Mappings of finite distortion on metric surfaces

joint work with Kai Rajala

Damaris Meier

University of Fribourg

Geometry and Analysis on Metric Surfaces
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Mappings of finite distortion on $\mathbb{R}^2$

Let $\Omega \subset \mathbb{R}^2$ be a domain and $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be **non-constant**.

- f has *finite distortion* if $\exists\ K \colon \Omega \to [1, \infty)$ measurable s.th.

$$||Df(x)||^2 \leq K(x) \cdot J_f(x) \quad \text{for a.e. } x \in \Omega.$$

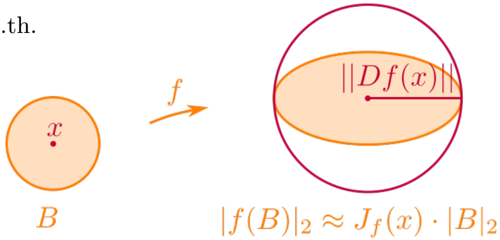
  The *distortion* K_f of f is defined by

$$K_f(x) = \begin{cases} \frac{||Df(x)||^2}{J_f(x)}, & \text{if } J_f(x) > 0, \\ 1, & \text{else.} \end{cases}$$

- f is *quasiregular* if $\exists\ K \geq 1$ s.th.

$$K_f \leq K \quad \text{a.e.}$$

- f is *quasiconformal* if f is quasiregular and a homeomorphism.



Mappings of finite distortion on $\mathbb{R}^2$

Remark: f is quasiregular with $K = 1$ iff. f is *complex analytic*.

Topological properties: *continuity, openness and discreteness*.

Question: Do same topological properties hold after relaxing conditions?

Stoïlow factorization Theorem: If f is quasiregular, then f admits a factorization $f = g \circ h$ with g analytic and h a quasiconformal homeomorphism.

$$\begin{tikzcd} \Omega \arrow[r, "f"] \arrow[d, "h"', "\text{qc}"] & \mathbb{R}^2 \\ \Omega' \arrow[ur, "g", "\text{analytic}"'] & \end{tikzcd}$$

In particular: *f is continuous, open and discrete.*

Iwaniec-Šverák Theorem: If f satisfies

$$||Df(x)||^2 \leq K(x) \cdot J_f(x) \qquad \text{for a.e. } x \in \Omega$$

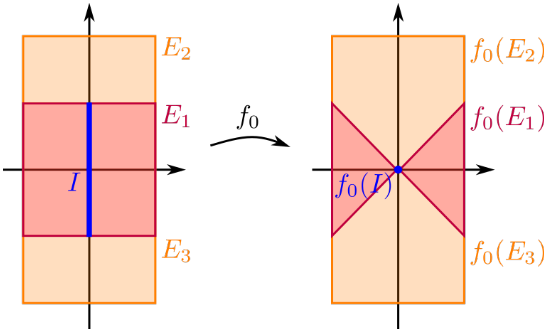
for some $K \in L^1_{\text{loc}}(\Omega)$, then f is *continuous, open and discrete.*

Theorem is sharp by following example.

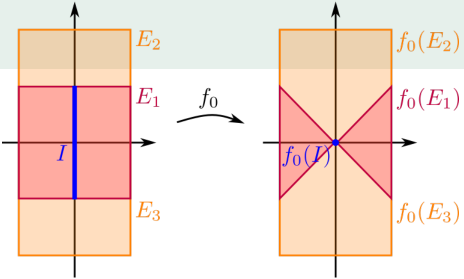
Example (Ball's map)

Define $f_0\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f_0(x,y) = (x, \eta(x,y))$, where

$$\eta(x,y) = \begin{cases} |x|y, & (x,y) \in E_1, \\ (2(|y|-1) + |x|(2-|y|))\frac{y}{|y|}, & (x,y) \in E_2 \cup E_3, \\ y, & \text{else.} \end{cases}$$



Example (Ball's map)



$$f_0\colon \mathbb{R}^2 \to \mathbb{R}^2,\ f_0(x,y) = (x, \eta(x,y)),$$

$$\eta(x,y) = \begin{cases} |x|y, \\ (2(|y|-1) + |x|(2-|y|))\frac{y}{|y|}, \\ y. \end{cases}$$

