∾ Knotted families from graspers ∾

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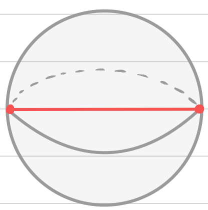
@ Spaces of manifolds:
Algebraic & Geometric Approaches

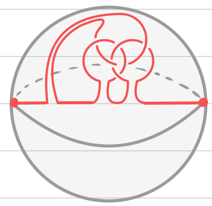
Banff

~ Introduction ~

$u: D^1 \hookrightarrow D^3$ the unknot

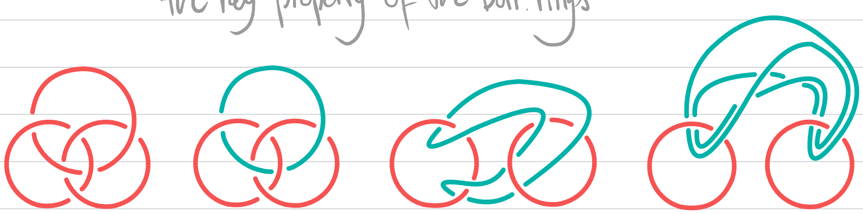


$T: D^1 \hookrightarrow D^3$ the trefoil



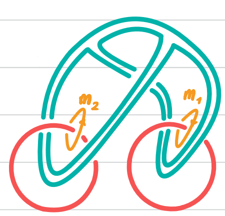
pick 3 subarcs of u
& connect sum them
into the Borr. rings

the key property of the Borr. rings



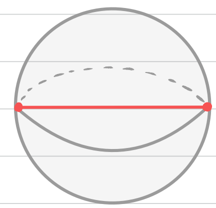
One of the components is the commutator of the meridians of the other two:

$$c_3 \simeq [m_1, m_2] = m_1 m_2 m_1^{-1} m_2^{-1}$$

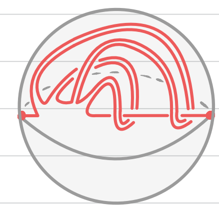


$\Longrightarrow$

$u: D^1 \hookrightarrow D^3$ the unknot



$T: D^1 \hookrightarrow D^3$ the trefoil



grab one subarc of u
and connect sum into the
commutator of
its own meridians.

This is an example of

Goussarov-Habiro clasper surgery of degree $n=2$.

Idea for general $n \geq 2$:

- fix n meridian circles $m_1, \ldots, m_n$ of u
- fix a bracketed word in letters $m_1, \ldots, m_n$ with each letter appearing exactly once
- connect sum a subarc of u into the corresponding iterated commutator of m_i

choices how to perform this ambient connect sum.

∼ Graspers ∼

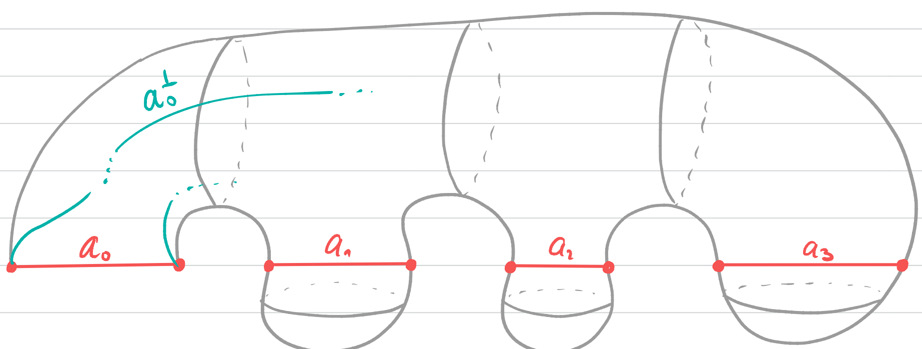
def. A degree $n \geq 1$ grasper relative to u is a smooth embedding

$$G: \mathbb{B}^3 \hookrightarrow \mathbb{R}^3$$

s.t. $G \circ a_i = u|_{J_i}$, $\quad 0 \leq i \leq n$

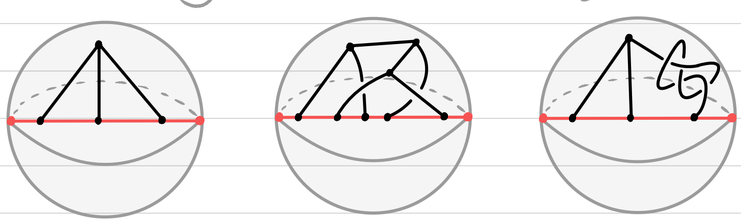
where we in advance fix intervals $J_i \subseteq \mathbb{D}^1$.
and a collection of arcs

