Uniformization of metric surfaces

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ETH Geometry Seminar

Outline

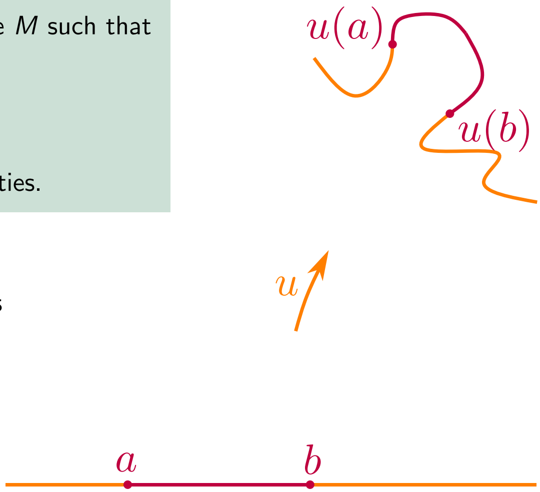
Uniformization of metric surfaces

Uniformization problem

Uniformization problem: Find conditions on a metric space X homeomorphic to a model space M such that there exists a mapping

$$u\colon M \to X$$

with good geometric and analytic properties.



Dimension 1:

- Every locally rectifiable curve admits a parametrization by **arclength**.
  - u is 1-Lipschitz, i.e.

$$d(u(a), u(b)) \leq L \cdot |a - b|$$

for $L = 1$.

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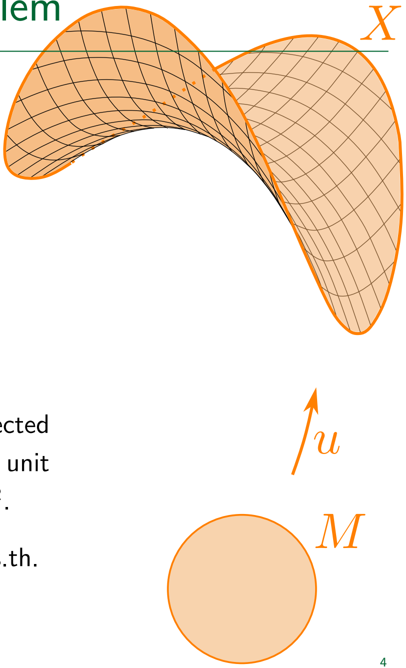
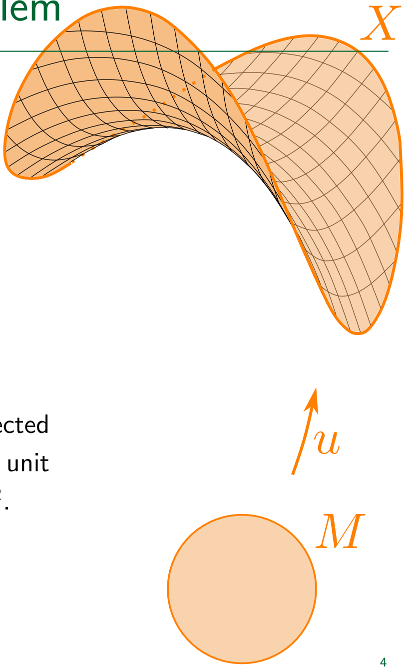


Dimension 2:

- **Classical uniformization theorem:** Every simply connected Riemann surface X is conformally equivalent to the open unit disc D, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$.
  - Conformal map is locally bi-Lipschitz, i.e. $\exists L \geq 1$ s.th.

$$L^{-1} \cdot |a-b| \leq d(u(a), u(b)) \leq L \cdot |a-b|.$$

Uniformization problem



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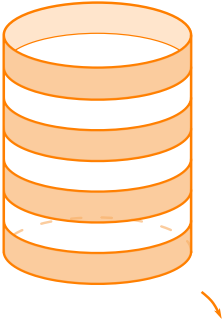
Dimension 2:

- **Classical uniformization theorem:** Every simply connected Riemann surface X is conformally equivalent to the open unit disc D, the complex plane $\mathbb{C}$, or the Riemann sphere $\mathbb{S}^2$.
  - Conformal map is locally bi-Lipschitz.
  - Maps infinitesimal balls to balls.

Metric surfaces

Definition: A metric space X is a *metric surface* if X is homeomorphic to a 2-dimensional manifold M.

- Non-smooth metric surfaces appear naturally as
  - deformations of smooth surfaces,
  - limits of sequences of Riemannian surfaces,
  - boundaries of Gromov hyperbolic groups.



Goal: Find conditions on X such that there exists a parametrization $u\colon M \to X$ satisfying certain properties.

- We are interested in non-smooth metric surfaces of **locally finite area** (Hausdorff 2-measure $\mathcal{H}^2$).

