The Turán number of the grid

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joint work with Oliver Janzer, Benny Sudakov and István Tomon

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- Poorly understood for general bipartite graphs, e.g. not known for $C_8, K_{4,4}, Q_3$.

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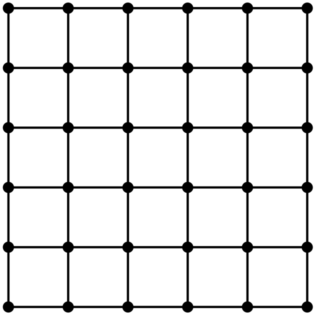
- Even $r = 2$ is open and interesting.
- Alon, Krivelevich, Sudakov 2003: ex$(n, F) = O(n^{2-\frac{1}{4r}})$.

Our result

Theorem (B., Janzer, Sudakov, Tomon 2022+)

For fixed $t \geq 2$, the $t \times t$ grid F_t satisfies ex$(n, F_t) = \Theta_t(n^{3/2})$.

F_6



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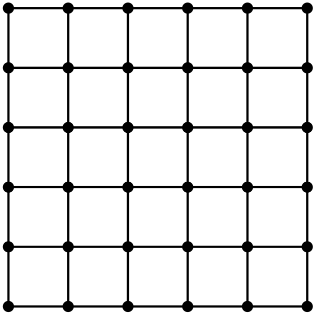

F_t contains a 4-cycle, so we only need to prove ex$(n, F_t) = O(n^{3/2})$.

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- Kim, Lee and Lee proved Sidorenko's conjecture for grids in arbitrary dimension.
- Füredi and Ruszinkó studied an extremal problem for a certain hypergraph grid graph.

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2. Get rid of the polylog factor using the tensor power trick.

Proof sketch of $\text{ex}(n, F_t) = O(n^{3/2}(\log n)^t)$

Setting:

- G is almost regular, i.e. $\Delta(G)/\delta(G) = O(1)$ with average degree d, where $d = \alpha n^{1/2}$ and $\alpha = (\log n)^t$.

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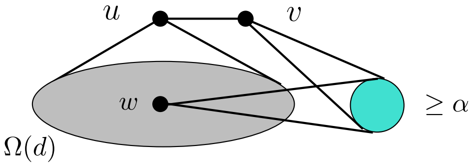

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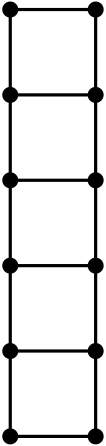

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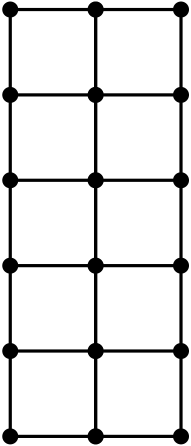

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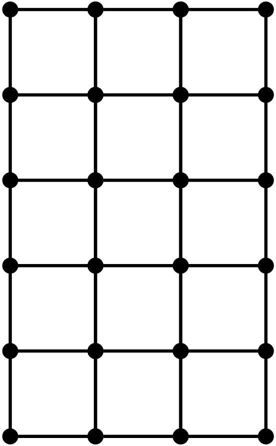

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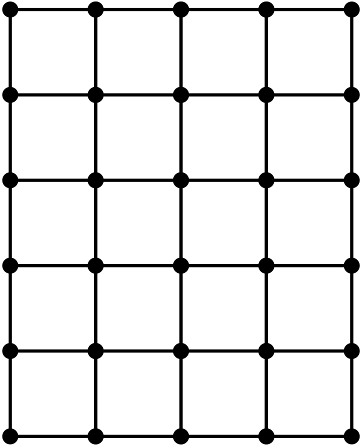

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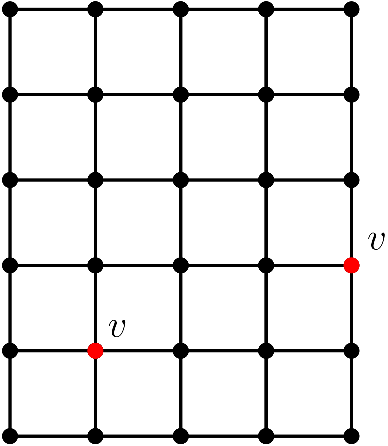

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- Find many *ladders*, i.e. $2 \times t$ grids.
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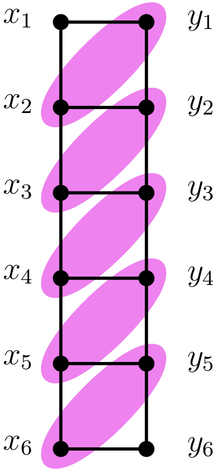

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Claim

We can find integers $s_1, s_2, \ldots s_{t-1} \geq \alpha$ *s.t. there are* $\Omega\left(\frac{nd^t \prod_{i=1}^{t-1} s_i}{(\log n)^{t-1}}\right)$ *ladders with* $d_i \in [s_i/2, s_i]$, $i \in [t-1]$.