Calculate distortion K_{f_0}:

$$Df_0(x,y) = \begin{pmatrix} 1 & \frac{x}{|x|}y \\ 0 & |x| \end{pmatrix}, \qquad \begin{aligned} &||Df_0(x,y)|| = 1, \\ &J_{f_0}(x,y) = |x|, \end{aligned} \qquad \Rightarrow \qquad K_{f_0}(x,y) = \frac{1}{|x|}$$

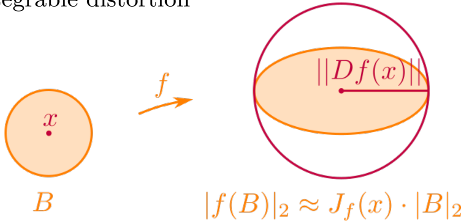
$$Df_0(x,y) = \begin{pmatrix} 1 & \frac{xy(2-|y|)}{|x|\,|y|} \\ 0 & 2-|x| \end{pmatrix}, \qquad \begin{aligned} &||Df_0(x,y)|| \in [1,2], \\ &J_{f_0}(x,y) \in [1,2], \end{aligned} \qquad \Rightarrow \qquad K_{f_0}(x,y) \leq 4$$

$$Df_0(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{aligned} &||Df_0(x,y)|| = 1, \\ &J_{f_0}(x,y) = 1, \end{aligned} \qquad \Rightarrow \qquad K_{f_0}(x,y) = 1$$

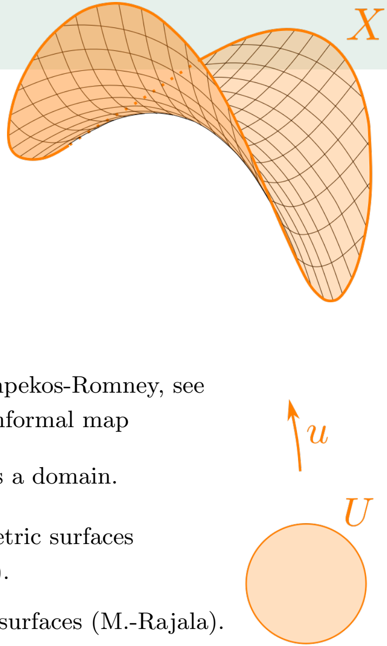
Conclusion: K_{f_0} is bounded outside E_1 and $K_{f_0}(x,y) = \frac{1}{|x|} \quad \forall (x,y) \in E_1$.

Theory of mappings of finite distortion has been extended to

- Higher dimensions (Euclidean $\mathbb{R}^n$)
- $W^{1,1}_{\mathrm{loc}}$-maps with exponentially integrable distortion
- Generalized n-manifolds
  - Ahlfors n-regular
  - Poincaré inequality
- Subriemannian manifolds



Question: How can we set up theory of mappings of finite distortion within a general metric space setting?



Definition: A metric space X is a *metric surface* if X is homeomorphic to a domain in $\mathbb{R}^2$ and of locally finite $\mathcal{H}^2_X$.

Appear naturally as boundaries, limits and deformations of smooth objects.

Tools available for metric surfaces:

- Uniformization of metric surfaces (Ntalampekos-Romney, see also M.-Wenger): $\exists$ weakly $(4/\pi)$-quasiconformal map

$$u\colon U \to X, \qquad \text{where } U \subset \mathbb{R}^2 \text{ is a domain.}$$

- Coarea inequality for Sobolev maps on metric surfaces (Esmayli-Ikonen-Rajala, M.-Ntalampekos).
- Area inequality on "good" part of metric surfaces (M.-Rajala).

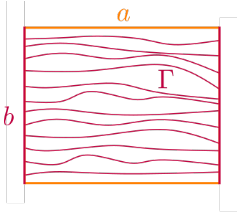
Conformal modulus

Let X be a metric surface and Γ a family of curves in X.

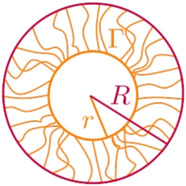
- A Borel function $\rho\colon X \to [0,\infty]$ is *admissible for* Γ if
$$\int_\gamma \rho \geq 1$$
holds for every locally rectifiable curve $\gamma \in \Gamma$.