Lipschitz uniformization

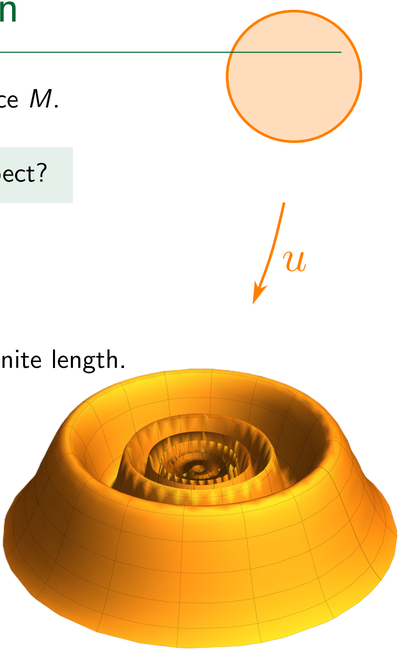
Let X be a metric surface homeomorphic to a Riemannian surface M.

Question: What type of parametrization $u\colon M \to X$ can we expect?

- If $u\colon M \to X$ is **Lipschitz**, then

$$\ell(u \circ \gamma) \leq L \cdot \ell(\gamma) \quad \text{for every curve } \gamma \text{ in } M.$$

$\Rightarrow$ Every pair of points in X can be joined by a curve of finite length.



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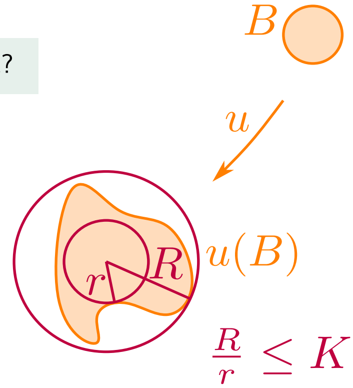
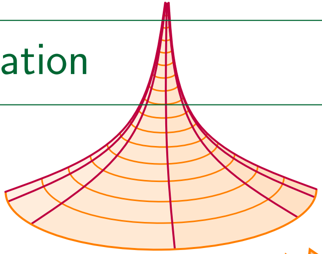


1. **Quasisymmetric uniformization:** A homeomorphism $u\colon M \to X$ is **quasisymmetric** if it distorts shapes of sets in a controlled manner on *all scales*.

2. **Quasiconformal uniformization:** A homeomorphism $u\colon M \to X$ is **quasiconformal** if it distorts shapes of sets in a controlled manner on *infinitesimal scales*.

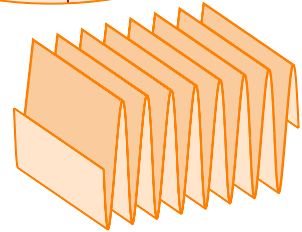
3. **Weakly quasiconformal uniformization:** An *almost* homeomorphism $u\colon M \to X$ is **weakly quasiconformal** if it distorts shapes of sets in a controlled manner on *infinitesimal scales*.

Quasisymmetric uniformization



Theorem [Bonk–Kleiner 2002]: Let $X \approx S^2$ be an *Ahlfors 2-regular* metric surface. There exists a quasisymmetric map $u\colon S^2 \to X$ if and only if X is *linearly locally contractible*.

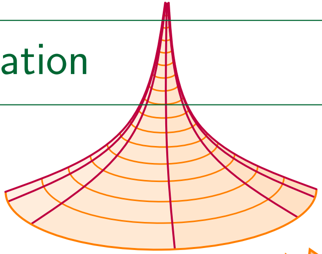
Ahlfors 2-regularity: $\mathcal{H}^2(B(x,r))$ is comparable to r^2.



Linear local contractibility (LLC): $B(x,r)$ is contractible in $B(x,\lambda r)$.

$\Rightarrow$ Prevent surface from having cusps, thin bottlenecks, dense wrinkles.

Quasisymmetric uniformization



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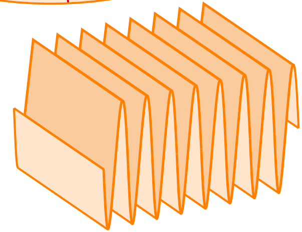


- Theorem does not generalize to higher dimensions (Semmes, Heinonen-Wu, Pankka-Wu).
- Ahlfors 2-regularity is not a quasisymmetric invariant.
  - $\mathrm{id}\colon S^2 \to (S^2, d^{\alpha}_{S^2})$ for $\alpha \in (0,1)$ is quasisymmetry.
- Same statement without Ahlfors 2-regularity would solve:

Cannon's conjecture: Let G be a Gromov hyperbolic group whose boundary at infinity $\partial_\infty G$ is homeomorphic to S^2. Then G is a Kleinian group, i.e. G admits an isometric, properly discontinuous, and cocompact action on $\mathbb{H}^3$.

Geometric quasiconformality

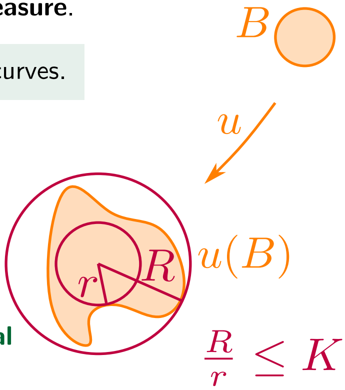
Let X and Y be a metric surfaces of **locally finite Hausdorff 2-measure**.

Observation: X and Y contain an abundance of locally rectifiable curves.