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- Let $\mathcal{H}' \subseteq \mathcal{H}$ satisfy

$$\delta(\mathcal{H}') \geq \bar{d}(\mathcal{H})/2 \geq \frac{|\mathcal{F}|}{O(nd^{t-1})} \geq \Omega\left(\frac{d \prod_{i=1}^{t-1} s_i}{(\log n)^{t-1}}\right).$$

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- Enough to show: for a $(t-1)$-path P and a fixed vertex $v \in V(G)$, the number of P' with $(P, P') \in E(\mathcal{H})$ containing v is $o(\delta(\mathcal{H}'))$.

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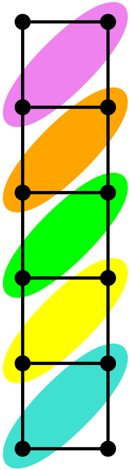

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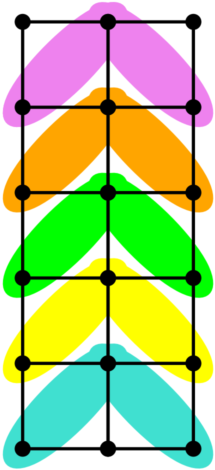

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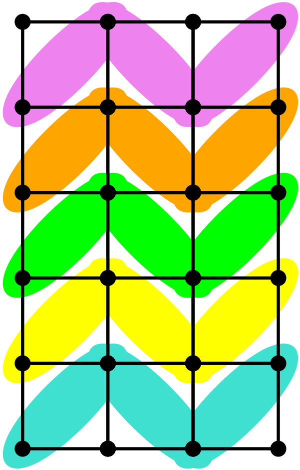

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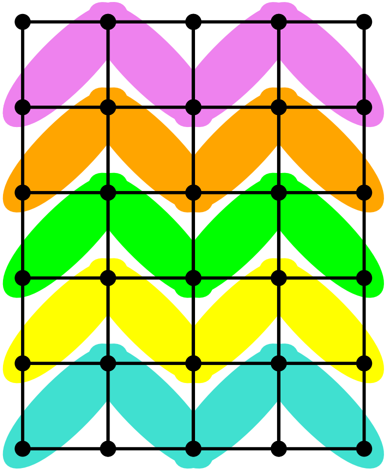

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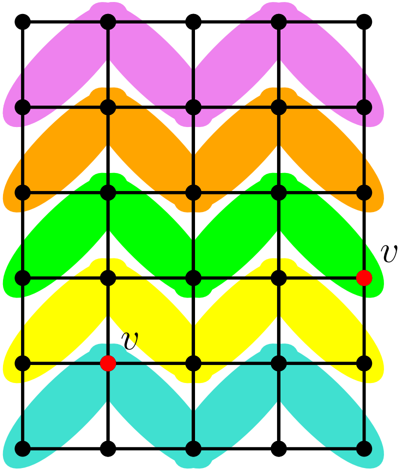

The number of extensions containing a fixed vertex

Want: for a path $P = (x_1, \ldots, x_t)$ and $v \in V(G)$, the number of P' with $(P, P') \in E(\mathcal{H})$ containing v is $o\left(\frac{d \prod_{i=1}^{t-1} s_i}{(\log n)^{t-1}}\right)$.

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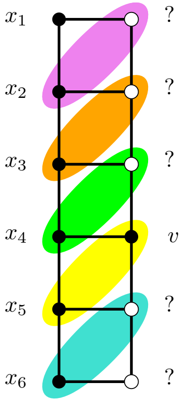

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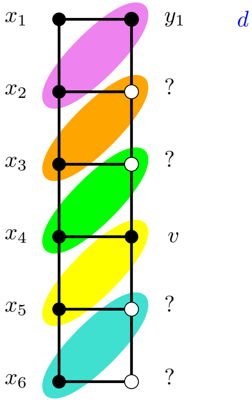

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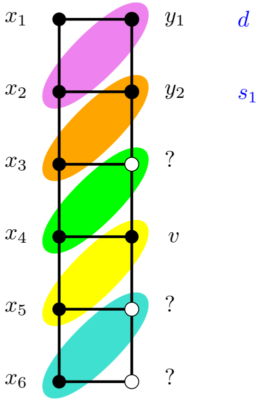

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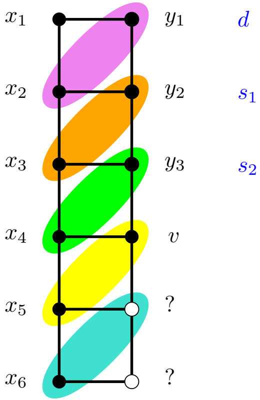