- The *(conformal) modulus* of Γ is
$$\operatorname{mod}(\Gamma) := \inf_{\rho \text{ admissible for } \Gamma} \int_X \rho^2 \, d\mathcal{H}^2.$$

- A property (P) holds for *almost every curve* in Γ if $\exists\, \Gamma' \subset \Gamma$ such that (P) holds for every $\gamma \in \Gamma'$ and
$$\operatorname{mod}(\Gamma \setminus \Gamma') = 0.$$



$\operatorname{mod}(\Gamma) = \frac{b}{a}$



$\operatorname{mod}(\Gamma) = 2\pi \left(\log\left(\frac{R}{r}\right)\right)^{-1}$

Let X and Y be metric surfaces.

- A Borel function $\rho^u \colon X \to [0, \infty]$ is a *(weak) upper gradient* of $f \colon X \to Y$ if

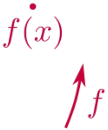
$$d_Y(f(x), f(y)) \leq \int_\gamma \rho^u \, ds$$

  $\forall\ x, y \in X$ and (almost) every rectifiable curve γ in X joining x, y.

- f is in the *Newton-Sobolev Space* $N^{1,2}_{loc}(X, Y)$ if

$$x \mapsto d_Y(y, f(x)) \in L^2_{\mathrm{loc}}(X)$$

  for some $y \in Y$ and f has an upper gradient $\rho^u \in L^2_{\mathrm{loc}}(X)$.



Properties: Every $f \in N^{1,2}_{\mathrm{loc}}(X, Y)$

- is absolutely continuous along a.e. curve γ in X.
- has a *minimal weak upper gradient* $\rho^u_f \in L^2_{\mathrm{loc}}(X)$.

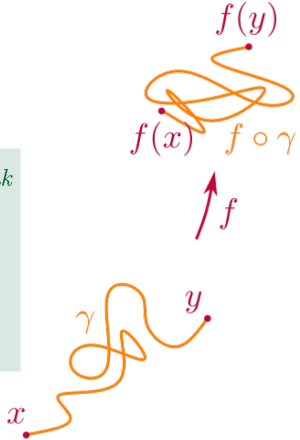
  $\to$ Corresponds to maximal stretch of f.

Lower Gradients

For $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ the weak upper gradient inequality is equivalent to:

$$\ell(f \circ \gamma) \leq \int_\gamma \rho^u \, ds$$

for almost every rectifiable curve γ in X.



Definition: A Borel function $\rho^l \colon X \to [0,\infty]$ is a *weak lower gradient* of $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ if $\rho^l \leq \rho^u_f$ a.e. and

$$\ell(f \circ \gamma) \geq \int_\gamma \rho^l \, ds$$

for almost every rectifiable curve γ in X.

- 0 is always a lower gradient.
- Every $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ has a *maximal weak lower gradient* $\rho^l_f \in L^2_{\mathrm{loc}}(X)$.

  $\rightarrow$ Corresponds to minimal stretch of f.

Sense-preservation

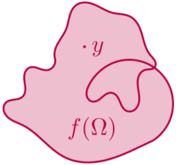
Note: Non-negativity of the Jacobian and integration by parts gives that $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ of finite distortion is sense-preserving.

Let X and Y be metric surfaces.

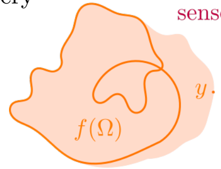
- A map $f\colon X \to Y$ is *sense-preserving* if for every domain $\Omega \subset X$, $\overline{\Omega} \subset X$ with $f|_{\partial\Omega}$ continuous

$$\deg(y, f, \Omega) \geq 1$$

for every $y \in f(\Omega) \setminus f(\partial\Omega)$.



sense-preserving

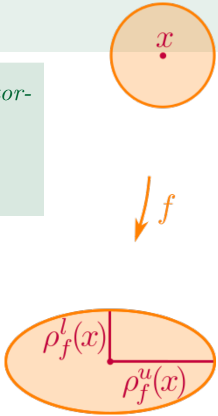


not sense-preserving

Proposition: Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving. Then

- f is continuous,
- f satisfies Lusin (N), i.e. if $E \subset X$ with $\mathcal{H}^2(E) = 0$, then $|f(E)|_2 = 0$.

Distortion along paths



Definition: $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ sense-preserving has *finite distortion along paths* if $\exists\, K\colon X \to [1,\infty)$ measurable s.th.

$$\rho_f^u(x) \le K(x) \cdot \rho_f^l(x) \qquad \text{for a.e. } x \in X.$$

- The *distortion along paths* K_f of f is

$$K_f(x) := \begin{cases} \frac{\rho_f^u(x)}{\rho_f^l(x)}, & \text{if } \rho_f^l(x) \neq 0, \\ 1, & \text{else.} \end{cases}$$

- f is *quasiregular* if K_f is uniformly bounded and *quasiconformal along paths* if f is also a homeomorphism.