- A homeomorphism $u\colon X \to Y$ is **quasiconformal** if it distorts *shapes of sets* in a controlled manner on infinitesimal scales.

$$\Uparrow$$

- A homeomorphism $u\colon X \to Y$ is **geometrically quasiconformal** if it distorts *families of curves* in a controlled manner.



Question: How can we measure "largeness" of families of curves?

Geometric quasiconformality

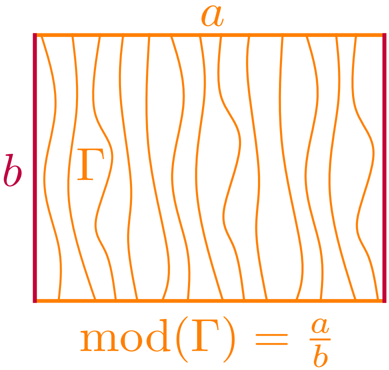
Let X, Y be metric surfaces and Γ a family of curves in X.

- The **(conformal) modulus** of Γ is

$$\mathrm{mod}(\Gamma) := \inf \int_X \rho^2 \, d\mathcal{H}^2,$$

  where the infimum is taken over all Borel functions $\rho \colon X \to [0, \infty]$ with

$$\int_\gamma \rho \geq 1 \quad \text{for every locally rectifiable } \gamma \in \Gamma.$$



- $u \colon X \to Y$ is **geometrically quasiconformal** if $\exists K \geq 1$ s.th.

$$K^{-1} \,\mathrm{mod}(\Gamma) \leq \mathrm{mod}(u \circ \Gamma) \leq K \,\mathrm{mod}(\Gamma)$$

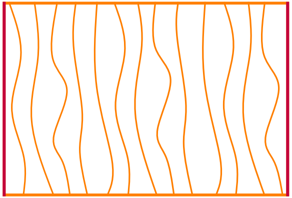
  for every family Γ of curves in X.

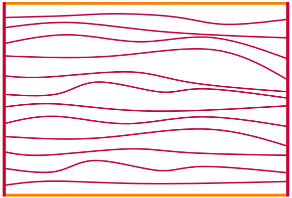
Geometric quasiconformality

Modulus in the plane:

(1) If $Q \subset \mathbb{R}^2$ is a quadrilateral, then

$$\mathrm{mod}(\Gamma(Q)) \cdot \mathrm{mod}(\Gamma^*(Q)) = \frac{a}{b} \cdot \frac{b}{a} = 1.$$





Quasiconformal uniformization

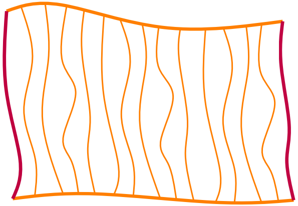
If $u\colon U \subset \mathbb{R}^2 \to X$ is geometrically quasiconformal, then

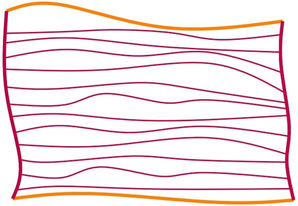
(1) For every quadrilateral $Q \subset X$

$$\mathrm{mod}(\Gamma(Q)) \cdot \mathrm{mod}(\Gamma^*(Q)) \leq \kappa.$$

For example: $\mathbb{R}^2$ and Ahlfors 2-regular metric spaces.

Theorem [Rajala 2017]: Let $X \approx \mathbb{R}^2$ be a metric surface of locally finite $\mathcal{H}^2$. There exists a geometrically quasiconformal map from a domain $U \subset \mathbb{R}^2$ onto X if and only if X satisfies (1).





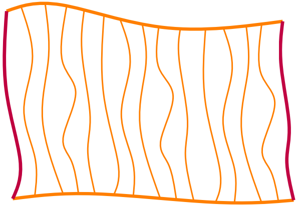
Quasiconformal uniformization

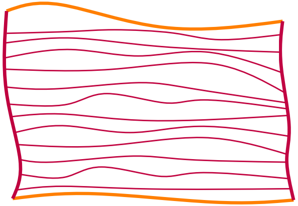
Definition: A metric surface X is **reciprocal** if X satisfies

(1) For every quadrilateral $Q \subset X$

$$\mathrm{mod}(\Gamma(Q)) \cdot \mathrm{mod}(\Gamma^*(Q)) \leq \kappa.$$

For example: $\mathbb{R}^2$ and Ahlfors 2-regular metric spaces.





Theorem [Rajala 2017]: Let $X \approx \mathbb{R}^2$ be a metric surface of locally finite $\mathcal{H}^2$. There exists a geometrically quasiconformal map from a domain $U \subset \mathbb{R}^2$ onto X if and only if X is *reciprocal*.

- X Ahlfors 2-regular and LLC: Quasiconformal maps are quasisymmetric $\Rightarrow$ recover Theorem of Bonk–Kleiner.
- In general, reciprocality condition is difficult to verify.
- There exist plenty of metric surfaces that are not reciprocal.

Quasiconformal uniformization

Example: Consider a small ball $T := \overline{B}(0, \varepsilon)$ for $0 < \varepsilon < 1$.

