Turán numbers of sunflowers

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ETH Zürich

joint work with Matija Bucić and Benny Sudakov

Slides based on a deck by Matija Bucić.

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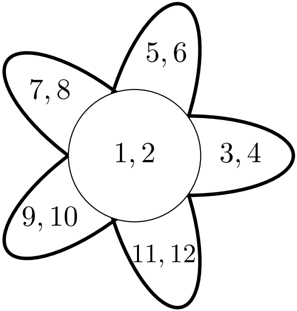

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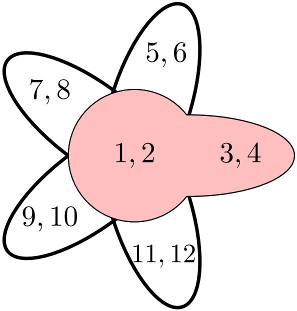

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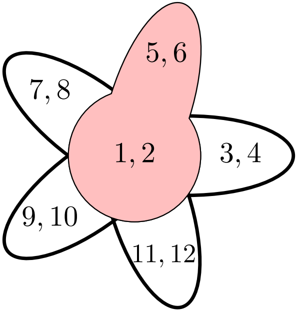

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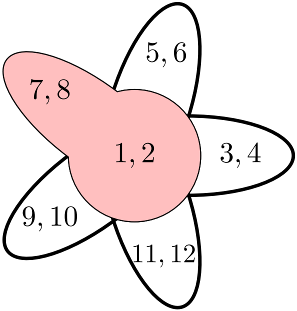

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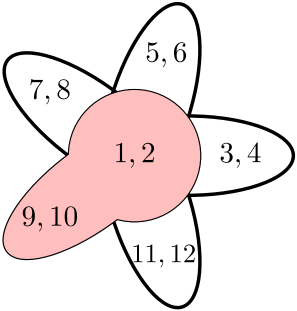

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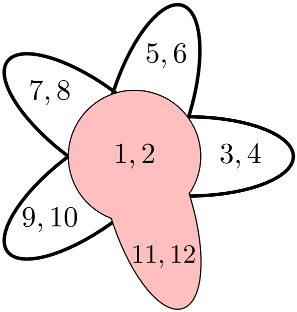

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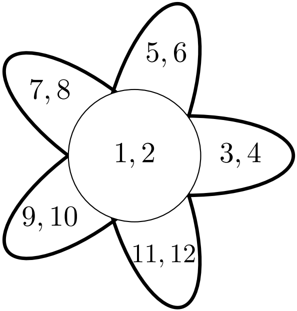

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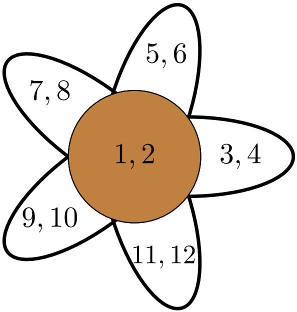

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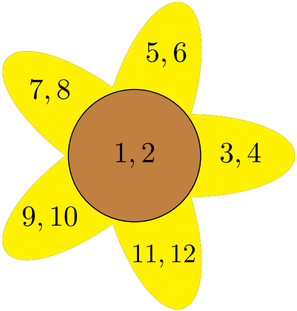

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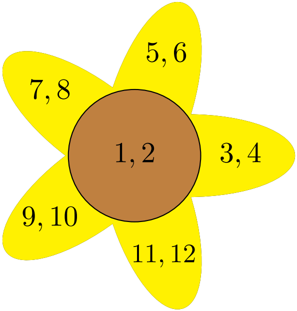

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- r*-uniform* if all sets have size r.

Erdős-Rado sunflower conjecture

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What is the max size of a family of r-sets without a k petal sunflower?

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For fixed k,

$$f_r(k) \leq C_k^r.$$

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- Relations to many topics in computer science and probability theory.

