Robust Algorithms for the Secretary Problem

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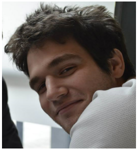

Goran Zuzic

Slides based on a deck by Goran Zuzic.

Classical Secretary Model

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- Items have positive real values.
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Probability maximization:

- Items only have a relative order.
- Objective: probability of choosing the maximum element.

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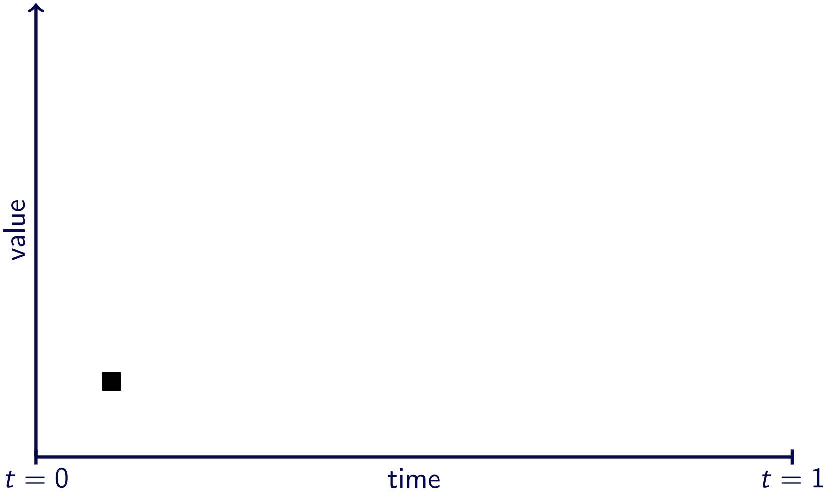

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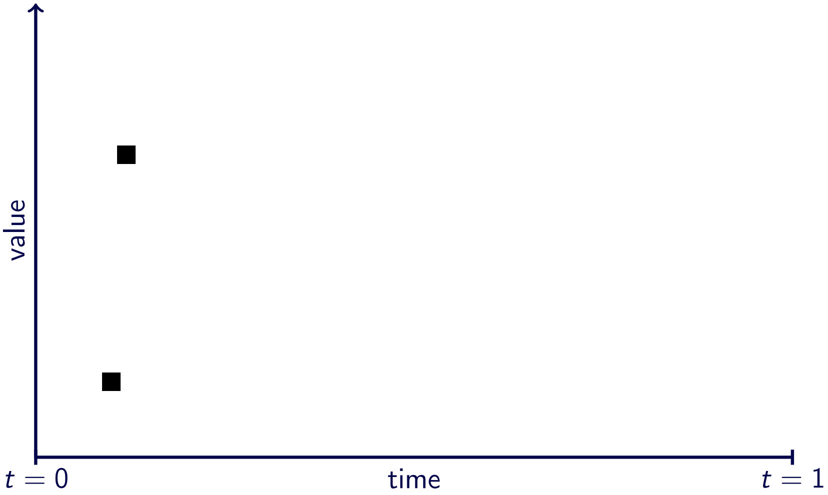

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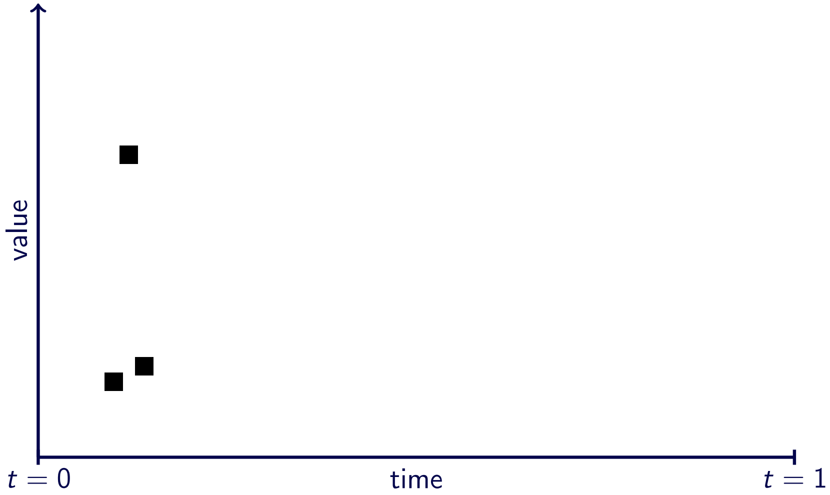