Let $X := \overline{D}/T$ be the **quotient space**.

- The natural projection $\pi\colon \overline{D} \to X$ is a local isometry on $\overline{D} \setminus T$.
- Assume there exists a quasiconformal parametrization

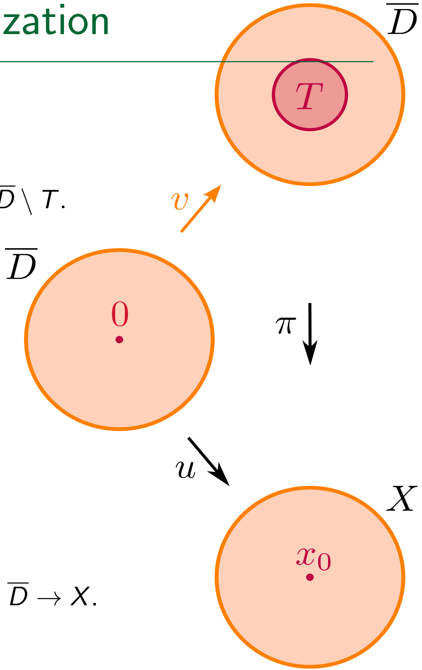
$$u\colon \overline{D} \to X \text{ with } u(0) = x_0 := \pi(T).$$

- The map $v\colon \overline{D} \setminus \{0\} \to \overline{D} \setminus T$ defined by

$$v = \pi^{-1} \circ u|_{\overline{D}\setminus\{0\}}$$

is quasiconformal. $\to$ **not possible!** (Grötzsch)

$\Rightarrow$ X does **not** possess a quasiconformal parametrization $u\colon \overline{D} \to X$.



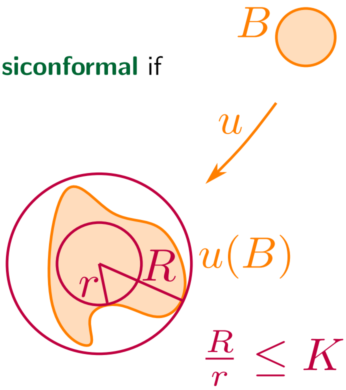
Weakly quasiconformal uniformization

Let $X \approx M$ be a compact metric surface of finite $\mathcal{H}^2$.

Definition: A continuous, surjective map $u\colon M \to X$ is **weakly quasiconformal** if

- u is a uniform limit of homeomorphisms $M \to X$, and
- there exists $K \geq 1$ s.th. for every family Γ of curves in M

$$\operatorname{mod}(\Gamma) \leq K \cdot \operatorname{mod}(u \circ \Gamma).$$



Question (Rajala–Wenger): Can X always be parametrized by a *weakly quasiconformal map* $u\colon M \to X$?

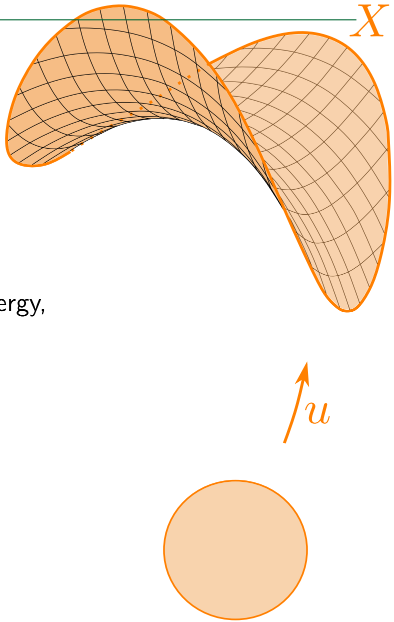
YES if X is **locally geodesic** (M.–Wenger, Ntalampekos–Romney, M.).

YES **always** (Ntalampekos–Romney).

Uniformization by minimizing energy

Uniformization by minimizing energy

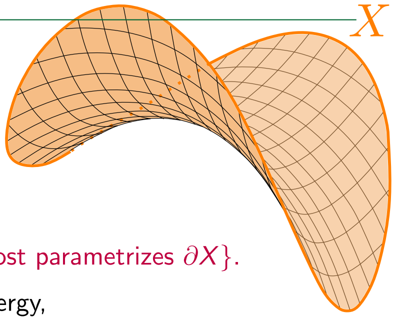
Theorem [M.–Wenger]: Let $X \approx \overline{D}$ be a locally geodesic metric surface. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$, then there exists a weakly quasiconformal map $u\colon \overline{D} \to X$.



Strategy of proof:

1. Define a class $\Lambda(X)$ of **candidates** $v\colon D \to X$.
   - $v \in \Lambda(X)$ is **regular enough** to define a notion of energy,
   - $v \in \Lambda(X)$ **spans** the metric surface X.

Uniformization by minimizing energy



Theorem [M.–Wenger]: Let $X \approx \overline{D}$ be a locally geodesic metric surface. If $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$, then there exists a weakly quasiconformal map $u\colon \overline{D} \to X$.