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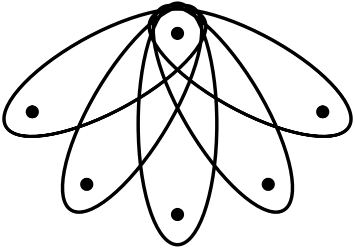

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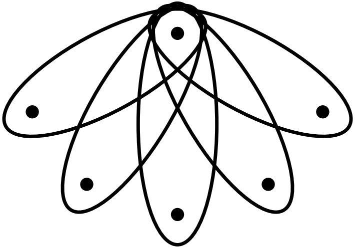

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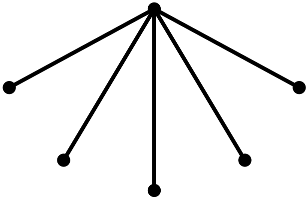

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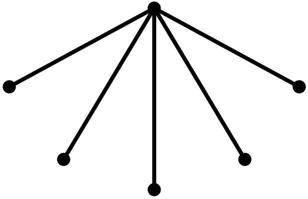

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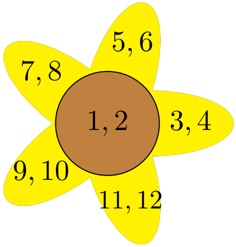

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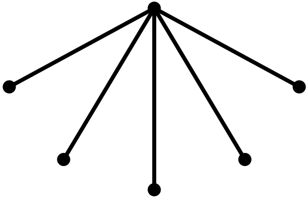

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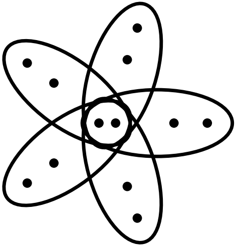

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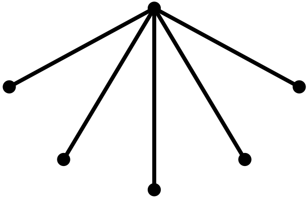

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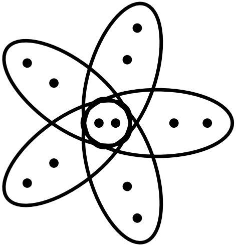

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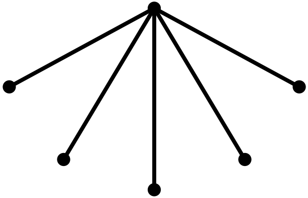

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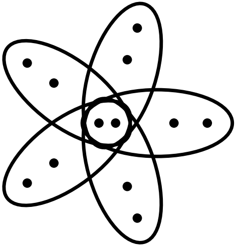

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- The $r = 4$ case solved approximately by Bucić, Draganić, Sudakov, Tran.

Main result

Theorem (B., Bucić. and Sudakov)

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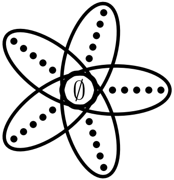

$\text{ex}(n, \mathcal{S}_0^{(5)}(k)) \approx n^4 k$

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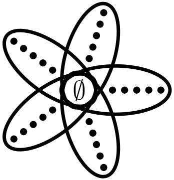
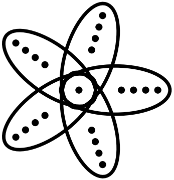
$$\text{ex}(n, \mathcal{S}_t^{(r)}(k)) \approx_r \begin{cases} n^{r-t-1}k^{t+1} & \textit{if } t \le \frac{r-1}{2}, \\ n^t k^{r-t} & \textit{if } t > \frac{r-1}{2}. \end{cases}$$



$\text{ex}(n, \mathcal{S}_0^{(5)}(k)) \approx n^4 k$



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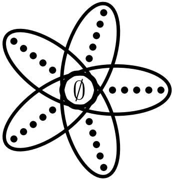

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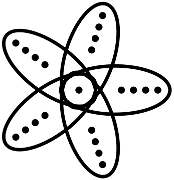

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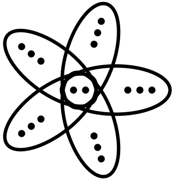

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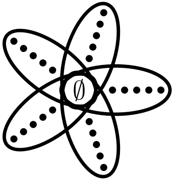

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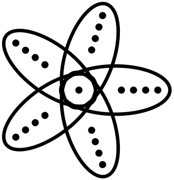

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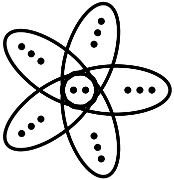

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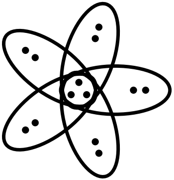

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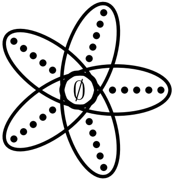

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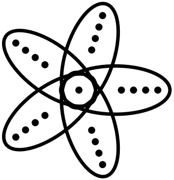

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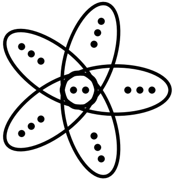

