MACKEY CONSTRAINTS FOR JAMES'S COMPACTNESS THEOREM AND RISK MEASURES

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ABSTRACT. *Let A be a closed, convex, bounded and non weakly compact subset of a Banach space E with $0 \notin A$. Let us fix a convex and weakly compact subset W of E, a functional $z_0^* \in E^*$ with $z_0^*(A) > 0$, and $\epsilon > 0$. Then there is a linear form $x_0^* \in B_{pW}(z_0^*, \epsilon)$, i.e. $x_0^* \in E^*$ and*

$$\sup_{w \in W} |x_0^*(w) - z_0^*(w)| < \epsilon,$$

which does not attain its infimum on A, and such that $x_0^*(A) > 0$. As a consequence we show that a Fatou coherent monetary utility function $u_1 \neq$ ess.inf has the Lebesgue property if and only if the convolution $u_1 \Box u_2$ is Fatou for every Fatou coherent monetary utility function u_2.

1. INTRODUCTION

The well-known James theorem [9] claims that *a bounded closed convex set A in a Banach space E is weakly compact if and only if every $x^* \in E^*$ attains its supremum on A.*

Since its appearance, the weak compactness theorem of James has become the subject of much interest. One of the concerns about it has been to obtain proofs which are simpler than the original one. Another, and we deal with it in this paper, is to generalise this central result, which in particular has led to new applications as the ones recently given, [1, 2, 15, 16]. A recent work by H. Pfitzner about James's theorem, which also closes a long standing open problem in the area [17], deserves a mention here. We shall deal with two main results in this direction:

Theorem 1. *Let E be a Banach space. Let A and B be bounded, closed and convex sets with distance $d(A, B) = \inf\{\|a - b\| : a \in A, b \in B\} > 0$. If every $x^* \in E^*$ with*

$$\sup(x^*, B) < \inf(x^*, A)$$

attains its infimum on A and its supremum on B, then A and B are both weakly compact.

And the second one is the following:

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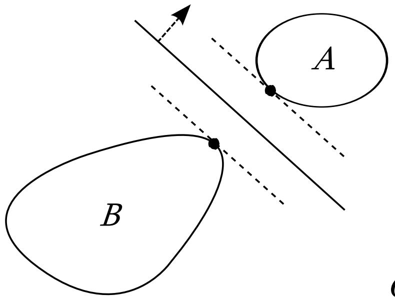
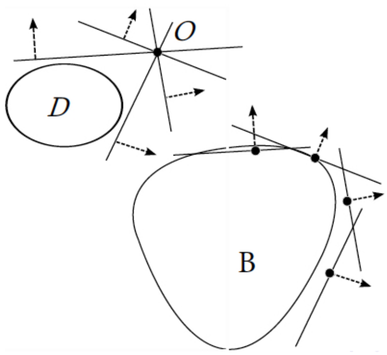


*One-sided James's Compactness Theorem*1

Theorem 2. *Let E be a Banach space and let D be a weakly compact subset of E which does not contain the origin. If B is a bounded subset of E such that every element of $x^* \in E^*$ with $x^*(D) < O$ attains its supremum on B, then B is weakly relatively compact.*

Both results has been proved in [1] for Banach spaces where the dual unit ball is sequentially $\sigma(E^*, E)$-compact or just block-convex compact. In this paper the authors asked for both results in arbitrary Banach spaces. Finally Theorem 1 in [13] using [12] and Theorem 2 in [14] have been proved in full generality.



*One-sided James's Compactness Theorem*2

One aim of this paper is to prove a new James's type result that includes both theorems as particular cases. Indeed we prove the following:

Theorem 3. *Let A be a nonempty, closed, convex, bounded and non weakly compact subset of a Banach space E such that $0 \notin A$. Let us fix a convex and weakly compact subset D of E, a functional $z_0^* \in E^*$ with $\inf z_0^*(A) > 0$ and $\epsilon > 0$. Then there is a continuous linear form $x_0^* \in B_{p_D}(z_0^*, \epsilon)$ i.e. $x_0^* \in E^*$ and*

$$\sup_{d \in D} |x_0^*(d) - z_0^*(d)| < \epsilon,$$

which does not attain its infimum on A and such that $\inf x_0^*(A) > 0$.

A direct consequence, just making a traslation of the unit ball to get a set A without the origin, is the following:

Theorem 4. *A Banach space E is reflexive if and only if the set NA(E) of norm attaining linear functionals of E^*, has non empty interior for the Mackey topology $\tau(E^*, E)$ on E^*.*

This can be rephrased as:

Corollary 5. *For a non reflexive Banach space E the set $E^* \setminus NA(E)$, of all linear forms which do not attain their supremum on the unit ball of E, is dense in E^* for the Mackey topology $\tau(E^*, E)$ of uniform convergence on weakly compact subsets of E.*

The last corollary extends to the Mackey topology previous results from [4, 10] for $\sigma(E^*, E)$.

