( ) the relative Dax invariant, given by a clever count of double point loops in a homotopy to K, detects the action:

$$\mathrm{Dax}(K + \mathrm{fm}(r)^G; K) = [r]:$$

Picture of LBT for 2-disks

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Note: A similar construction by Gabai in “Self-Referential Discs and the Light Bulb Lemma”.

Other results

Special case: spheres with a common dual

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- a framed embedded sphere $G\colon S^2 \hookrightarrow N$.

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Proposition

There is a bijection

$$\cup\, \nu_x G\colon \; \mathrm{Emb}_\partial[D^2; N \smallsetminus \nu G] \xrightarrow{\cong} \mathrm{Emb}^G[S^2; N],$$

where $s = \partial(\nu_x G)\colon S^1 \hookrightarrow \partial(N \smallsetminus \nu G)$ is a meridian circle of G at $x \in G$, and its dual is a push-off of G into $\partial(N \smallsetminus \nu G)$.

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Observe: $\partial(N \setminus \nu G) = \partial N \sqcup \partial(\nu G)$ and $\partial(\nu G) = \mathbb{S}^1 \times \mathbb{S}^2$. Conversely, if a 4-manifold M has a boundary component $\mathbb{S}^1 \times \mathbb{S}^2$, attaching $\mathbb{D}^2 \times \mathbb{S}^2$ to it takes us to the setup of spheres with a fixed dual.

Theorem

If $M = N r \; G$ for a framed $G\colon \mathbf{S}^2 \; , ! \; N$, then $hr + \bar{r}i \quad \mathbf{dax}(\; {}_3M)$.

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Corollary [Gabai when $T_N = 0$, Schneiderman–Teichner in general]

The set of spheres homotopic to $[F] \in \mathrm{Emb}^G[\mathbb{S}^2; N] = \mathrm{Emb}_{@}[\mathbb{D}^2; M]$ is given by

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- Group structures on sets of isotopy classes, see the next slide.

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Moreover, the sequence of Theorem A becomes *an exact sequence of groups*, with the bijection ª [: $\mathrm{Map}_{@}[\mathrm{D}^2; M] = {}_2M$ inducing a nonstandard group structure ? on ${}_2M$:

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Note:

$\mathrm{Emb}_@[\mathrm{D}^2; M]$ is almost never abelian (we have seen $\mathrm{dax}({}_3M) \quad \mathbb{Z}[\ \uparrow 1]$ and is rarely symmetric, so $\sim$ not in the image of dax).

Thank you!