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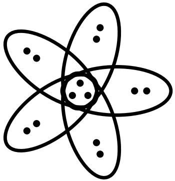

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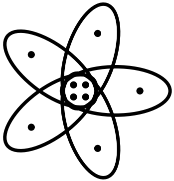

$\mathrm{ex}(n, \mathcal{S}_4^{(5)}(k)) \approx n^4 k$

Upper bounds: overview

Theorem (B., Bucić and Sudakov)

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We use induction to reduce to the *balanced* case:

$$\text{ex}(n, \mathcal{S}_t^{(2t+1)}(k)) \leq O(n^t k^{t+1}).$$

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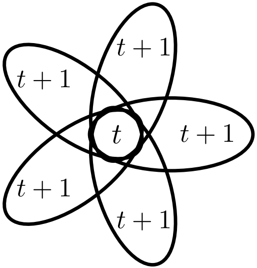
A balanced sunflower:



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Upper bounds: ex$(n, \mathcal{S}_t^{(2t+1)}(k)) = O(n^t k^{t+1})$

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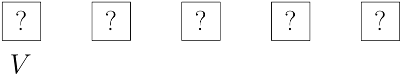

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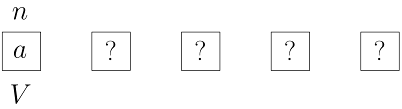

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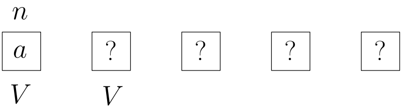

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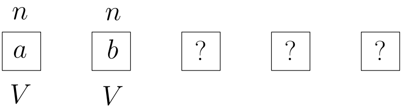

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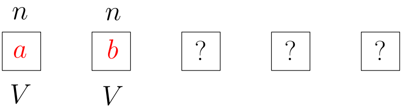

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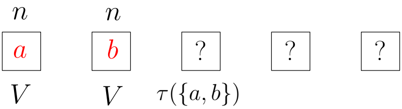

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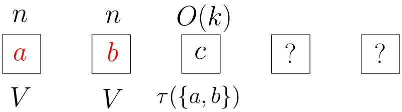

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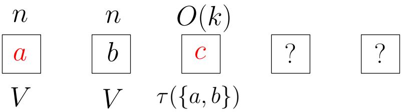

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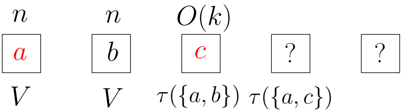

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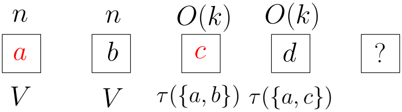

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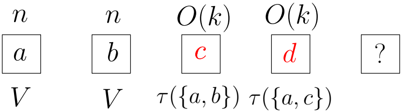

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- For every set S of t vertices, fix such a cover $\tau(S)$.
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a	b	c	d	?
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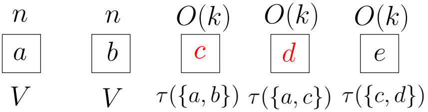

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Our list has $O(n^t k^{t+1})$ $(2t+1)$-tuples, but does it contain all edges?

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So we cannot reach $X = e$ with this start. However, maybe if we started with e.g. $a = v_3, b = v_4 \ldots$

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A $(t+1,t)$ system on $[2t+1]$ does not exist so we counted all edges.

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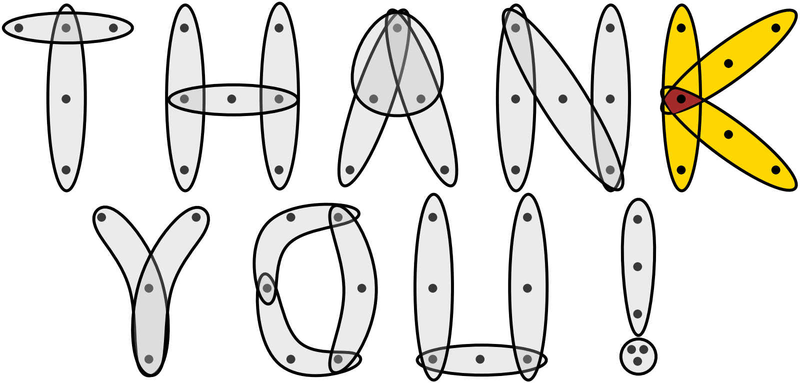
- Known for $r \leq 4$, up to constant factor.


THANK YOU!