Theorem 1: If $f \in N^{1,2}_{\mathrm{loc}}(X,\mathbb{R}^2)$ is a mapping of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$, then f is open and discrete.

Generalization of Iwaniec-Šverák Theorem

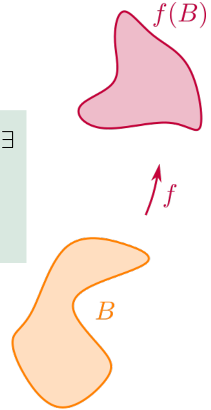
Theorem 2: If $f \in N^{1,2}_{\mathrm{loc}}(X,\mathbb{R}^2)$ is an injective mapping of finite distortion with $K_f \in L^1_{\mathrm{loc}}(X)$, then $f^{-1} \in N^{1,2}_{\mathrm{loc}}(f(X),X)$.

Generalization of Theorem of Hencl-Koskela

Analytic distortion

We define the *Jacobian*

$$J_f(x) = \limsup_{r\to 0} \frac{\mathcal{H}^2_Y(f(\overline{B}(x,r)))}{\pi r^2}.$$



Definition: $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$ has *finite analytic distortion* if $\exists$ $C\colon X \to [1,\infty)$ measurable s.th.

$$\rho^u_f(x)^2 \le C(x)\cdot J_f(x) \quad \text{for a.e. } x \in X,$$

- The *analytic distortion* C_f of f is

$$C_f(x) := \begin{cases} \frac{\rho^u_f(x)^2}{J_f(x)}, & \text{if } J_f(x) \neq 0,\\ 1, & \text{if } J_f(x) = 0. \end{cases}$$

- If $X = U \subset \mathbb{R}^2$ or $Y = \mathbb{R}^2$ and f is a homeomorphism onto its image, then

$$J_f = \frac{d\nu}{d\mathcal{H}^2_X}, \qquad \textit{Radon-Nikodym derivative of } \nu \textit{ w.r.t. } \mathcal{H}^2_X.$$

where $\nu(B) = \mathcal{H}^2_Y(f(B))$ is the pullback measure of $\mathcal{H}^2_Y$ under f.

Equivalence of definitions of distortion

Theorem 3: Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving.

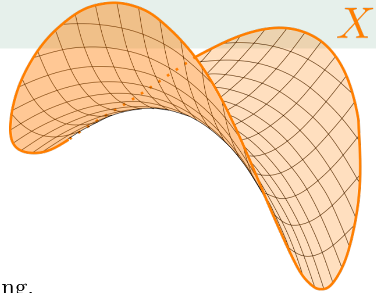
1. If f is of finite distortion along paths and $K_f \in L^1_{\mathrm{loc}}(X)$, then f is of finite analytic distortion and
$$C_f(x) \leq 4\sqrt{2}\, K_f(x) \quad \text{for a.e. } x \in X.$$
2. If f is of finite analytic distortion, then f is of finite distortion along paths and
$$K_f(x) \leq 4\sqrt{2}\, C_f(x) \quad \text{for a.e. } x \in X.$$

Corollary: If $f\colon X \to f(X) \subset \mathbb{R}^2$ is a homeomorphism, then t.f.a.e.

1. f is quasiconformal along paths,
2. f is analytically quasiconformal,
3. f is *geometrically quasiconformal*, i.e. $\exists\ K \geq 1$ s.th.
$$\frac{1}{K} \operatorname{mod}(\Gamma) \leq \operatorname{mod}(f \circ \Gamma) \leq K \operatorname{mod}(\Gamma)$$
$\forall$ family Γ of curves in X.

Moreover, if f satisfies any of these conditions, then so does f^{-1}.

Proof of Theorem 3



Theorem (Ntalampekos-Romney):
$\exists\, u \in N^{1,2}_{\mathrm{loc}}(U, X)$, $U \subset \mathbb{R}^2$, s.th.

- u is sense-preserving,
- u is $\sqrt{2}$-quasiregular.

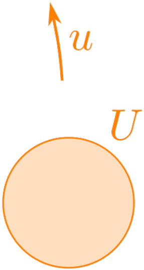
Consider: $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ sense-preserving.

