Energy minimizing harmonic 2-spheres in metric spaces

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Oberwolfach 2025

Existence problem for harmonic maps

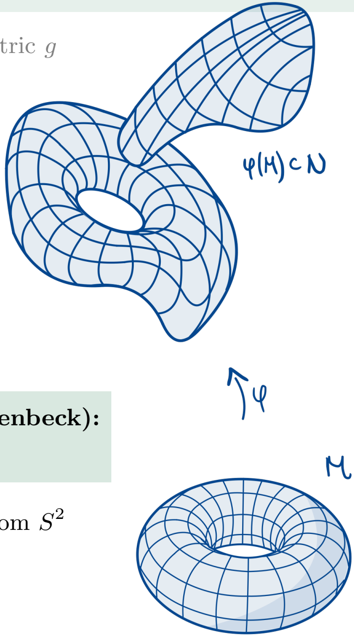
M - closed surface equipped with Riemannian metric g

N - compact Riemannian manifold

Question: Is every continuous map $\varphi\colon M \to N$ homotopic to a harmonic map $u\colon M \to N$?

A map $u \in W^{1,2}(M,N)$ is *harmonic*, if u is a critical point of the Dirichlet energy functional

$$E(u) = \frac{1}{2}\int_M |Du|^2 \, d\mathcal{H}^2_g.$$



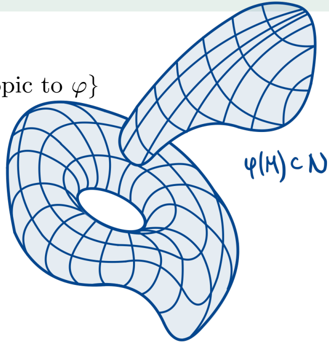
Theorem (Lemaire, Schoen-Yau, Sacks-Uhlenbeck): YES whenever $\pi_2(N) = 0$.

- *Recall:* $\pi_2(N) = 0$ iff. every continuous map from S^2 to N is homotopic to a constant map.
- Theorem is not true if $\pi_2(N) \neq 0$.

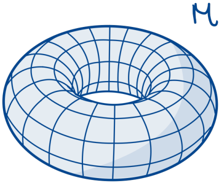
Let $\varphi\colon M \to N$ be continuous. Define

- $\Lambda(\varphi) := \{u \in W^{1,2}(M,N) : u \text{ cont. and homotopic to } \varphi\}$
- $e(\varphi) := \inf\{E(u) : u \in \Lambda(\varphi)\}$

Goal: Find $u \in \Lambda(\varphi)$ satisfying $E(u) = e(\varphi)$.



$\uparrow \varphi$



Direct variational method:

- Show that $\Lambda(\varphi) \neq \emptyset$.
- Let $(u_n) \subset \Lambda(\varphi)$ be energy minimizing, i.e. $E(u_n) \to e(\varphi)$.
- Subsequence of (u_n) converges to $u\colon M \to N$.
- Show that $u \in \Lambda(\varphi)$.
- Lower semi-continuity of energy gives that u is energy minimizer in $\Lambda(\varphi)$.
- Show that u satisfies further regularity properties.

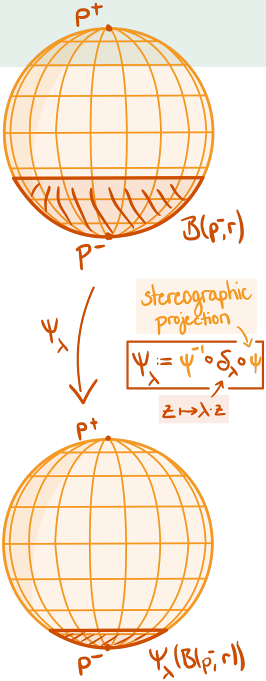
Example: Scaling map $\psi_\lambda \colon S^2 \to S^2$

Let $\psi_\lambda \colon S^2 \to S^2$ be the *scaling map* of factor $\lambda > 0$.

- ψ_λ is conformal,
- $\psi_\lambda \in \Lambda(\mathrm{id})$, and
- $E(\psi_\lambda) = 4\pi = e(\mathrm{id})$.

For $\lambda_n \to 0$, the sequence $(\psi_{\lambda_n}) \subset \Lambda(\mathrm{id})$ is *energy minimizing* but converges in L^2 to a *constant map* $u \colon S^2 \to S^2$.

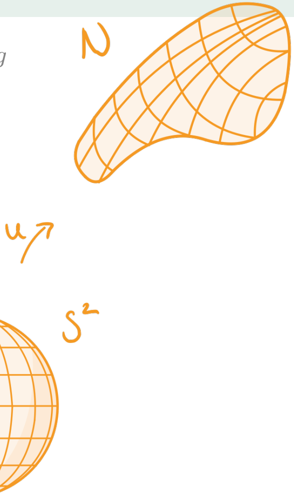
$$\implies \quad u \notin \Lambda(\mathrm{id})$$



Non-trivial harmonic spheres

M - closed surface equipped with Riemannian metric g
N - compact Riemannian manifold

Theorem (Sacks-Uhlenbeck): If $\pi_2(N) \neq 0$, then there exists a non-trivial $u: S^2 \to N$ minimizing energy within its homotopy class.
Every such u is a conformal branched immersion.



Non-trivial harmonic spheres

Theorem (Sacks-Uhlenbeck): If $\pi_2(N) \neq 0$, then there exists a non-contractible $u: S^2 \to N$ minimizing energy within its homotopy class. Every such u is a conformal branched immersion.

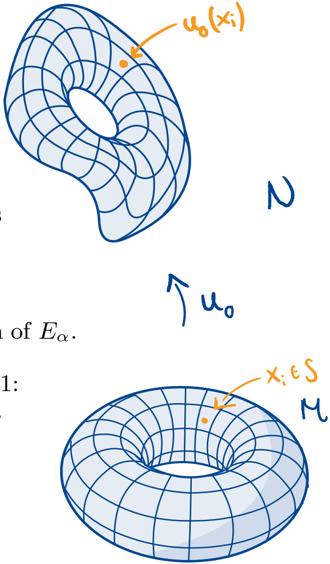
Let $\varphi: M \to N$ be continuous.