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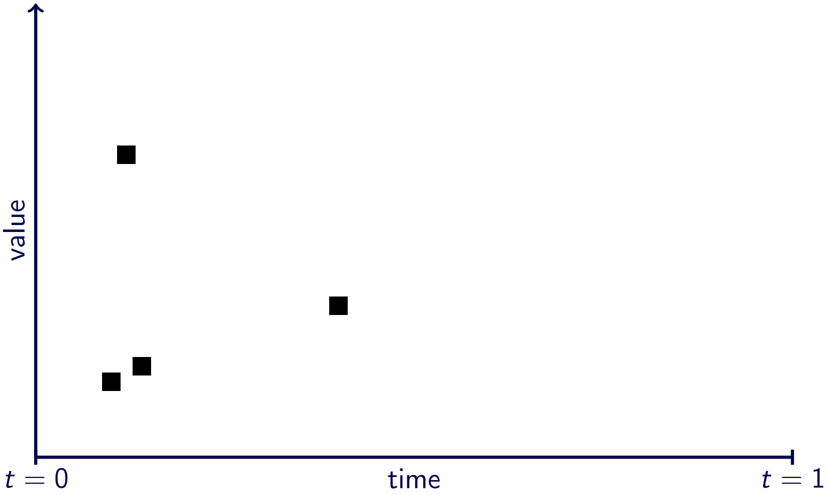

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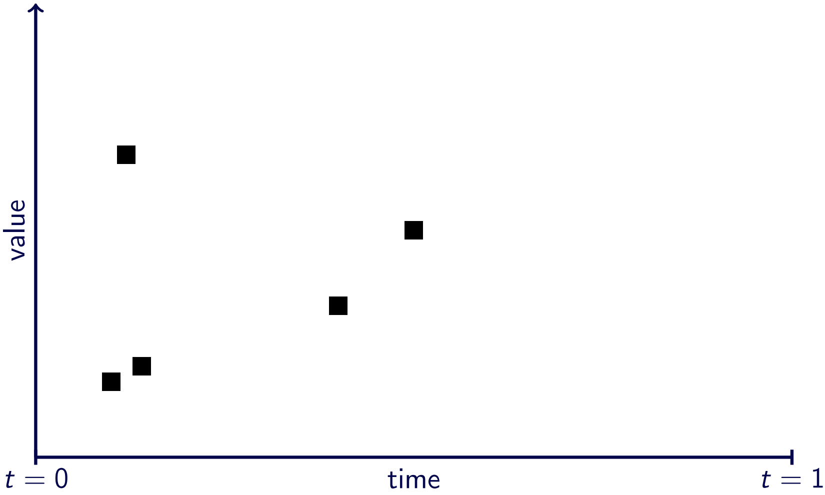