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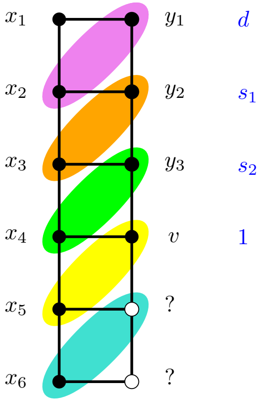

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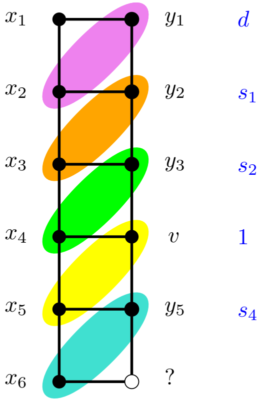

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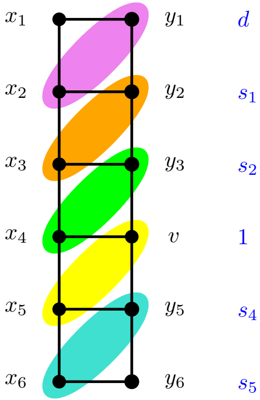

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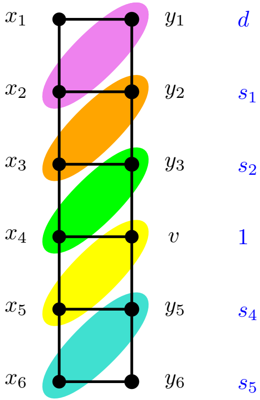

In total: $\frac{d \prod_{i=1}^{t-1} s_i}{s_j} = o\left(\frac{d \prod_{i=1}^{t-1} s_i}{(\log n)^{t-1}}\right)$ since $s_j \geq \alpha = (\log n)^t$.

Finding many ladders

Claim

$\exists s_1, \dots s_{t-1} \geq \alpha$ *s.t. there are* $\Omega\left(\frac{nd^t \prod_{i=1}^{t-1} s_i}{(\log n)^{t-1}}\right)$ *ladders with* $d_i \in [s_i/2, s_i]$.

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Proof. Consider intervals $[1, 2], [2, 4], [4, 8] \dots$ and take the best one.

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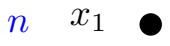

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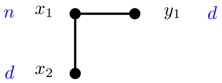

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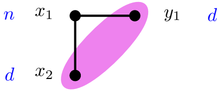

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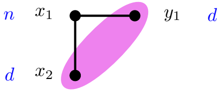

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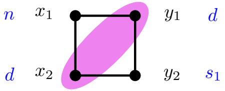

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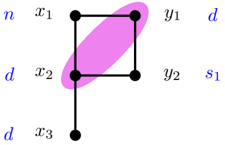

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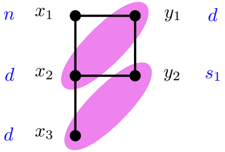

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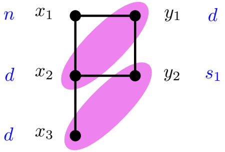

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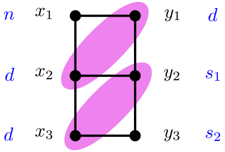

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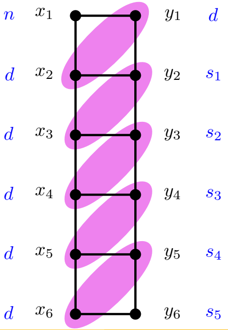

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$d_5 = d(x_6, y_5) \in [s_5/2, s_5]$ $\quad \frac{1}{\log n}$

Using the tensor power trick

The k^{th} tensor power G^k of a graph G is the graph with vertices $(v_1, \ldots, v_k) \in V(G)^k$ and where $(v_1, \ldots, v_k) \sim (u_1, \ldots, u_k)$ iff $v_i \sim u_i, \forall i \in [k]$.

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- When finding extensions, we make sure not to ruin too many coordinates.

Concluding remarks

Theorem (B., Janzer, Sudakov, Tomon 2022+)

Let P and T be a path and a tree with at least one edge each. Then $\text{ex}(n, P\square T) = \Theta(n^{3/2})$.

Open problems:

- Prove $\text{ex}(T_1\square T_2) = O(n^{3/2})$ for any two trees T_1, T_2.
- Prove $\text{ex}(n, F_t^{(d)}) = O(n^{2-1/d})$ for the d-dimensional grid $F_t^{(d)}$.
- Determine the correct dependence of $\text{ex}(n, F_t)$ on t. We can show $ct^{1/2}n^{3/2} \leq \text{ex}(n, F_t) \leq e^{O(t^5)}n^{3/2}$.
- Erdős' conjecture.


THANK
YOU