- For $\alpha > 1$, consider perturbed energy functionals

$$E_\alpha(v) = \int_M \left(|Dv|^2 + 1\right)^\alpha \, d\mathcal{H}_g^2.$$

- A priori estimates from Euler-Lagrange equation of E_α.
- Convergence of E_α-minimizer $v_\alpha \in \Lambda(\varphi)$ as $\alpha \to 1$:
    - $\exists \; S \subset M$ finite s.th. $v_\alpha|_{M \setminus S}$ converge in C^∞ and limit extends to smooth harmonic map

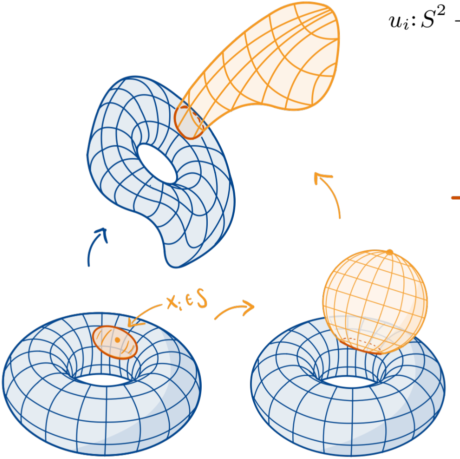
$$u_0: M \to N.$$

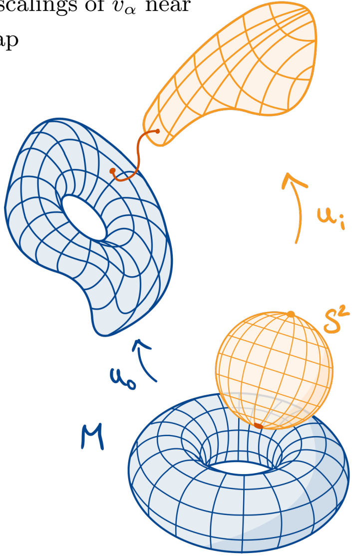


Non-trivial harmonic spheres

- Convergence of v_α as $\alpha \to 1$:
  - For every $x_i \in S$ a suitable sequence of rescalings of v_α near x_i converge to a non-trivial harmonic map

$$u_i \colon S^2 \to N.$$





Non-trivial harmonic spheres

- Convergence of v_α as $\alpha \to 1$:
  - For every $x_i \in S$ a suitable sequence of rescalings of v_α near x_i converge to a non-trivial harmonic map

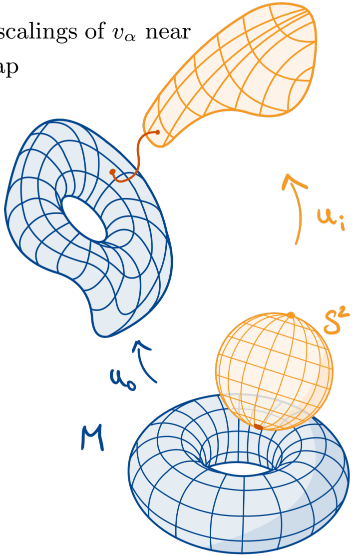
$$u_i\colon S^2 \to N.$$

- **Energy gap [Sacks-Uhlenbeck]:**
  There exists $\varepsilon > 0$ s.th.

$$E(u_i) > \varepsilon \qquad \text{for all } i \geq 1.$$

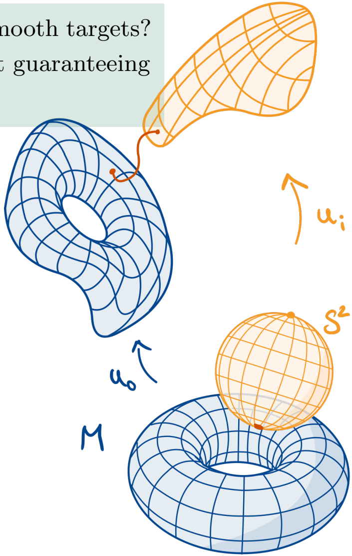
- **Energy identity [Jost]:**

$$\begin{aligned} e(\varphi) &= E(u_0) + E(u_1) + \ldots + E(u_m) \\ &= e(u_0) + e(u_1) + \ldots + e(u_m). \end{aligned}$$



Questions: Do these results generalize to non-smooth targets? What are the essential assumptions on the target guaranteeing the existence of non-trivial harmonic spheres?

Hope: Find a conceptually simpler proof (not depending on PDE-Methods).



Sobolev maps into metric spaces

X - compact metric space

M - closed surface equipped with Riemannian metric g

Ω - open subset of M

A measurable map $u\colon \Omega \to X$ is in $L^2(\Omega, X)$ if for some (any) $x \in X$

$$u_x(z) := d(x, u(z)) \in L^2(\Omega).$$

Definition: $u \in L^2(\Omega, X)$ is in the *Sobolev space $W^{1,2}(\Omega, X)$* if

- $u_x \in W^{1,2}(\Omega)$ for every $x \in X$, and
- $\exists\ h \in L^2(\Omega)$ s.th. for all $x \in X$ we have

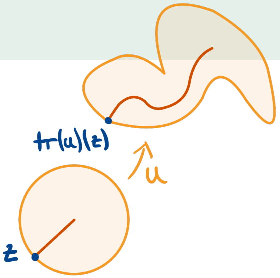
$$|\nabla u_x|_g \leq h \quad \text{a.e. on } \Omega.$$

$$\begin{tikzcd} \Omega \arrow[r, "u"] \arrow[dr, "u_x"'] & X \arrow[d, "d(x,\cdot)"] \\ & \mathbb{R} \end{tikzcd}$$