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{R}^2 \\ U \subset \mathbb{R}^2 \arrow[u, "u"] \arrow[ur, "h"'] & \end{tikzcd}$$

$\implies$ $h := f \circ u \in N^{1,2}_{\mathrm{loc}}(U, \mathbb{R}^2)$
is sense-preserving.

$\implies$ $U = \bigcup_{j \geq 0} G_j$ with $|G_0|_2 = 0$ and
$u|_{G_j}$, $h|_{G_j}$ j-Lipschitz $\forall\, j \geq 1$.

$\overset{\text{Prop}}{\implies}$ f and h satisfy Lusin (N):
$|f(X_0)|_2 = |h(G_0)|_2 = 0$ for $X_0 := u(G_0)$



Lemma 1: Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving, then

$$J_f = 0 \quad \text{a.e. in } X_0.$$

Proof of Theorem 3

- $U = \bigcup_{j \geq 0} G_j$ with $|G_0|_2 = 0$ and $u|_{G_j}$, $h|_{G_j}$ j-Lipschitz $\forall\, j \geq 1$.
- Set $X_0 := u(G_0)$ and $X' := X \setminus X_0$. $\Rightarrow$ X' is countably 2-rectifiable.

Theorem (Kirchheim): $\exists E \subset X$, $\mathcal{H}^2(E) = 0$, s.th.

$$\lim_{r \to 0} \frac{\mathcal{H}^2(B(x,r) \cap X')}{\pi r^2} = 1 \qquad \forall x \in X' \setminus E.$$

$$\begin{tikzcd} X \arrow[r, "f"] & \mathbb{R}^2 \\ U \subset \mathbb{R}^2 \arrow[u, "u"] \arrow[ur, "h"'] & \end{tikzcd}$$

Linear approximation of J_f and Kirchheim's Theorem give:

Lemma 2: Let $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ be sense-preserving. Then for $E \subset X$ Borel

$$\int_E J_f(x)\, d\mathcal{H}^2_X \leq \int_{\mathbb{R}^2} N(y, f, E)\, dy,$$

with equality if f is furthermore open and discrete.

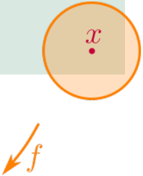
$\Longrightarrow$ For a homeo $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ we have $J_f = \dfrac{d\nu}{d\mathcal{H}^2_X}$, where $\nu(B) = |f(B)|_2$.

Area inequality: Let $f \in N^{1,2}_{\mathrm{loc}}(X,Y)$. Then for $E \subset X'$ Borel

$$\int_E \rho_f^u(x)\rho_f^l(x)\, d\mathcal{H}^2_X \leq 4\sqrt{2} \int_Y N(y,f,E)\, d\mathcal{H}^2_Y.$$

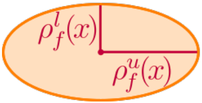
If f also satisfies Lusin's condition (N), then

$$\int_E \rho_f^u(x)\rho_f^l(x)\, d\mathcal{H}^2_X \geq \frac{1}{4\sqrt{2}} \int_Y N(y,f,E)\, d\mathcal{H}^2_Y.$$



If $f \in N^{1,2}_{\mathrm{loc}}(X,\mathbb{R}^2)$ is of finite analytic distortion:

- Lemma 2 + Area ineq: $J_f(x) \leq 4\sqrt{2}\,\rho_f^u(x)\rho_f^l(x)$ for a.e. $x \in X'$.
- Lemma 1: $\rho_f^u = 0$ a.e. on X_0.
  $\Rightarrow$ By definition: $\rho_f^l = 0$ a.e. on X_0.
  $\Rightarrow$ $\rho_f^u \leq 4\sqrt{2}\,\rho_f^l$ a.e. on X_0.



If $f \in N^{1,2}_{\mathrm{loc}}(X,\mathbb{R}^2)$ is of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$:

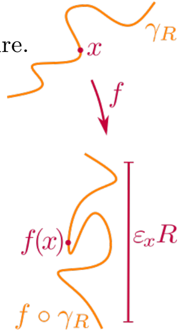
- Theorem 1: f is open and discrete.
- Lemma 2 + Area ineq: $\rho_f^u(x)\rho_f^l(x) \leq 4\sqrt{2}\, J_f(x)$ for a.e. $x \in X'$.

Proof of Theorem 3

Proposition: If $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ is of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$, then $\rho^l_f = 0$ a.e. in X_0.