$a_0 : \mathbb{D}^1 \hookrightarrow \partial\mathbb{B}^3$ and $a_i : (\mathbb{D}^1, \partial\mathbb{D}^1) \hookrightarrow (\mathbb{B}^3, \partial\mathbb{B}^3)$.



Moreover, there is a fixed arc $a_0^\perp : \mathbb{D}^1 \hookrightarrow \mathbb{B}^3$.
Up to homotopy, $a_0^\perp \cdot a_0^{-1}$ is an iterated commutator of the meridians of a_i, $1 \leq i \leq n$.

$\hookrightarrow$ we need to pick a bracketed word or equivalently, a rooted planar binary tree Γ



def. Surgery along G on u is the knot

$$r_u(G) := u|_{\mathbb{D}^1 \setminus J_0} \cup G|_{a_0^\perp}$$

Rem. This is a version of surgery along a grope, which is closely related to surgery along a clasper.

Gusarov 1998
Habiro 2000

Conant – Teichner 2004

∼ Main result ∼

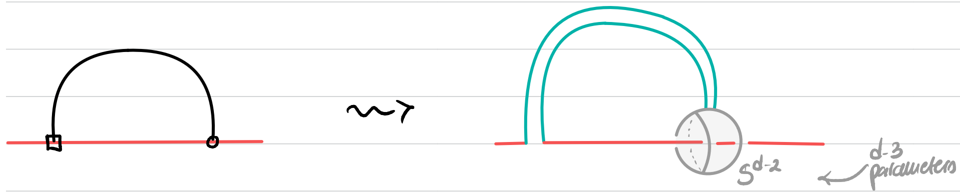
Thm. [K. 2023]

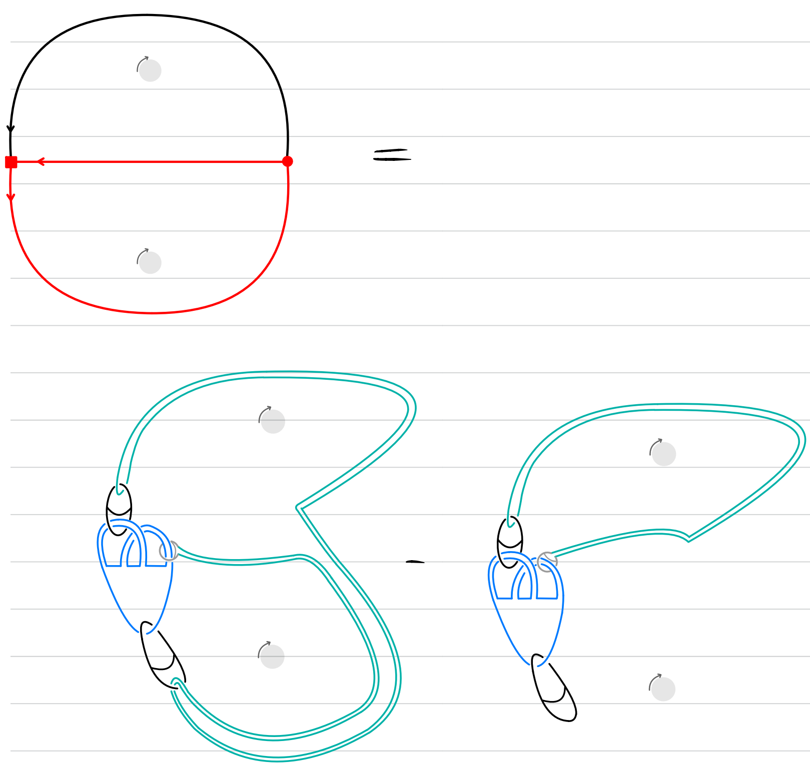
Analogous construction on $u : \mathbb{D}^1 \hookrightarrow M$ for an oriented compact d-manifold M, $d \geq 3$ gives classes

$$r(\Gamma^{g^n}) \in \pi_{n(d-3)}(\mathrm{Emb}_\partial(\mathbb{D}^1, M), u)$$

that are first detected in the $(n+1)$-st stage of the Goodwillie - Weiss Taylor tower.