The *Reshetnyak energy E_+^2* of $u \in W^{1,2}(\Omega, X)$ is defined by

$$E_+^2(u) := \inf\left\{ \|h\|^2_{L^2(\Omega)} : h \text{ as in the definition above} \right\}.$$

Setting



Let X be a compact metric space satisfying:

- X admits a *local quadratic isoperimetric inequality (local qii)*, i.e. $\exists\ C, l_0 > 0$ s.th. every Lipschitz curve

$$\gamma : S^1 \to X \quad \text{of length} \quad \ell(\gamma) \le l_0$$

  is the trace of a Sobolev map $u \in W^{1,2}(D, X)$ with

$$\mathrm{Area}(u) \le C \cdot \ell(\gamma)^2.$$

Theorem [Lytchak-Wenger]: For every Sobolev map $u \in W^{1,2}(D, X)$ there exists $v \in W^{1,2}(D, X)$ with

$$E_+^2(v) = \inf\{E_+^2(w) : w \in W^{1,2}(D, X),\ \mathrm{tr}(w) = \mathrm{tr}(u)\}$$

and $\mathrm{tr}(v) = \mathrm{tr}(u)$. Any such v has a locally Hölder continuous representative $\bar{v}$, which extends continuously to the boundary whenever $\mathrm{tr}(u)$ is continuous.

We call the continuous map $\bar{v}$ *Dirichlet solution*.

Setting

Let X be a compact metric space satisfying:

- X admits a *local quadratic isoperimetric inequality (local qii)*, i.e. $\exists\ C, l_0 > 0$ s.th. every Lipschitz curve

$$\gamma\colon S^1 \to X \quad \text{of length} \quad \ell(\gamma) \leq l_0$$

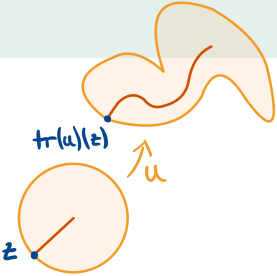
  is the trace of a Sobolev map $u \in W^{1,2}(D, X)$ with

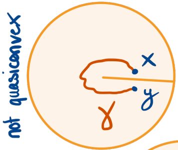
$$\operatorname{Area}(u) \leq C \cdot \ell(\gamma)^2.$$

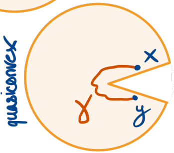
- X is *quasiconvex*, i.e. $\exists\ \lambda \geq 1$ s.th. every pair of points $x, y \in X$ can be joined by a curve γ in X with

$$\ell(\gamma) \leq \lambda \cdot d(x, y).$$

- Every continuous map from S^2 to X of sufficiently small diameter is null-homotopic.





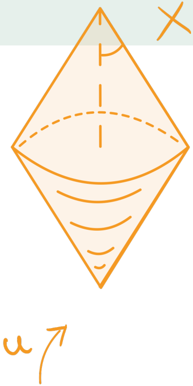


Examples: Closed Riemannian manifolds, compact Lipschitz manifolds, compact locally CAT(κ)-spaces, some compact sub-Riemannian manifolds, ...

M - closed surface equipped with Riemannian metric g

X - compact quasiconvex metric space satisfying a local qii,

every continuous $S^2 \to X$ of small diam is null-homotopic



Proposition: Every $u \in W^{1,2}(M, X)$ minimizing energy in its homotopy class is *harmonic* (i.e. locally energy minimizing), and thus Hölder continuous.
If $M = S^2$, then u is *infinitesimally isotropic*.

- Hölder continuity is best we can hope for.
  - **Example:** X double cone of small cone angle and $u: S^2 \to X$ radial stretch function $t \mapsto t^\alpha$ for some $\alpha \in (0, 1)$.

Regularity

M - closed surface equipped with Riemannian metric g

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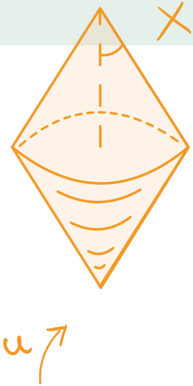

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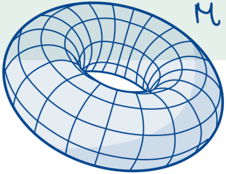
If $M = S^2$, then u is *infinitesimally isotropic*.

- Hölder continuity is best we can hope for.
- Infinitesimal isotropy implies
  - infinitesimal $\sqrt{2}$-quasiconformality, i.e.

$$\frac{\text{max stretch at } z}{\text{min stretch at } z} \leq \sqrt{2} \quad \text{for a.e. } z \in S^2,$$

  - weak conformality if X is Riemannian or locally CAT(1).

Existence



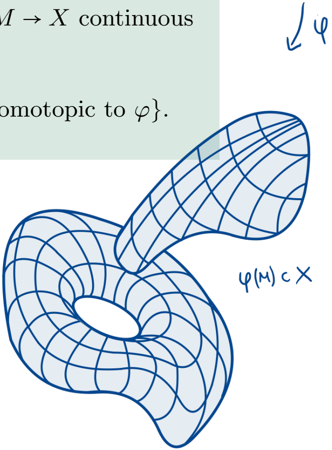
M - closed surface equipped with Riemannian metric g

X - compact quasiconvex metric space satisfying a local qii,
every continuous $S^2 \to X$ of small diam is null-homotopic

Theorem: If $\pi_2(X) = 0$, then for every $\varphi\colon M \to X$ continuous there exists an energy minimizer in

$$\Lambda(\varphi) := \{u \in W^{1,2}(M, X) : u \text{ cont. and homotopic to } \varphi\}.$$

- Theorem fails if $\pi_2(X) \neq 0$.



Example: Define $X := S^2 \sqcup [0,1] \sqcup S^2 /_{\sim}$, then

- $\pi_2(X) \neq 0$, and
- X satisfies all standing assumptions.