Strategy of proof:

1. Define $\Lambda(X) := \{v\colon D \to X \text{ Sobolev: } \operatorname{tr}(v)\colon S^1 \to X \text{ almost parametrizes } \partial X\}$.
   - $v \in \Lambda(X)$ is **regular enough** to define a notion of energy,
   - $v \in \Lambda(X)$ **spans** the metric surface X.

Metric space valued Sobolev maps: A map $v\colon D \to X$ is Sobolev if $\forall x \in X$

- postcomposition v_x with distance function $d(\cdot, x)$ is in $W^{1,2}(D)$,
- $\exists\ h \in L^2(D)$ s.th. $|\nabla v_x| \leq h$ a.e. on D.

$$
\begin{tikzcd}
D \arrow[r, "v"] \arrow[dr, "v_x"'] & X \arrow[d, "{d(\cdot, x)}"] \\
& \mathbb{R}
\end{tikzcd}
$$

Reshetnyak energy: $E_+^2(v) := \inf\left\{\|h\|_{L^2(D)}^2 : h \text{ as above}\right\}$.

Uniformization by minimizing energy

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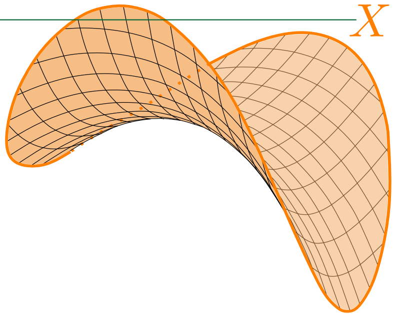
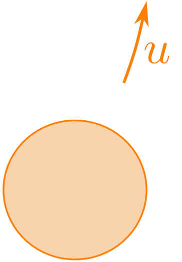


Strategy of proof:

1. Define a class $\Lambda(X)$ of **candidates** $v\colon \overline{D} \to X$.
2. Show that $\Lambda(X)$ is **not empty**. $\rightarrow$ highly non-trivial
   - X might contain a purely 2-unrectifiable part that is dense in X.
   - In general, $\exists$ only few Lipschitz maps from open subsets of D to X.



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1. Define a class $\Lambda(X)$ of **candidates** $v\colon \overline{D} \to X$.
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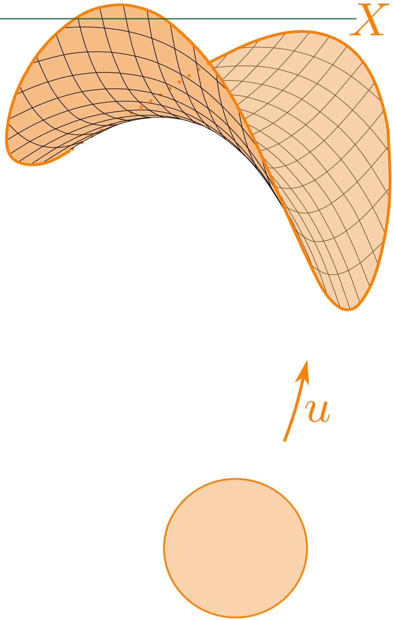
3. Show that there exists an **energy minimizer** $u \in \Lambda(X)$.
   - Direct variational method.



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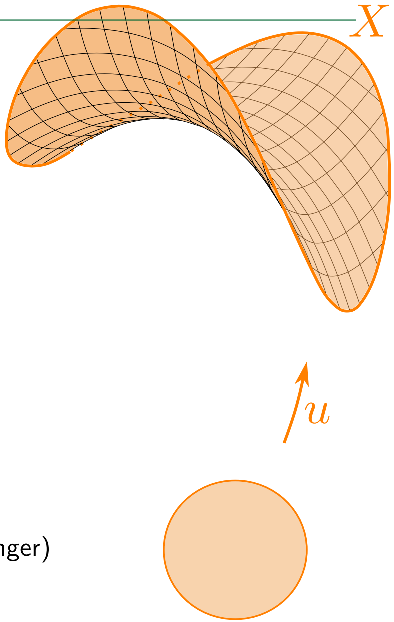
2. Show that $\Lambda(X)$ is **not empty**. $\rightarrow$ highly non-trivial
3. Show that there exists an **energy minimizer** $u \in \Lambda(X)$.
4. Use the fact that u is energy minimizing to derive further **regularity** and **distortion** properties of u.
   - Show that u has a continuous representative $\bar{u}$.
   - $\bar{u}$ is uniform limit of homeomorphisms. (Lytchak–Wenger)
   - $\bar{u}$ has desired distortion property. (Lytchak–Wenger)



Existence of Sobolev maps

Let $X \approx \overline{D}$ be locally geodesic, $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$.

Goal: Construct $v \in \Lambda(X)$ as limit of Lipschitz mappings

$$v_n \colon \overline{D} \to N_{1/n}(X) \subset E(X)$$

of uniformly bounded area and $v_n|_{S^1}$ parametrizing ∂X.