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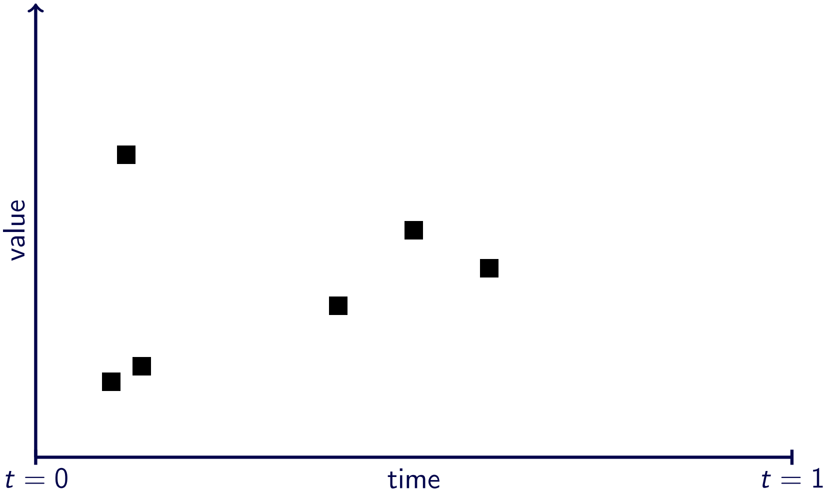

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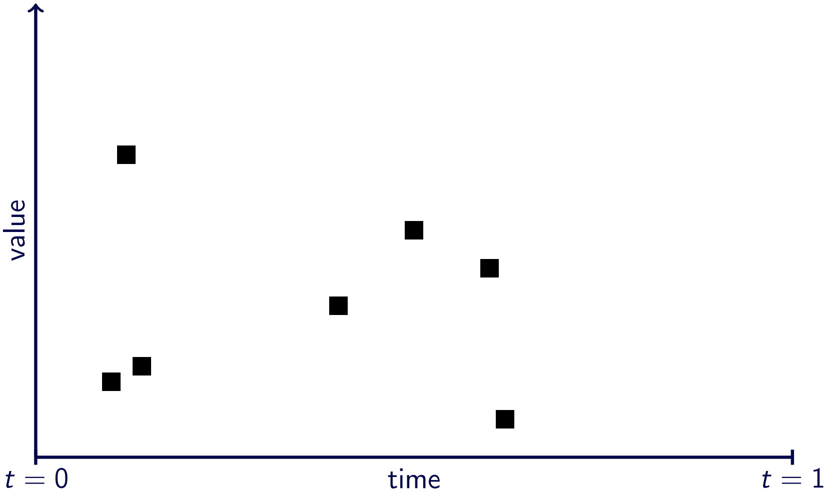

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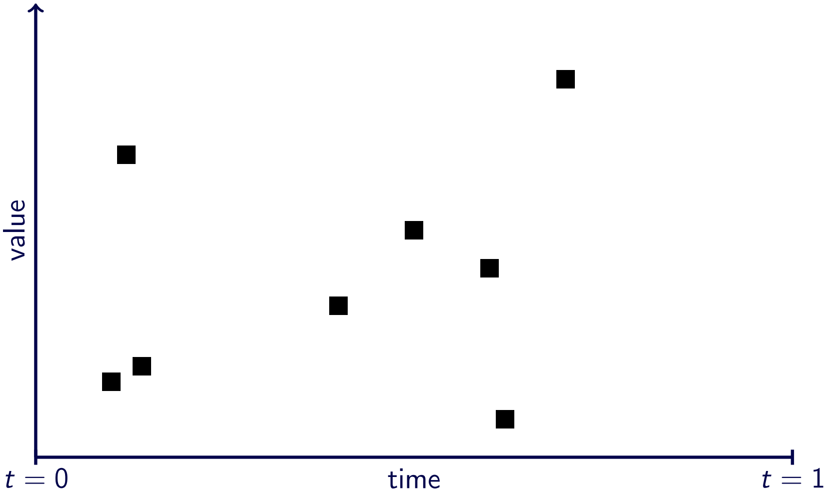