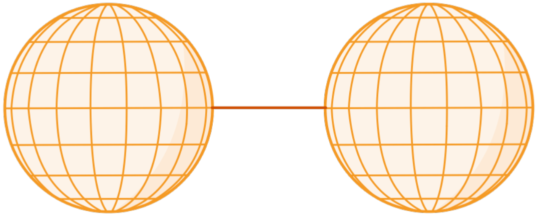
Let $\varphi\colon S^2 \to X$ be as illustrated.
Assume there exists an energy minimizer $u \in \Lambda(\varphi)$.

$\Rightarrow$ u is infinitesimally quasiconformal

$\Rightarrow$ energy of $u|_{u^{-1}((0,1))}$ is zero

$\Rightarrow$ u is locally constant on $u^{-1}((0,1))$

$\rightarrow$ **not possible!**



Existence

M - closed surface equipped with Riemannian metric g

X - compact quasiconvex metric space satisfying a local qii,
every continuous $S^2 \to X$ of small diam is null-homotopic

For $\varphi: M \to X$ continuous, we define

$$e_+(\varphi) := \inf\{E_+^2(u) : u \in \Lambda(\varphi)\}.$$

Main Theorem: Every continuous map $\varphi: M \to X$ has an iterated decomposition into $\varphi_0: M \to X$ and finitely many $\varphi_1, \dots, \varphi_k: S^2 \to X$ such that

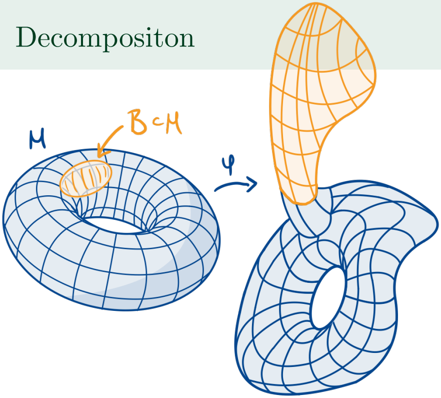
$$e_+(\varphi_0) + e_+(\varphi_1) + \cdots + e_+(\varphi_k) = e_+(\varphi)$$

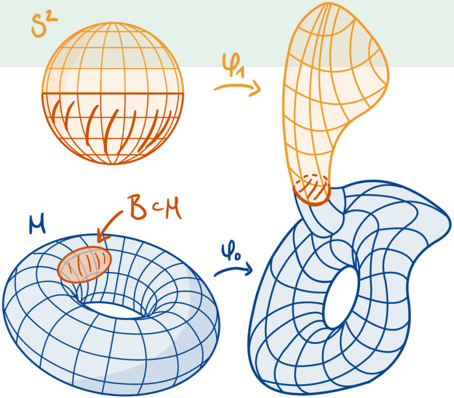
and such that every φ_i contains an energy minimizer in its homotopy class.

Theorem also holds for a general definition of energy E. We recover:

- Theorems of Lemaire, Schoen-Yau and Sacks-Uhlenbeck for $X = N$.
- Result of Breiner-Fraser-Huang-Mese-Sargent-Zhang for X locally CAT(1).

Decompositon





Definition: $\varphi_0 \colon M \to X$ and $\varphi_1 \colon S^2 \to X$ *decompose* $\varphi \colon M \to X$ if

- φ_0 agrees with φ on $M \setminus B$,
- φ_1 obtained by gluing $\varphi|_B$ and $\varphi_0|_B$ along ∂B,
- φ_1 is essential, and if $M = S^2$ also φ_0.

Iterated decomposition:

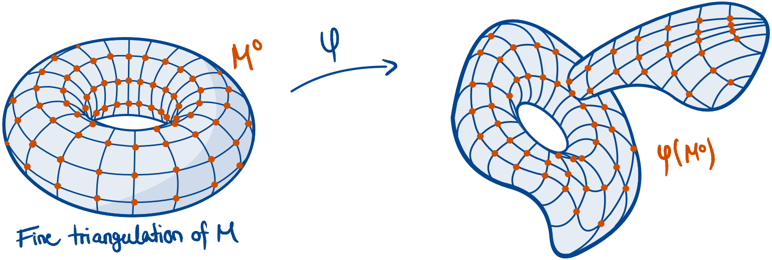
0-step: $\varphi_0 = \varphi$,

k-step: $\varphi_0 \colon M \to X$ and $\varphi_1, \dots, \varphi_k \colon S^2 \to X$ obtained from decomposing a map in a $(k-1)$-step iterated decomposition of φ.

(A) **Existence of homotopic Sobolev mappings:** For every $\varphi\colon M \to X$ continuous, the set $\Lambda(\varphi)$ is not empty.

- Choose fine enough triangulation of M and set

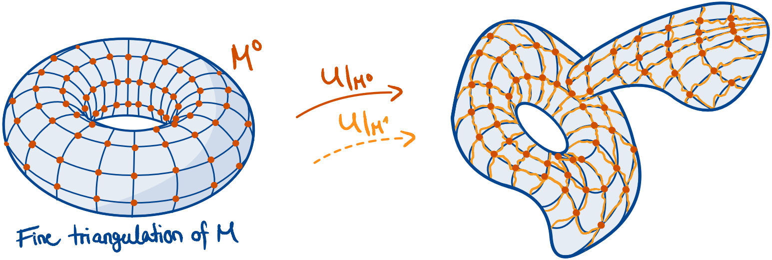
$$u|_{M^0} = \varphi|_{M^0}.$$



(A) **Existence of homotopic Sobolev mappings:** For every $\varphi\colon M \to X$ continuous, the set $\Lambda(\varphi)$ is not empty.