Proof: By Theorem 1, f is a local homeomorphism on $X \setminus \mathcal{B}_f$, where $\mathcal{B}_f$ is a discrete set of branch points.

Assume: The set $A := \{x \in X_0 : \rho^l_f(x) > 0\}$ has positive measure.



Lemma 3: $\exists A' \subset A \setminus \mathcal{B}_f$ of positive measure s.th. $\forall x \in A'$

- $\exists \gamma_x$ parametrized by arclength, $x = \gamma_x(t)$ for $t \in (0, \ell(\gamma_x))$,
- $\exists \delta_x, \varepsilon_x \in (0,1)$ s.th. $\forall R \in (0, \delta_x)$

$$\mathrm{diam}(|f \circ \gamma_R|) \geq \varepsilon_x R,$$

where $\gamma_R = \gamma_x|_{[t-R,t+R]}$.

$\implies$ $\exists \varepsilon > 0$ s.th. $A_\varepsilon := \{x \in A' : \varepsilon_x \geq \varepsilon\}$ is of positive measure.

Claim: $J_f(x) > 0$ for a.e. $x \in A_\varepsilon$.

Contradicts Lemma 1.

Proof of Theorem 3

Claim: $J_f(x) > 0$ for a.e. $x \in A_\varepsilon$.

Let M be large enough and $R > 0$ s.th.

$$5MR < \delta_x.$$

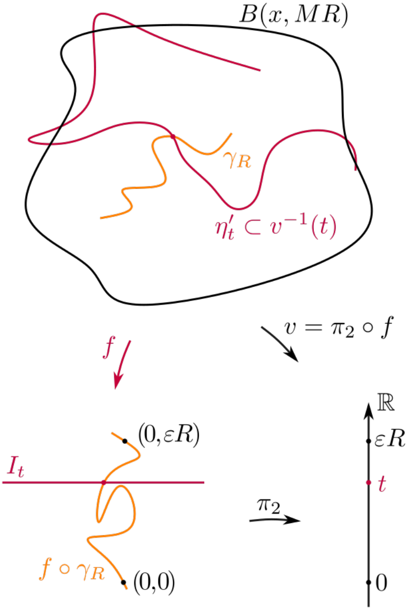
$$F_M(R) := \{t \in (0, \varepsilon R) : \eta'_t \subset B(x, MR)\}$$

Lemma: For a.e. $x \in A_\varepsilon$ $\exists M < \infty$ s.th.

$$|F_M(R)|_1 \geq \frac{\varepsilon R}{2}.$$

$$G_M(R) := \bigcup_{t \in F_M(R)} \underbrace{f(\eta'_t)}_{=I_t} \subset f(B(x, MR))$$

$$\varepsilon R^2 \leq 2R \cdot |F_M(R)|_1 = |G_M(R)|_2$$
$$\leq |f(B(x, MR))|_2$$

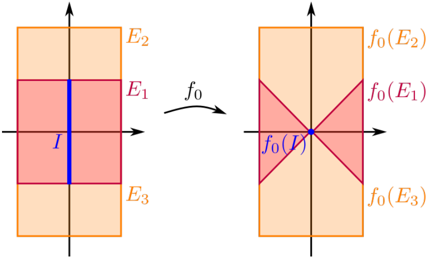


Quasiconformal uniformization of metric surfaces

Theorem A: If X admits a quasiregular map $f : X \to \mathbb{R}^2$, then X admits a quasiconformal homeomorphism $\varphi : X \to U \subset \mathbb{R}^2$.

Theorem A is sharp:

- $\exists$ a metric surface X that does not admit a quasiconformal homeomorphism $\varphi\colon X \to U \subset \mathbb{R}^2$ but admits $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$.



Generalization of Stoïlow factorization Theorem:

Theorem B: If $f\colon X \to \mathbb{R}^2$ is quasiregular, then f admits a factorization $f = g \circ v$ for $v\colon X \to V \subset \mathbb{R}^2$ quasiconformal and $g\colon V \to \mathbb{R}^2$ analytic.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "v"', "\text{qc}"] & \mathbb{R}^2 \\ V \subset \mathbb{R}^2 \arrow[ur, "g", "\text{analytic}"'] & \end{tikzcd}$$

- Follows from Theorem A and measurable Riemann Mapping Theorem.