Area formula for Lipschitz maps:

$$\mathrm{Area}(v_n) = \int_{E(X)} \underbrace{|v_n^{-1}\{x\}|}_{\text{multiplicity of } x} \, d\mathcal{H}^2(x)$$

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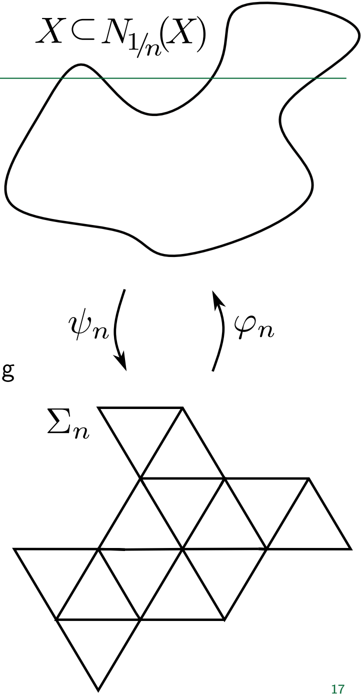
Idea: Factorize through a 2-dim simplicial complex Σ_n consisting of Euclidean cells of sidelength r/n to construct v_n.

- There exist Lipschitz maps

  $$\psi_n \colon X \to \Sigma_n \quad \text{and} \quad \varphi_n \colon \Sigma_n \to N_{1/n}(X) \subset E(X)$$

  that are almost inverse to each other.

  (Jørgensen-Lang, Basso-Wenger-Young)



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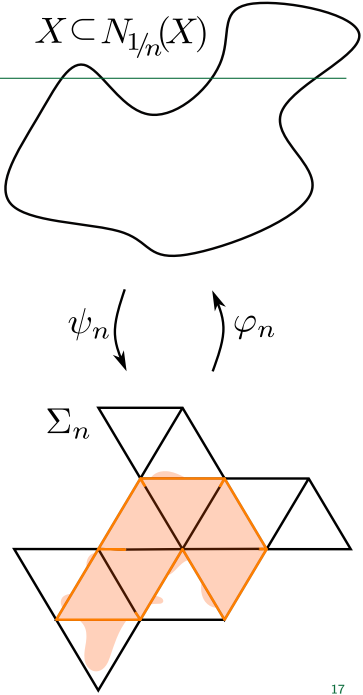
of uniformly bounded area and $v_n|_{S^1}$ parametrizing ∂X.

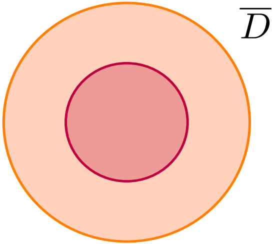
1. Construct a **continuous** map $\varrho_n \colon \overline{D} \to \Sigma_n$ of **small "area"** such that $\varrho_n|_{S^1} \colon S^1 \to \Sigma_n^{(1)}$ is Lipschitz and close to $\psi_n(\partial X)$.

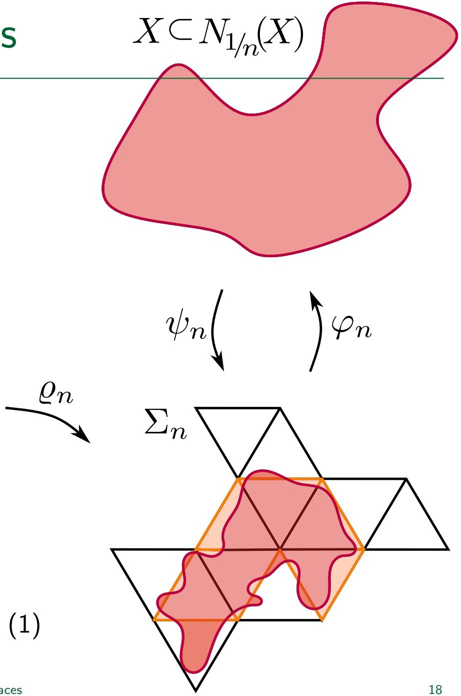


Existence of Sobolev maps

1) Construct **continuous** ϱ_n of small "area":

- Let $\eta \colon \overline{D} \to X$ be homeomorphism extending a constant speed parametrization of ∂X. *(Jordan-Schoenflies)*
- "Push" $\psi_n \circ \eta|_{S^1}$ to 1-skeleton $\Sigma_n^{(1)}$ by a Lipschitz homotopy H of small area.
- ϱ_n obtained by gluing $\psi_n \circ \eta$ and H and reparametrizing satisfies





$$\int_{\Sigma_n} \left|\varrho_n^{-1}\{z\}\right| \, d\mathcal{H}^2(z) \leq C(\mathcal{H}^2(X) + \ell(\partial X)). \quad (1)$$

Existence of Sobolev maps

Let $X \approx \overline{D}$ be locally geodesic, $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$.

Goal: Construct $v \in \Lambda(X)$ as limit of Lipschitz mappings

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of <u>uniformly bounded area</u> and $v_n|_{S^1}$ parametrizing ∂X.