- Choose fine enough triangulation of M and set $u|_{M^0} = \varphi|_{M^0}$.
- Use quasiconvexity to extend $u|_{M^0}$ to a Lipschitz map

$$u|_{M^1}\colon M^1 \to X.$$



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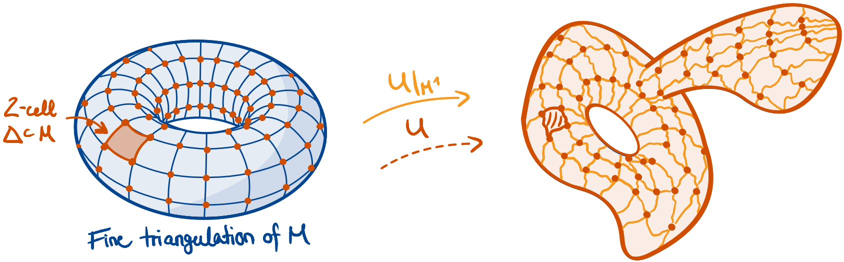
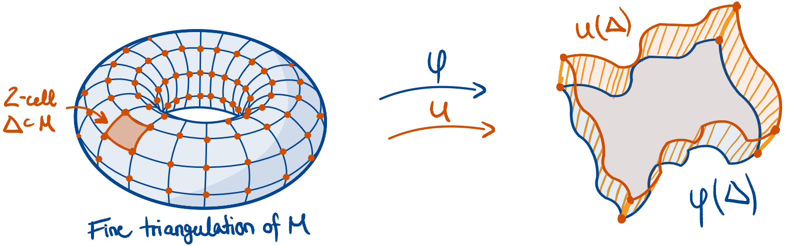
- Choose fine enough triangulation of M and set $u|_{M^0} = \varphi|_{M^0}$.
- Use quasiconvexity to extend $u|_{M^0}$ to a Lipschitz map $u|_{M^1}\colon M^1 \to X$.
- Local qii + [Lytchak-Wenger]: $u|_{M^1}$ extends to a

$$u \in W^{1,2}(M, X) \text{ continuous.}$$



(A) **Existence of homotopic Sobolev mappings:** For every $\varphi\colon M \to X$ continuous, the set $\Lambda(\varphi)$ is not empty.

- **We have shown:** $\forall\ \varepsilon > 0\ \exists\ u \in W^{1,2}(M, X)$ with $\operatorname{dist}(u, \varphi) < \varepsilon$.
- Construct homotopy between u and φ by using quasiconvexity, local qii and contractibility of spheres of small diameter.

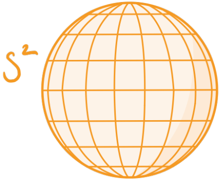


(A) **Existence of homotopic Sobolev mappings:**

For every $\varphi\colon M \to X$ continuous, the set

$$\Lambda(\varphi) := \{u \in W^{1,2}(M, X) : u \text{ continuous and homotopic to } \varphi\}$$

is not empty.

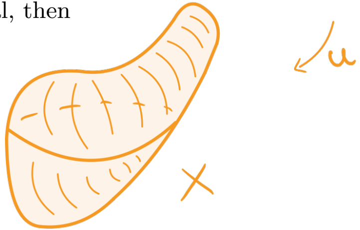


(B) **Spheres of small area are null-homotopic:**

There exists $\varepsilon_0 > 0$ s.th. every $u \in W^{1,2}(S^2, X)$ with $\text{Area}(u) < \varepsilon_0$ is null-homotopic.

- **Energy gap for essential maps:**
  If $u \in W^{1,2}(S^2, X)$ is essential, then

$$E_+^2(u) \geq \text{Area}(u) \geq \varepsilon_0.$$



(1) **Convergence of energy distributed minimizing sequence:** A sequence (u_n) of continuous mappings in $W^{1,2}(M,X)$ of uniformly bounded energy is *minimizing* if

$$E_+^2(u_n) - e_+(u_n) \to 0 \quad \text{for } n \to \infty.$$

- **Rellich-Kondrachov:** A subsequence of (u_n) converges in L^2 to $u \in W^{1,2}(M,X)$, i.e.

$$\int_M d^2(u(z), u_n(z))\, d\mathcal{H}^2(z) \xrightarrow{n\to\infty} 0.$$

- A priori, there is no reason that u has a continuous representative.
- Even if $(u_n) \subset \Lambda(\varphi)$ and u has a continuous representative $\bar{u}$ we might have

$$\bar{u} \notin \Lambda(\varphi).$$

(1) Convergence of energy distributed minimizing sequence: A sequence (u_n) of continuous mappings in $W^{1,2}(M,X)$ of uniformly bounded energy is *minimizing* if

$$E_+^2(u_n) - e_+(u_n) \to 0 \quad \text{for } n \to \infty.$$

Theorem: Let (u_n) be a minimizing sequence converging in L^2 to $u \in W^{1,2}(M,X)$. If $\exists\, r_0 > 0$ s.th.

$$E_+^2(u_n|_{B(p,r_0)}) \le \frac{\varepsilon_0}{5} \qquad \forall\, p \in M,\ \forall\, n \in \mathbb{N},$$

then u has a continuous representative $\bar{u} \in W^{1,2}(M,X)$ with

- $\bar{u}$ satisfies $E_+^2(\bar{u}) = e_+^2(\bar{u})$, and
- $\bar{u}$ is homotopic to u_n for large n.

Idea: Build homotopy between u_n and u as in (A) while using (B).

Problems: Lack of continuity of u and "only" L^2-convergence.

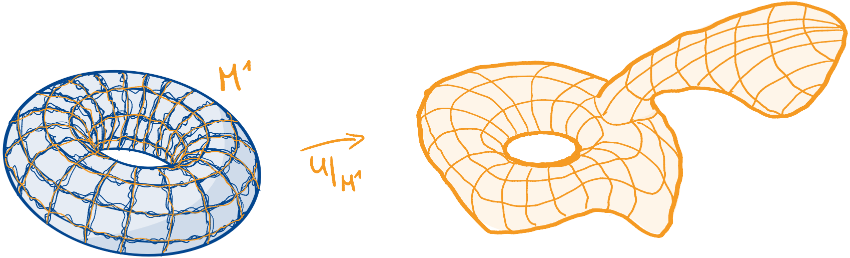
Energy distributed minimizing sequences

Fix fine enough triangulation of M.