Theorem A: If X admits a quasiregular map $f : X \to \mathbb{R}^2$, then X admits a quasiconformal homeomorphism $\varphi : X \to U \subset \mathbb{R}^2$.

Proof: Assume $f\colon X \to \mathbb{R}^2$ is K-quasiregular.

Ntalampekos-Romney: $\exists\, u \in N^{1,2}_{\mathrm{loc}}(U, X)$, $U \subset \mathbb{R}^2$, s.th.

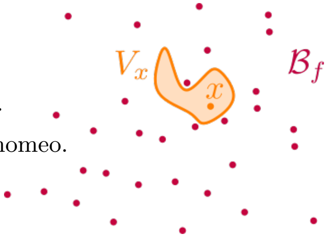
- u is $\sqrt{2}$-quasiregular,
- u is monotone, i.e. $u^{-1}(x)$ is connected $\forall\, x \in X$.

$$\begin{tikzcd}
X \arrow[r, "f"] & \mathbb{R}^2 \\
U \subset \mathbb{R}^2 \arrow[u, "u"] \arrow[ur, "h"] &
\end{tikzcd}$$

$\Longrightarrow \quad h := f \circ u \in N^{1,2}_{\mathrm{loc}}(U, \mathbb{R}^2)$ is $\sqrt{2}K$-quasiregular.

$\overset{\text{Thm 1}}{\Longrightarrow} \quad f$ and h are discrete $\quad \Rightarrow \quad u$ is discrete

$\overset{u \text{ mon}}{\Longrightarrow} \quad u$ is a homeomorphism



Let B_f be the (discrete) set of branch points of f.

$\Longrightarrow \quad \forall\, x \in X \setminus \mathcal{B}_f\ \exists$ nbhd $V_x \subset X$ s.th. $f|_{V_x}$ is homeo.

$\overset{\text{Cor}}{\Longrightarrow} \quad f|_{V_x}$ and $h|_{u^{-1}(V_x)}$ are geometrically qc

$\Longrightarrow \quad u^{-1}|_{V_x}$ is geometrically qc.

Theorem A: If X admits a quasiregular map $f : X \to \mathbb{R}^2$, then X admits a quasiconformal homeomorphism $\varphi : X \to U \subset \mathbb{R}^2$.

Proof: We have established that $u^{-1}|_{X \setminus \mathcal{B}_f}$ is geometrically qc.

$\overset{\text{[Wil12]}}{\Longrightarrow}$ $u^{-1} \in N^{1,2}_{\mathrm{loc}}(X \setminus \mathcal{B}_f, \mathbb{R}^2)$ is analytically qc

Set $\Gamma^* := \{\gamma \in \Gamma(X) : u^{-1}$ is not abs. cont. along $\gamma\}$.

- $\mathrm{mod}(\Gamma_0) = 0$ for $\Gamma_0 := \Gamma^* \cap \Gamma(X \setminus \mathcal{B}_f)$
- $\forall \gamma \in \Gamma^*$: $|\gamma| \cap \mathcal{B}_f$ is finite

  $\Rightarrow$ γ has a subcurve in Γ_0

  $\Rightarrow$ $\mathrm{mod}\,\Gamma^* \leq \mathrm{mod}\,\Gamma_0 = 0$

Conclusion: $\varphi := u^{-1} \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ satisfies

$$\rho^u_\varphi(x)^2 \leq C \cdot J_\varphi(x) \quad \text{a.e.}$$

□

Example

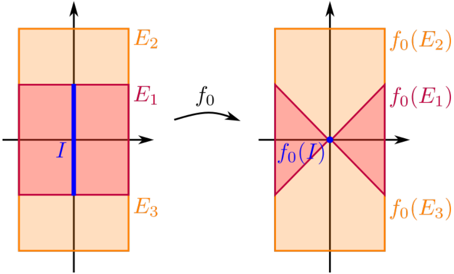
Proposition (M.-Rajala 23): $\exists$ a metric surface X that does not admit a quasiconformal homeomorphism $\varphi\colon X \to U \subset \mathbb{R}^2$ but admits $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$ of finite distortion along paths with $K_f \in L^1_{\mathrm{loc}}(X)$.