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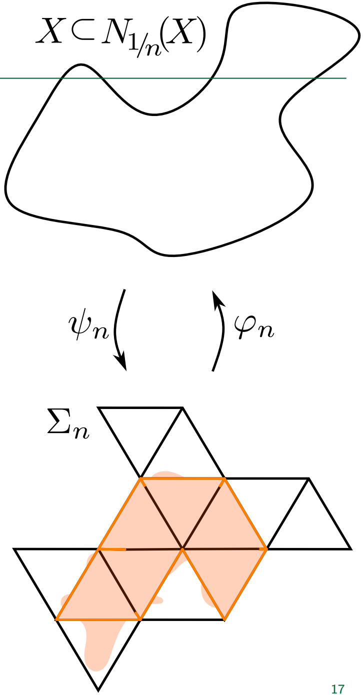
2. Transform ϱ_n into a **Lipschitz** map $\overline{\varrho_n} \colon \overline{D} \to \Sigma_n$.



Existence of Sobolev maps

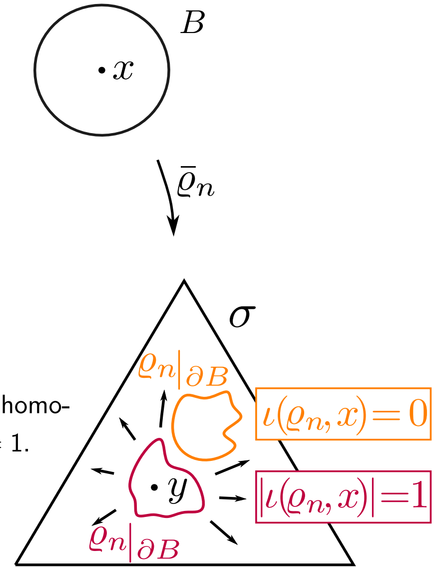
2) Make ϱ_n **Lipschitz:**

- For every 2-cell σ in Σ_n choose $y \in \mathrm{int}(\sigma)$ with

$$\underbrace{\left|\varrho_n^{-1}\{y\}\right|}_{\text{multiplicity of } y} \leq \underbrace{\frac{1}{|\sigma|_2} \int_\sigma \left|\varrho_n^{-1}\{z\}\right| \, d\mathcal{H}^2(z)}_{\text{average multiplicity in } \sigma}$$

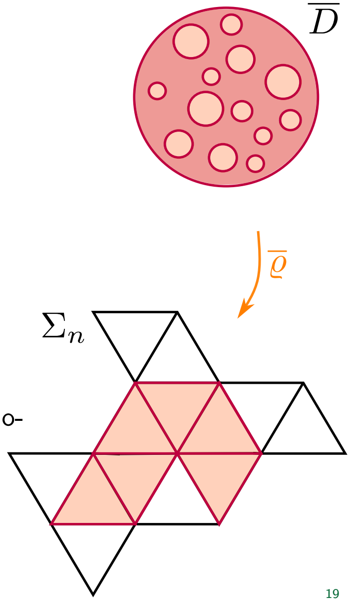
and $\underbrace{|\iota(\varrho_n, x)|}_{\text{winding number}} \leq 1$ for any $x \in \varrho_n^{-1}(y)$ (Radó).

- Define $\overline{\varrho_n}$ on small balls B such that
  - $\varrho_n|_B$ is constant with image in $\partial\sigma$ if $\iota(\varrho, x) = 0$,
  - $\overline{\varrho_n}|_B$ is a biLipschitz homeomorphism and $\overline{\varrho_n}|_{\partial B}$ is homotopic to the projection of $\varrho_n|_{\partial B}$ to $\partial\sigma$ if $|\iota(\varrho, x)| = 1$.



Existence of Sobolev maps

2) Make ϱ_n **Lipschitz:**



- For every 2-cell σ in Σ_n choose $y \in \operatorname{int}(\sigma)$ with

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- Use extension properties of $\Sigma_n^{(1)}$ to extend $\overline{\varrho_n}|_{\bigcup B \cup S^1}$ to Lipschitz map $\overline{\varrho_n}\colon \overline{D} \to \Sigma_n$ satisfying (1).

Existence of Sobolev maps

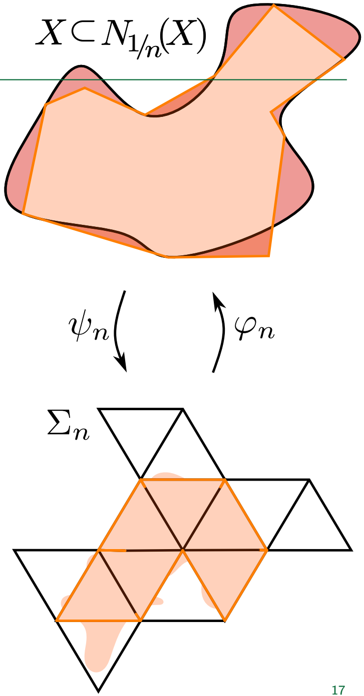
Let $X \approx \overline{D}$ be locally geodesic, $\mathcal{H}^2(X) < \infty$ and $\ell(\partial X) < \infty$.

Proposition: For every $n \in \mathbb{N}$ there exists a Lipschitz map

$$v_n \colon \overline{D} \to N_{1/n}(X) \subset E(X)$$

of <u>uniformly bounded area</u> and $v_n|_{S^1}$ parametrizing ∂X.