- **Control along 1-skeleton:** After "wiggling" we find good triangulation M_ξ of M s.th. (up to taking subsequence)
  - $(u_n|_{M_\xi^1})$ has uniformly bounded length and

$$u_n|_{M_\xi^1} \xrightarrow{\text{uniformly}} \text{cont. rep. of } u|_{M_\xi^1}.$$

→ Uses methods introduced in [Soultanis-Wenger].



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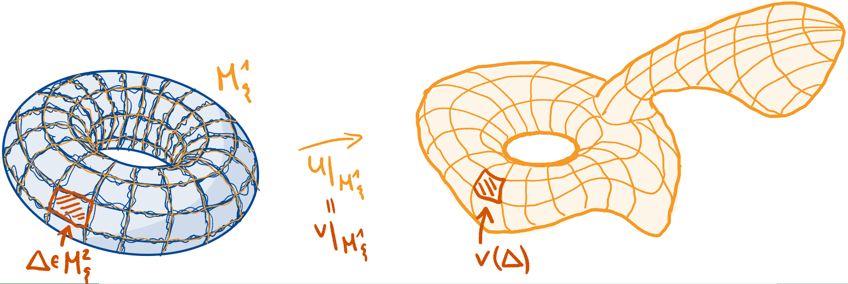
$$u_n|_{M_\xi^1} \xrightarrow{\text{uniformly}} \text{cont. rep. of } u|_{M_\xi^1}.$$

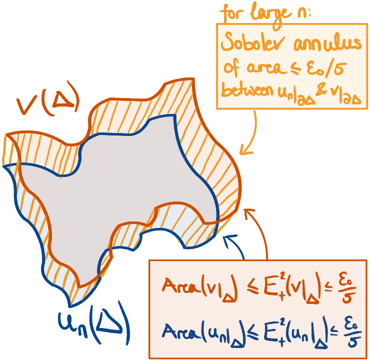
- Compare u to the continuous map $v \in W^{1,2}(M, X)$ defined as follows:
  - $v|_{M_\xi^1}$ agrees with the cont. rep. of $u|_{M_\xi^1}$, and
  - $v|_\Delta$ is Dirichlet solution with trace $u|_{\partial\Delta}$ for all $\Delta \in M_\xi^2$.



Energy distributed minimizing sequences

- **u_n is homotopic to v for large $n \in \mathbb{N}$:** Use quasiconvexity and local qii to construct Sobolev annulus of small area between $u_n|_{\partial\Delta}$ and $v|_{\partial\Delta}$ for every $\Delta \in M^2_\xi$.
  - Gluing this annulus to $u_n|_\Delta$ and $v|_\Delta$ gives Sobolev sphere of small area.

  (B): Induces homotopy between u_n and v.



- LSC of energy + minimality of (u_n):

$$e_+(v) \leq E^2_+(v) \leq E^2_+(u) \leq \liminf_{n\to\infty} E^2_+(u_n) = e_+(v).$$

$\Rightarrow$ $u|_\Delta$ is an energy minimizer for every $\Delta \in M^2_\xi$.

$\Rightarrow$ [Lytchak-Wenger]: u has a continuous representative $\bar{u}$.

Repeat arguments to build homotopy between $\bar{u}$ and u_n for large n.

(1) **Convergence of energy distributed minimizing sequence:** A sequence (u_n) of continuous mappings in $W^{1,2}(M,X)$ of uniformly bounded energy is *minimizing* if

$$E_+^2(u_n) - e_+(u_n) \to 0 \quad \text{for } n \to \infty.$$

Theorem: Let (u_n) be a minimizing sequence converging in L^2 to $u \in W^{1,2}(M,X)$. If $\exists\, r_0 > 0$ s.th.

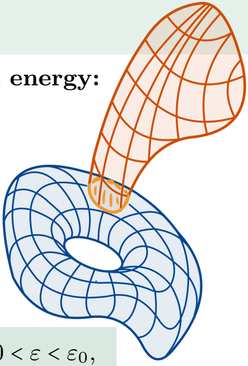
$$E_+^2(u_n|_{B(p,r_0)}) \le \frac{\varepsilon_0}{5} \qquad \forall\, p \in M,\ \forall\, n \in \mathbb{N},$$

then u has a continuous representative $\bar{u} \in W^{1,2}(M,X)$ with

- $\bar{u}$ satisfies $E_+^2(\bar{u}) = e_+^2(\bar{u})$, and
- $\bar{u}$ is homotopic to u_n for large n.

Idea: Build homotopy between u_n and u as in (A) while using (B).

Problems: Lack of continuity of u and "only" L^2-convergence.



(2) **ε-indecomposability implies uniformly distributed energy:** (up to precomposition with conformal diffeomorphisms)

Condition: A continuous map $\varphi: M \to X$ is *ε-indecomposable* if for any decomposition $\varphi_0: M \to X$ and $\varphi_1: S^2 \to X$ of φ we have

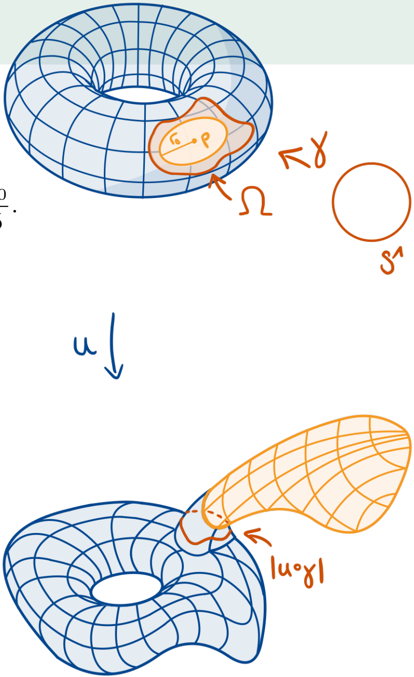
$$e_+(\varphi_0) + e_+(\varphi_1) \geq e_+(\varphi) + \varepsilon.$$

Proposition: If $\varphi: M \to X$ is ε-indecomposable for some $0 < \varepsilon < \varepsilon_0$, then there exists $r_0 > 0$ s.th. the following holds:
If $u \in \Lambda(\varphi)$ is almost energy minimizing, then there exists a conformal diffeomorphism $\eta: M \to M$ s.th.

$$E_+^2(u \circ \eta|_{B(p,r_0)}) \leq \frac{\varepsilon_0}{5} \qquad \forall\, p \in M.$$

- We can choose $\eta = \mathrm{id}$ if $M \neq S^2$.