Recall Ball's map: Let $f_0\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f_0(x,y) = (x, \eta(x,y))$ with

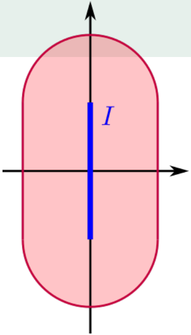
$$\eta(x,y) = \begin{cases} |x|y, & (x,y) \in E_1, \\ (2(|y|-1) + |x|(2-|y|))\frac{y}{|y|}, & (x,y) \in E_2 \cup E_3, \\ y, & \text{else.} \end{cases}$$

- f_0 is not open and discrete: $f_0^{-1}(0,0) = I$,
- K_{f_0} is bounded outside E_1 and

$$K_{f_0}(x,y) = \frac{1}{|x|} \quad \forall (x,y) \in E_1.$$



Example



Let $p > 1$. Define a weight $\omega\colon \mathbb{R}^2 \to [0,1]$ by

$$\omega(z) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, I) \geq 1, \\ \operatorname{dist}(x, I)^p, & \text{else.} \end{cases}$$

and set

$$d_\omega(x, y) = \inf \int_\gamma \omega \, ds,$$

where inf is taken over all rect $\gamma\colon [0,1] \to \mathbb{R}^2$, $\gamma(0) = x$, $\gamma(1) = y$.

Define: $X := (\mathbb{R}^2/I, d_\omega)$ and $\pi_\omega := \mathrm{id}_\omega \circ \pi$, where

- $\pi\colon \mathbb{R}^2 \to \mathbb{R}^2/I$ is the natural projection, and
- $\mathrm{id}_\omega\colon \mathbb{R}^2/I \to X$ is the identity map.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[d, "\pi"'] \arrow[dr, "\pi_\omega := \mathrm{id}_\omega \circ \pi"] & \\ \mathbb{R}^2/I \arrow[r, "\mathrm{id}_\omega"] & X \end{tikzcd}$$

- X is homeo to $\mathbb{R}^2$ and of locally finite $\mathcal{H}^2$.
- [Raj17]: X does not admit a quasiconformal homeo $\varphi\colon X \to U \subset \mathbb{R}^2$.

Example

Define: $f\colon X \to \mathbb{R}^2$ by $f := f_0 \circ \pi_\omega^{-1}$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "f_0"] \arrow[dr, "\pi_\omega"'] & \mathbb{R}^2 \\ & X \arrow[u, "f:=f_0\circ\pi_\omega^{-1}"'] \end{tikzcd}$$

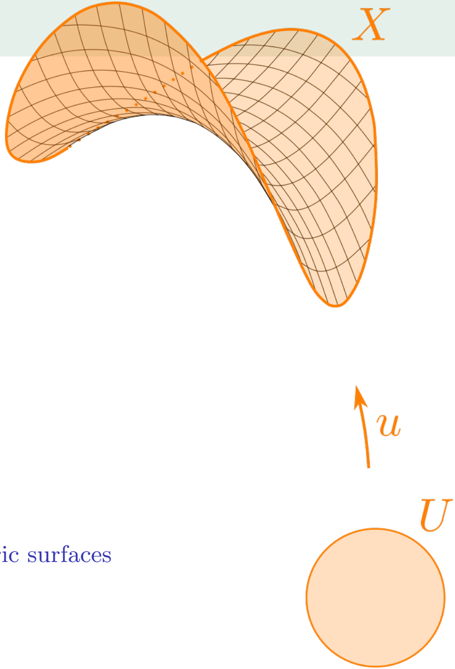
- $f \in N^{1,2}_{\mathrm{loc}}(X, \mathbb{R}^2)$:
  - f is absolutely continuous on a.e. curve,
  - $\rho_f^u(z) \leq L \cdot (\omega(z))^{-1}$ for a.e. $z \in X$ with $L = \mathrm{Lip}(f_0) = 2$,

  $\Rightarrow$ For every Borel set $E \subset X$

$$\int_E (\rho_f^u)^2 \, d\mathcal{H}^2_\omega \leq L^2 \int_E \omega^{-2} \, d\mathcal{H}^2_\omega = L^2 |\pi_\omega^{-1}(E)|_2.$$

- $K_f \in L^1_{\mathrm{loc}}(X)$:
  - $K_{f_0}(z) = K_f(\pi_\omega^{-1}(z))$ for a.e. $z \in X$,
  - K_{f_0} is bounded outside E_1 and

$$\int_{\pi_\omega(E_1)} K_f \, d\mathcal{H}^2_\omega = \int_{E_1} K_{f_0}(z)\omega^2 \, dz = \int_{E_1} |x|^{2p-1} \, dx \, dy < \infty.$$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