~ Examples ~





~ Related work ~

$n=1$ Haefliger, Dax, Gabai, K.–Teichner, K.

$n=2$ Turchin, Budney: generator of $\pi_{2(d-3)} \mathrm{Emb}_\partial(D^1, D^d) \cong \mathbb{Z}$ and iso to $\pi_0 \mathrm{Emb}_\partial(D^{4k-1}, D^{6k})$ for $d=2k+2$ generated by the Haefliger trefoil
Budney–Gabai: $\pi_2 \mathrm{Emb}_\partial(D^1, S^1 \times D^3)$.

$d=3$ K. (thesis)

(co)homology Cattaneo–Cotta-Ramusino–Longoni, Longoni

$\mathrm{Emb}_\partial(D^1, D^d)$ Scannell–Sinha, Conant, Lambrechts–Turchin, Arone–Lambrechts–Turchin–Volić, Boavida de Brito–Horel

relation to BDiff. Botvinnik–Watanabe:

$$\pi_{n(d-3)} \mathrm{Emb}^{fr}(S^1, X \# S^{d-1} \times S^1) \xrightarrow{ps} \pi_{n(d-3)} B\mathrm{Diff}_\partial(X)$$

Watanabe's clasper classes are in the image of this map.

Q: [work in progress] How are they related to grasper classes?

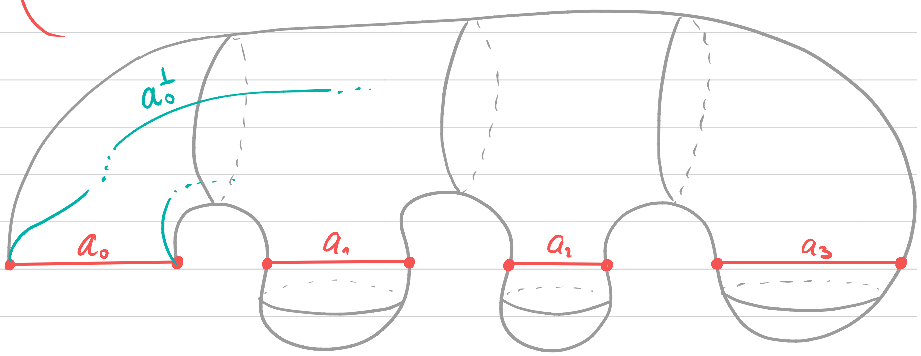
~ Graspers in d-manifolds ~

def. A degree $n \geq 1$ grasper relative to u is a smooth embedding
$$G : B^d \hookrightarrow M$$
s.t. $G \circ a_i = u|_{J_i}$, $\quad 0 \leq i \leq n$

where we in advance fix intervals $J_i \subseteq \mathbb{D}^1$ and a collection of arcs

$a_0 : \mathbb{D}^1 \hookrightarrow \partial B^3$ and $a_i : (\mathbb{D}^1, \partial \mathbb{D}^1) \hookrightarrow (B^3, \partial B^3)$.

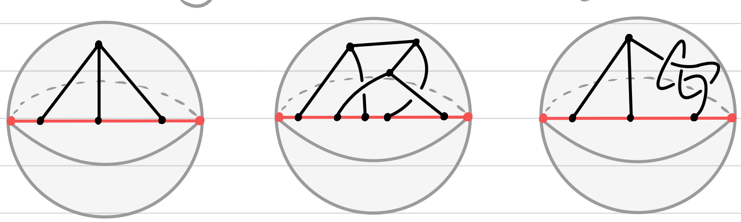
only for $d=3$ this is a choice



Moreover, there is a fixed arc $a_0^\perp : \mathbb{D}^1 \hookrightarrow B^3$.
Up to homotopy, $a_0^\perp \cdot a_0^{-1}$ is an iterated commutator of the meridians of a_i, $1 \leq i \leq n$.

only $d=3$

↪ we need to pick a bracketed word or equivalently, a rooted planar binary tree Γ.