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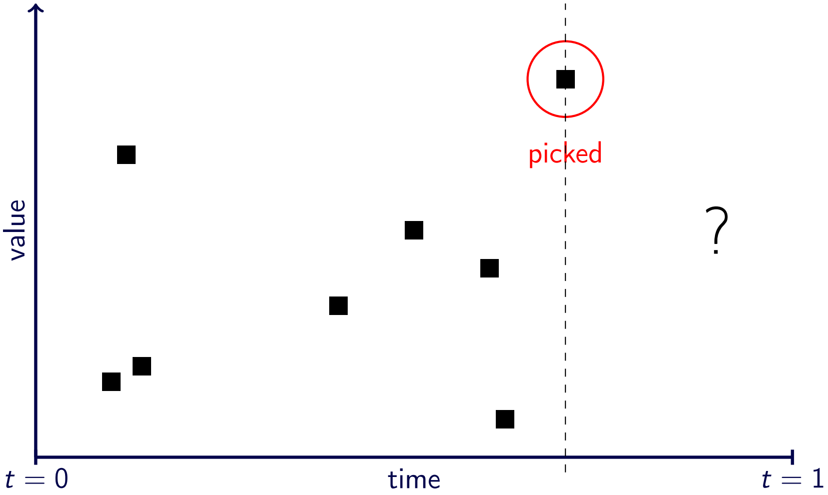

Theorem

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Algorithm:

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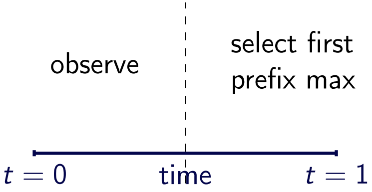
Algorithm:

- Do not pick anything until time $t = \frac{1}{2}$.
- Accept the first item that is larger than everything seen so far.

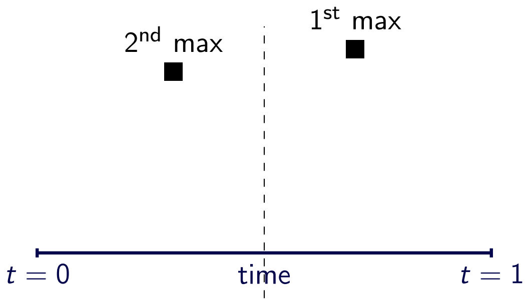


Classical Secretary Model

Proof of Dynkin's algorithm

Theorem

In the classic secretary setting there is an algorithm that gets the highest bid with probability at least $1/4$.



$$\Pr[\text{we select } 1^{st} \text{ max}] \geq \Pr[2^{\text{nd}} \text{ max in left half}] \cdot \Pr[1^{\text{st}} \text{ max in right half}]$$
$$= 1/2 \cdot 1/2 = 1/4$$

Classical secretary model:

- The $1/4$ can be improved to $1/e \approx 0.37$. [Dynkin'63]
- Simple model.
- Lots of generalizations: choosing multiple items, choosing matroid-independent elements, ...
- Application: online ad auctions.

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- We assume that all the elements are perfectly uniform.
- Even <u>one</u> non-uniform element can mess up everything.

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- g green items arriving uniformly random in $[0, 1]$
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Green max is unattainable.

Results

Value maximization:

Theorem (BGSZ ITCS '20)

Single-item RobSec admits $\mathbb{E}[\mathrm{ALG}] \geq \frac{1}{O(\log^* n)^2}\mathbb{E}[2^{nd}\ \textit{green}\ \max]$.

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Knapsack RobSec admits $\mathbb{E}[\mathrm{ALG}] \geq (1-\varepsilon)\mathbb{E}[\mathrm{OPT}]$ *when* $\frac{\textit{knapsack size}}{\textit{item size}} \geq \mathrm{poly}(\varepsilon^{-1}\log n)$

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No dependency on r (number of red items). This is very robust!