Uniformly distributing energy

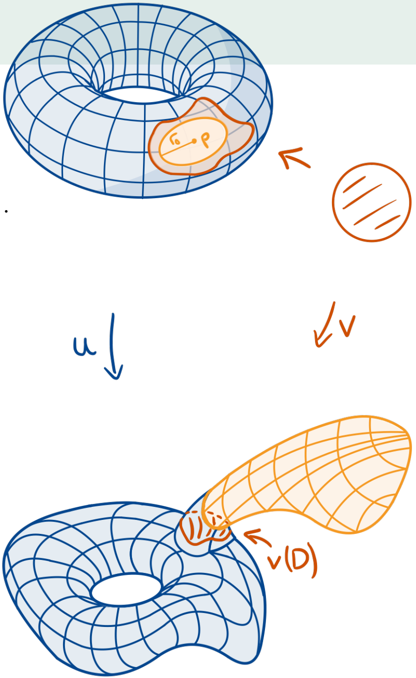


$M \neq S^2$: Let $r_0 > 0$ be well-chosen (decreases as $e_+(\varphi)$ increases).

- Assume $\exists\, p \in M$ s.th. $E_+^2(u|_{B(p,r_0)}) > \frac{\varepsilon_0}{5}$.
- We find Jordan curve $\gamma\colon S^1 \to M$ "surrounding" $B(p, r_0)$ with

$$E^2(u \circ \gamma) < \delta \quad \text{and} \quad \ell(u \circ \gamma) < l_0.$$

Uniformly distributing energy



$M \neq S^2$: Let $r_0 > 0$ be well-chosen (decreases as $e_+(\varphi)$ increases).

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- We find Jordan curve $\gamma\colon S^1 \to M$ "surrounding" $B(p, r_0)$ with

$$E^2(u \circ \gamma) < \delta \quad \text{and} \quad \ell(u \circ \gamma) < l_0.$$

- Let $v \in W^{1,2}(D, X)$ be the Dirichlet solution with trace $u \circ \gamma$.

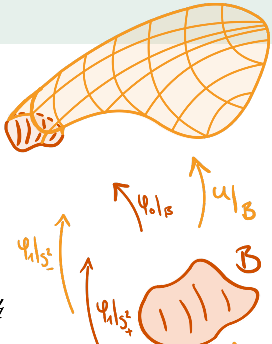
Use v to construct two continuous maps
$\varphi_0 \in W^{1,2}(M,X)$ and $\varphi_1 \in W^{1,2}(S^2,X)$.

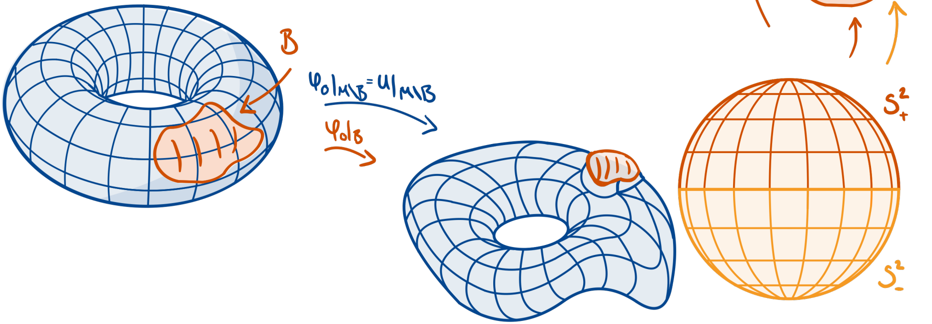
- If φ_1 is null-homotopc, then φ_0 is homotopic to φ and

$$E_+^2(\varphi_0) = E_+^2(u) - E_+^2(u|_B) + E_+^2(v) < e_+(\varphi). \quad ↯$$

$\Rightarrow$ φ_0 and φ_1 form a decomposition of φ and

$$E_+^2(\varphi_0) + E_+^2(\varphi_1) = E_+^2(u) + 2E_+^2(v) < e_+(\varphi) + \varepsilon. \quad ↯$$





Uniformly distributing energy

Use v to construct two continuous maps $\varphi_0 \in W^{1,2}(M,X)$ and $\varphi_1 \in W^{1,2}(S^2,X)$.

- If φ_1 is null-homotopc, then φ_0 is homotopic to φ and

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$\Rightarrow$ φ_0 and φ_1 form a decomposition of φ and

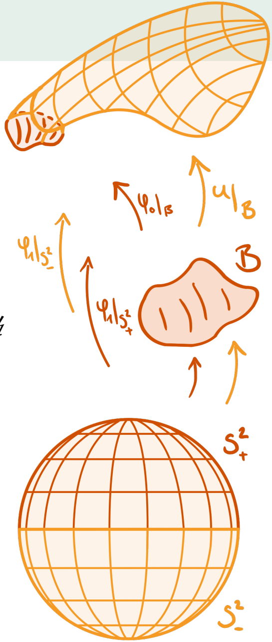
$$E_+^2(\varphi_0) + E_+^2(\varphi_1) = E_+^2(u) + 2E_+^2(v) < e_+(\varphi) + \varepsilon. \quad \lightning$$

$M = S^2$: Contradiction to ε-indecomposability only if φ_1 **and** φ_0 are essential.

- Precompose u with a certain diffeomorphism

$$\eta: S^2 \to S^2.$$

- Use a similar construction as above.