def. Surgery along G on u is the knot
$$r_u(G)(\vec{t}) := u|_{\mathbb{D}^1 \setminus J_0} \cup G|_{M_{\Gamma,d}(\vec{t})}$$

where $M_{\Gamma,d} : S^{n(d-3)} \longrightarrow \mathrm{Emb}_\partial(\mathbb{D}^1, B^d)$

is an embedded version of the Samelson product $x_{\Gamma,d-2}$ of the meridian spheres $m_1, \dots, m_n$ of the arcs $a_1, \dots, a_n$ according to the word given by Γ.

close relative of Whitehead product

def. For a tree Γ with n leaves define the Samelson product
$$x_{\Gamma,d-2} : S^{n(d-3)} \longrightarrow \Omega \bigvee_n S^{d-2}$$
inductively as follows. For $n=1$ and $\Gamma = |$ let
$$x_{|,d-2} : S^{d-3} \longrightarrow \Omega S^{d-2}$$

be the adjoint of the identity
(think: foliate by $(d-3)$-parameter family of based loops).

For $n = n_1 + n_2$ and $\Gamma = [\Gamma_1, \Gamma_2]$ where Γ_i has $n_i \geq 1$ leaves consider
$$S^{n(d-3)} \cong I^{n_1(d-3)} \times I^{n_2(d-3)} / \partial$$
and let
$$x_{[\Gamma_1,\Gamma_2],d-2}(\vec{t}_1, \vec{t}_2) := [x_{\Gamma_1,d-2}(\vec{t}_1), x_{\Gamma_2,d-2}(\vec{t}_2)]$$
together with a canonical nullhomotopy on ∂.

Key Lemma. The Samelson product $x_{\Gamma,d-1}$ can be obtained by gluing together maps

$$
\begin{tikzcd}
{[-1,0]^S \times [0,1]^{\{1\dots n\}\setminus S} \times S^{n(d-3)}} \arrow[d, phantom, "\cong" rotate=90] & \subseteq & {[-1,1]^n \times S^{n(d-3)}} \arrow[d] \\
I^n \times S^{n(d-3)} \arrow[d, "x^*_{\Gamma,d-1}"] & & S^n \wedge S^{n(d-3)} \arrow[d, phantom, "\cong" rotate=90] \\
\Omega \bigvee_n B^{d-1} \arrow[d, "\Omega(\bigvee_{i\in S} j_E \vee \bigvee_{i\notin S} j_W)"] & & S^{n(d-2)} \arrow[dll, bend left, "x_{\Gamma,d-1}"] \\
\Omega \bigvee_n S^{d-1} & &
\end{tikzcd}
$$

Key Thm. There is a map $M^*_{\Gamma,d}$ making the following diagram commute

$$
\begin{tikzcd}
{[0,1]^n \times S^{n(d-3)}} \arrow[r, "M^*_{\Gamma,d}"] \arrow[d] & \mathrm{Emb}_\partial(D^1, B^d) \arrow[d, "a_0 \cup -"] \\
\Omega \bigvee_n B^{d-1} \arrow[r, "m^*"', "\text{based meridian balls}" below] & \Omega B^d
\end{tikzcd}
$$

∼ Main result ∼
(more details)

Thm. [K. 2023]

Analogous construction on $u: D^1 \hookrightarrow M$ for an oriented compact d-manifold M, $d \geq 3$ gives classes

$$r(\Gamma^{g_{\underline{n}}}) \in \pi_{n(d-3)}(\mathrm{Emb}_\partial(D^1, M), u)$$

that are detected in the $(n+1)$-st stage of the Goodwillie–Weiss Taylor tower.

reduced punctured knots model:

$$T_n \simeq \{\Delta^n \xrightarrow{f} \mathrm{Emb}_\partial(J_0, M_{0n}) : f|_{\Delta_S} \subseteq \mathrm{Emb}_\partial(J_0, M_{0S})\}$$

$$F_{n+1} \simeq \{I^n \longrightarrow \mathrm{Emb}_\partial(J_0, M_{0\underline{n+1}}) : 0^S \longmapsto M_{0S_{n+1}},\ t_i = 1 \longmapsto u|_{J_0}\}$$

So $G \circ M^*_{\Gamma,d}$ defines a point in F_{n+1}.