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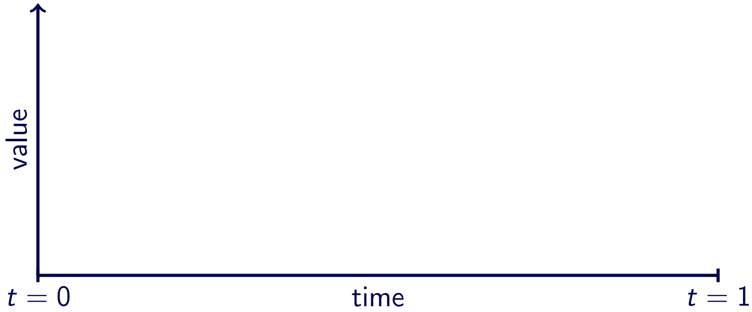

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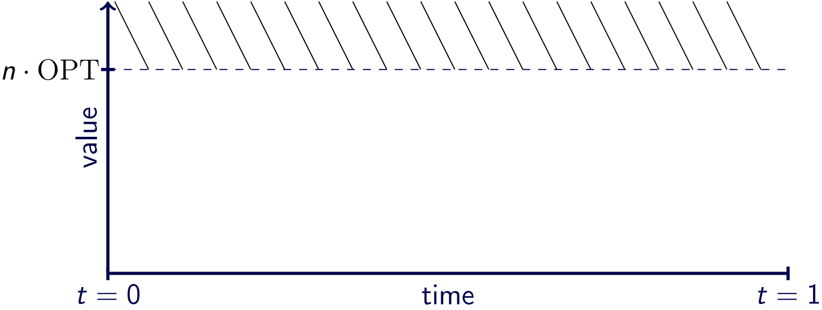

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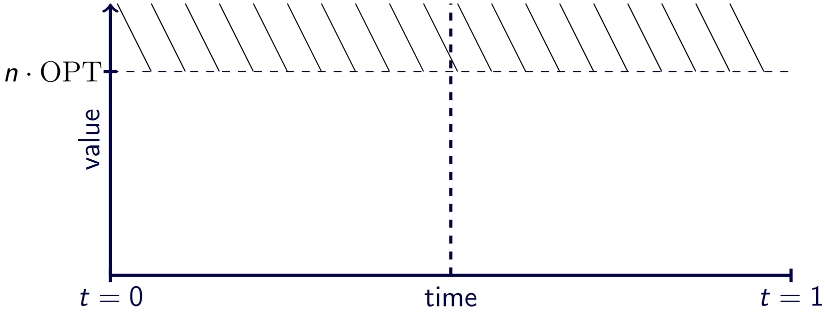

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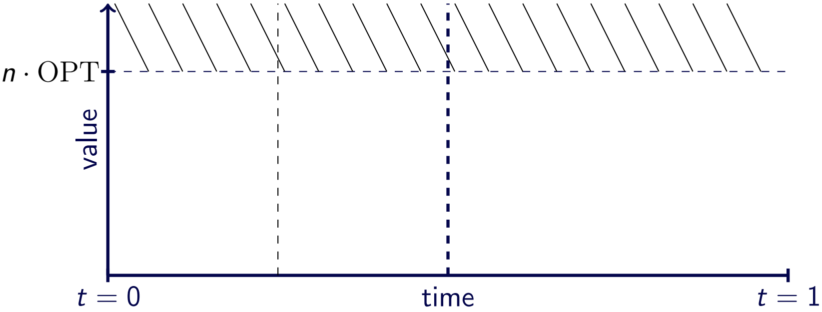

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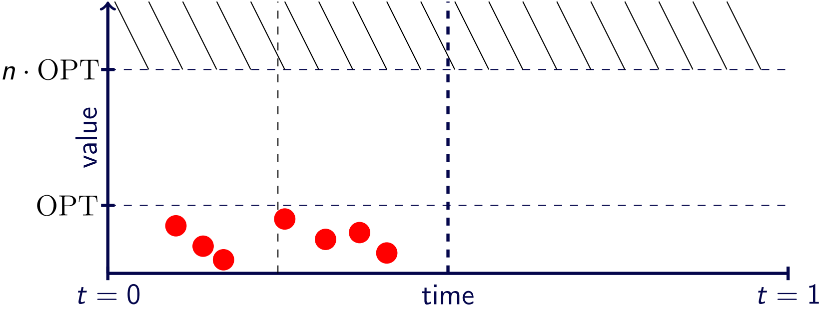