Existence

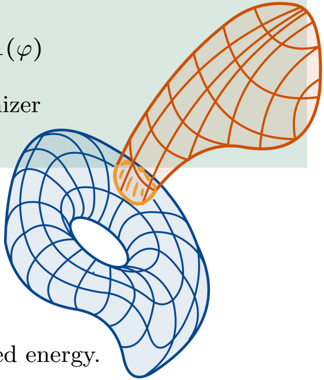
M - closed surface equipped with Riemannian metric g

X - compact quasiconvex metric space satisfying a local qii,
every continuous $S^2 \to X$ of small diam is null-homotopic

Main Theorem: Every continuous map $\varphi : M \to X$ has an iterated decomposition into $\varphi_0 : M \to X$ and finitely many $\varphi_1, \dots, \varphi_k : S^2 \to X$ such that

$$e_+(\varphi_0) + e_+(\varphi_1) + \cdots + e_+(\varphi_k) = e_+(\varphi)$$

and such that every piece contains an energy minimizer in its homotopy class.



We have established:

(1) Convergence of energy distributed minimizing sequences.

(2) ε-indecomposability implies uniformly distributed energy.

Proof of main theorem

Note: Every iterated decomposition of φ satisfies

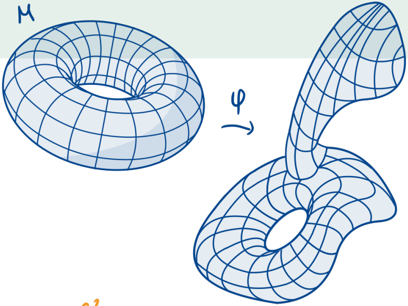
$$e_+(\varphi) \leq e_+(\varphi_0) + \underbrace{e_+(\varphi_1)}_{\overset{(B)}{\geq}\ \varepsilon_0} + \cdots + \underbrace{e_+(\varphi_k)}_{\overset{(B)}{\geq}\ \varepsilon_0}. \quad (1)$$

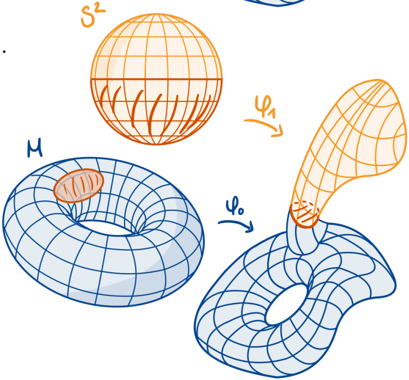
Let m be largest integer s.th. "=" holds in (1).

Take: Sequences of m-step iterated decompositions of φ satisfying

$$e_+(\varphi_0^n) + e_+(\varphi_1^n) + \cdots + e_+(\varphi_m^n) \xrightarrow{n\to\infty} e_+(\varphi).$$

Then φ_i^n is ε-indecomposable for suitable ε.





Take: Sequences of m-step iterated decompositions of φ satisfying

- φ_i^n is ε-indecomposable,
- $e_+(\varphi_0^n) + e_+(\varphi_1^n) + \cdots + e_+(\varphi_m^n) \xrightarrow{n\to\infty} e_+(\varphi)$.

Choose: $u_i^n \in \Lambda(\varphi_i^n)$ s.th.

$$E_+^2(u_i^n) \leq e_+(\varphi_i^n) + \frac{1}{n}.$$

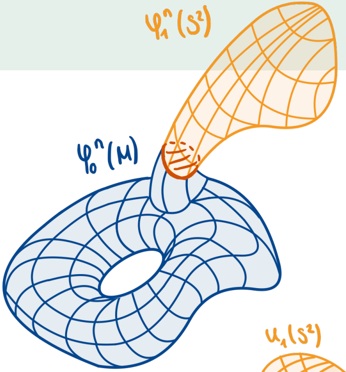
(2): $\exists\ r_i > 0$ and conformal diffeos η_i^n s.th.

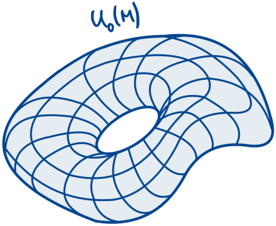
$$E_+^2(u_i^n \circ \eta_i^n|_{B(p,r_i)}) \leq \frac{\varepsilon_0}{5}.$$

(1): $u_i^n \circ \eta_i^n$ converges in L^2 to a continuous map

$$u_i \in W^{1,2}(M_i, X) \quad \text{with} \quad E_+^2(u_i) = e_+^2(u_i),$$

and u_i is homotopic to $u_i^n \circ \eta_i^n$ (and thus to φ_i^n) for sufficiently large n. $\square$

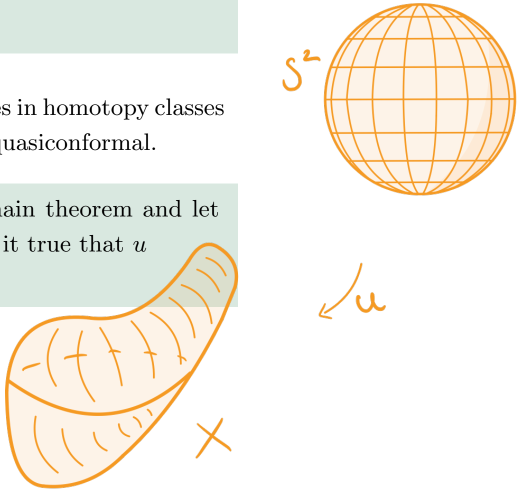




Question 1: Let X be a compact metric space with non-trivial k-th homotopy group for some $k \geq 2$. Under what additional conditions does X admit a non-trivial harmonic 2-sphere?

Recall: Energy minimizing spheres in homotopy classes are harmonic and infinitesimally quasiconformal.

Question 2: Let X be as in main theorem and let $u\colon S^2 \to X$ be a harmonic map. Is it true that u is infinitesimally quasiconformal?



Outline