1. Construct a **continuous** map $\varrho_n \colon \overline{D} \to \Sigma_n$ of **small "area"** such that $\varrho_n|_{S^1} \colon S^1 \to \Sigma_n^{(1)}$ is Lipschitz and close to $\psi_n(\partial X)$.
2. Transform ϱ_n into a **Lipschitz** map $\overline{\varrho_n} \colon \overline{D} \to \Sigma_n$.
3. Use extension properties of $N_{1/n}(X) \subset E(X)$ to change $\varphi_n \circ \overline{\varrho_n}$ into the desired Lipschitz map v_n.



Applications

Applications of weakly quasiconformal uniformization

Let $f\colon X \to Y$ be a Sobolev map between metric surfaces X and Y of locally finite area.

- Without more assumptions on X and/or Y, there is no notion of derivative of f.

General idea: Let $u\colon M \to X$ be the weakly quasiconformal uniformization map, where M is a smooth surface.

- $u \in W^{1,2}(M, X)$ and $h = f \circ u \in W^{1,2}(M, Y)$.
- We can "differentiate" u and h.
- Allows to prove statements about f.

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ M \arrow[u, "u"] \arrow[ur, "h=f\circ u"'] & \end{tikzcd}$$

Lipschitz-volume rigidity

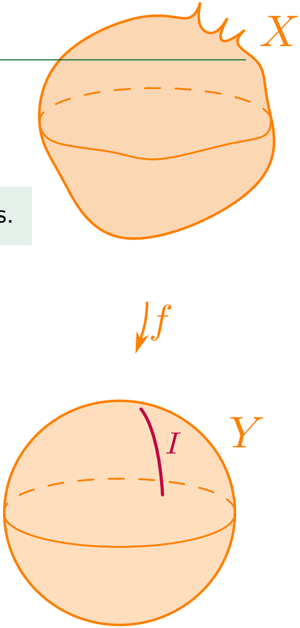
Question: Let $f\colon X \to Y$ be a 1-Lipschitz and surjective map between metric spaces that have the same volume. Is f an isometry?

Theorem [Folklore]: YES, if X and Y are closed Riemannian n-manifolds.

- Proofs by (Burago–Ivanov) and (Besson–Courtois–Gallot).

Question: Does the same hold for non-smooth metric surfaces?

NO, for example $\pi\colon \underbrace{S^2}_{=X} \to \underbrace{S^2/I}_{=Y}.$



Lipschitz-volume rigidity

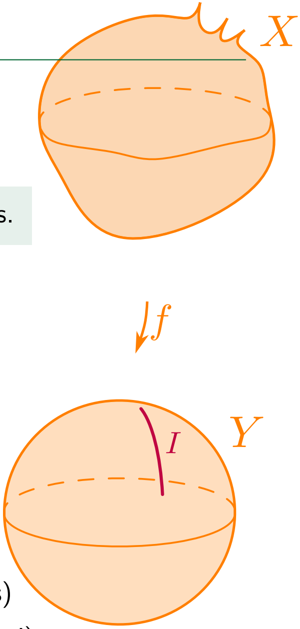
Question: Let $f : X \to Y$ be a 1-Lipschitz and surjective map between metric spaces that have the same volume. Is f an isometry?

Theorem [Folklore]: YES, if X and Y are closed Riemannian n-manifolds.

- Proofs by (Burago–Ivanov) and (Besson–Courtois–Gallot).

Theorem [M.–Ntalampekos 2024, Basso–Marti–Wenger 2024]: Let X be a closed metric surface and Y a closed Riemannian surface with $\mathcal{H}^2(X) = \mathcal{H}^2(Y)$. Then every 1-Lipschitz and surjective map $f : X \to Y$ is an isometry.

- Proof highly depends on weakly quasiconformal uniformization.
- Intermediate results depending on regularity of Y. (M.-Ntalampekos)
- Higher dimensional variant under additional assumptions on X. (Marti)



Application: Lipschitz-volume rigidity

Let X, Y be metric surfaces with $\mathcal{H}^2(X) = \mathcal{H}^2(Y) < \infty$.
Let $f\colon X \to Y$ be 1-Lipschitz and surjective.

- f is *area-preserving*, i.e. $\mathcal{H}^2_X(A) = \mathcal{H}^2_Y(f(A))$ for every $A \subset X$.

Theorem [M.–Ntalampekos 2024]:

X	Y	f	Conclusions about f
Reciprocal	-	(1-)Lip.	(1-)BLD on a.e. curve
-	Reciprocal	(1-)Lip.	(1-)QC homeom., and
			(1-)BLD on a.e. curve
-	Upper regular	Lip.	QC homeom., BLD
-	Riemannian	1-Lip.	Isometry

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ M \arrow[u, "u"] \arrow[ur, "h=f\circ u"'] & \end{tikzcd}$$

f is of **bounded length distortion (BLD)** (along a.e. curve) if $\exists K \geq 1$ s.th.

$$K^{-1} \cdot \ell(\gamma) \leq \ell(f \circ \gamma) \leq K \cdot \ell(\gamma) \quad \text{for (a.e.) curve } \gamma \text{ in } X.$$