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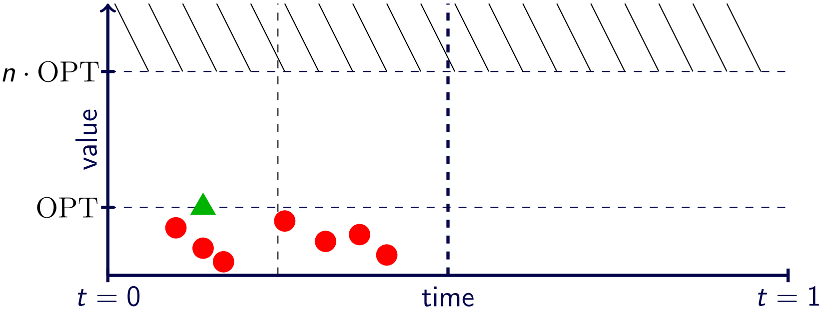

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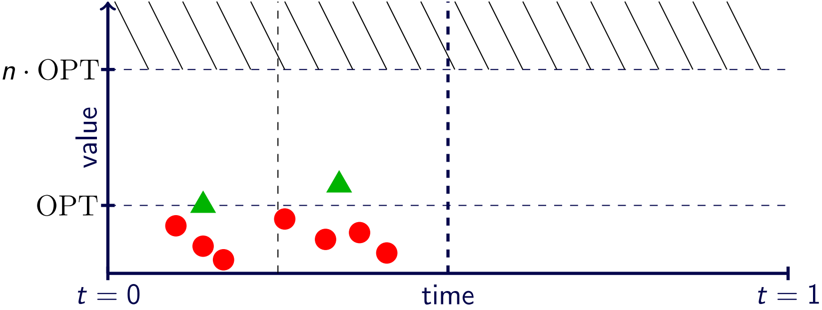

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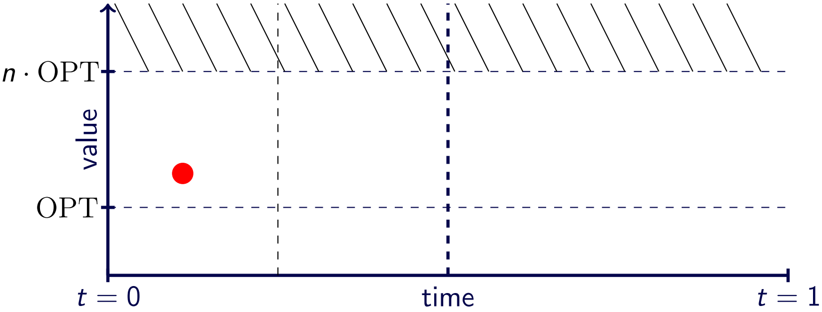

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  $\longrightarrow$ max in 1^{st} half $\in [\mathrm{OPT}, n \cdot \mathrm{OPT}]$

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$$\mathbb{E}[\mathrm{ALG}] \geq \Pr[\text{correct bucket}] \cdot \Pr[2^{nd} \text{ max in } 2^{nd} \text{ half}] \cdot \frac{OPT}{2}$$

$$= \frac{1}{\log n} \cdot \frac{1}{2} \cdot \frac{OPT}{2} = \frac{1}{O(\log n)} \cdot \mathrm{OPT}$$

Improving to $1/O(\log^* n)^C$-competitive ALG

Tool #4: Done or refine.

Observation:

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Idea: partition $[0, 1]$ into $O(\log^* n)$ equal intervals. In each interval we either:

- get high value
- or refine our estimate of OPT.

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