A second aim for us is to apply our results to obtain new characterizations of *convex risk measures having the Lebesgue property.* The Lebesgue property of a risk measure (i.e the validity of the dominated convergence theorem for the risk measure), is equivalent to the weak compactness of the sublevel sets of its Fenchel conjugate restricted to $\mathbb{L}^1$, as was established by Jouini, Schachermayer and Touzi, [11] and [6]. The known proof of this result and his extensions requires the use of James's compactness theorem. The work of J. Saint Raymond deserves special mention here, see[20]. For a complete description of historical facts realted with [20][15, 16] see p.205 in [2].

With the one-sided James theorem we now get new characterizations, for instance *we prove that given a Fatou coherent risk measure $\rho_1 \neq$ ess.sup (i.e. it verifies the classical Fatou lemma), it is has the Lebesgue property if and only if the inf-convolution $\rho_1 \Box \rho_2$ with any Fatou coherent risk measure ρ_2 is again a measure with the Fatou property,* see Theorem 12. The opportunity to deal with the Mackey topology provide us a natural context for proving that *any convex risk measure ρ has the Lebesgue property if and only if it is $\tau(\mathbb{L}^\infty, \mathbb{L}^1)$-continuous*, see Theorem 8 and its corollary.

1.1. **Notation and terminology.** Most of our notation and terminology is standard and can be found in standard references on Banach spaces [7].

Unless otherwise stated, E will denote a Banach space with norm $\|\cdot\|$. Given a subset S of a vector space, we write $\operatorname{co}(S)$, to denote its convex hull. If $(E, \|\cdot\|)$ is a normed space then E^* denotes its topological dual. If S is a subset of E^*, then $\sigma(E, S)$ denotes the topology of pointwise convergence on S. Dually, if S is a subset of E, then $\sigma(E^*, S)$ is the topology on E^* of pointwise convergence on S. In particular $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak (ω) and weak* (ω^*) topologies respectively. We denote by $\tau(E^*, E)$ the Mackey topology on E^*, i.e. the topology of uniform convergence on weakly compact sets of the Banach space E.

Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle = \langle x, x^*\rangle = x^*(x)$ for the evaluation of x^* at x. If $x \in E$ and $\delta > 0$ we denote by $B(x, \delta)$ (resp. $B[x, \delta]$) the open (resp. closed) ball centred at x of radius δ. We will simply write $B_E := B[0, 1]$ and the unit sphere $\{x \in E \colon \|x\| = 1\}$ will be denoted by S_E. When dealing with the Mackey topology $\tau(E^*, E)$ we denote by $B_{p_D}(x^*, \delta)$ (or $B_{p_D}[x^*, \delta]$) the open (resp. closed) ball centred at x of radius δ for the seminorm

$$p_D(x^*) = \sup\{|x^*(x)| : x \in D\}$$

of uniform convergence on the weakly compact set $D \subset E$. An element $x^* \in E^*$ is *norm-attaining* if there is $x \in B_E$ with $x^*(x) = \|x^*\|$. The set of norm-attaining functionals of E is usually denoted by $NA(E)$. The famous Bishop-Phelps theorem asserts that $NA(E)$ is norm dense in E^*.

If (x_n) is a bounded sequence in the Banach space E we denote by

$$\mathrm{co}_\sigma(x_n; n \geq 1) = \left\{ \sum_{n=1}^{\infty} \xi_n x_n : \xi_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \xi_n = 1 \right\},$$

the σ-convex hull of the sequence (x_n).

For an absolutely convex subset W of the vector space E we shall denote by E_W the linear span of W: i.e. $E_W = \cup_{n=1}^{\infty} nW$.

2. The Mackey topology in one-sided James's Compactness Theorem

In this section we present our main construction for proving the result stated in our abstract. A careful analysis of the ideas involved in James's theorem leads us to find appropriate non attaining linear forms for the one-sided James Theorem too, even when the Mackey topology $\tau(E^*, E)$ is involved. Indeed we have the following general statement, which is nothing else that a more detailed formulation of our Theorem 3:

Theorem 6. *Let A be a convex, closed and bounded subset of a Banach space E which is assumed to be non weakly compact. Let us fix a convex and weakly compact subset D of E, a functional $z_0^* \in E^*$ with $\inf z_0^*(A) > \eta > 0$. For $\epsilon > 0$ we can find a linear form $z^* \in E^*$ such that*

$$a \to \langle z_0^* + z^*, a\rangle \textit{ does not attain its infimum on } A,$$

$$\sup_{d \in D} |z^*(d)| < \epsilon,$$

$$\|z^*\| < \eta \textit{ and } \inf(z_0^* + z^*)(A) > 0.$$

Proof. Without loss of generality we may and do assume that our set A is a subset of the unit ball B_E and $\eta < 1$. Since A is weakly closed but not weakly compact, Eberlein's theorem allows us to find a sequence $\{x_n \mid n \geq 1\} \subset A$ which has no weak cluster point in E. We then find $x_0^{**} \in \overline{\{x_n : n \geq 1\}}^{\sigma(E^{**}, E^*)}$ such that

$$x_0^{**} \in \overline{A}^{\sigma(E^{**}, E^*)} \setminus E, \quad \|x_0^{**}\| \leq 1 \quad dist(x_0^{**}, E) > 0. \tag{1}$$

We will replace D by a translation $u_0 + D$ where $u_0 \in E$ will be chosen so that $\|u_0\|$ is big enough to satisfy properties that will be made precise later on. Let $\delta > \sup_{d \in D} \|d\|$. Then we define B as the $\sigma(E^{**}, E^*)$-closed convex hull of the line segment $[-x_0^{**}, x_0^{**}] = \{\lambda x_0^{**} : -1 \leq \lambda \leq 1\}$ and our bounded set A. Clearly $B \subset B_{E^{**}}$. Let us now take $\gamma > \frac{2(\|z_0^*\| + \eta)}{\eta} > 2$ (a quantity that will be explained later on). A first condition is that the ball $B^{**}(u_0, \|u_0\| - \delta) \subset E^{**}$ and $\gamma B_{E^{**}}$

are disjoint sets. We take $\|u_0\| = 1 + \delta + \gamma$. Because the sets are weak* compact we can separate them by a functional $x^* \in E^*$, $\|x^*\| = 1$. We then have

$$\sup x^*(B) \leq 1 \leq \gamma < \inf x^*(B^{**}(u_0, \|u_0\| - \delta)) = x^*(u_0) - \delta \leq \inf x^*(u_0 + D)$$

Hence we have

$$\|u_0\| - 1 - \delta \leq x^*(u_0) - \delta,$$

which implies that $x^*(u_0) \geq \|u_0\| - 1$. The selection of x^* then implies that:

$$x^*(B) \subset [-\mu, +\mu], \quad 0 \leq \mu \leq 1$$

$$x^*(u_0 + D) = [a, b],$$

where

$$a - \mu > x^*(u_0) - \delta - \mu > \gamma - 1 = \|u_0\| - 1 - \delta - 1$$

is big enough so that α, β can be found with

(2)

$$-\mu \leq 0 \leq \mu < \beta < \alpha < a < b; \alpha - \beta = \|u_0\| - \delta - 2 \text{ and } (\alpha - \beta) > x_0^{**}(z_0^*) > \eta.$$

where the last inequality follows from $\gamma - 1 > \|z_0^*\|$ which itself follows from $\eta < 1$. Let us observe that we have:

(3) $$\text{for every } n \in \mathbb{N} : x^*(x_n) \leq \mu < \beta.$$

We now use the Hahn–Banach theorem to separate x_0^{**} from E. The *open* ball (in E^{**}) with centre x_0^{**} and radius $dist(x_0^{**}, E)$ is disjoint from E. By the Hahn-Banach theorem there exists an element $z^{***} \in E^{***}$ with $\|z^{***}\| = 1$ so that $z^{***}(E) = 0$ and $z^{***}(x_0^{**}) = dist(x_0^{**}, E) > 0$. Let us choose $\lambda > 0$ such that $\lambda z^{***}(x_0^{**}) + x^*(x_0^{**}) > \alpha$ and consider the linear functional

$$x^{***} := x^* + \lambda z^{***}.$$

We will have $\|x^{***}\| \leq \lambda + 1$ and

(4) $$x^{***}(u_0 + D) = x^*(u_0 + D) \geq a > \alpha > 0,$$

$$x^{***}(x_0^{**}) = \lambda z^{***}(x_0^{**}) + x^*(x_0^{**}) > \alpha > 0.$$

Let us remind the reader that we also have by (3)

(5) $$x^{***}(x_n) = x^*(x_n) < \beta$$

for every $n \in \mathbb{N}$.

We now fix a positive real number r with

$$0 < \sqrt{r} < \eta/2 < 1/2,$$

(6) $$(r\alpha - r\beta) > \sqrt{r} x_0^{**}(z_0^*),$$

and recall that

$$z_0^*(A) > \eta > 0.$$

This choice for r is always possible since

$$\frac{x_0^{**}(z_0^*)}{\alpha - \beta} < \frac{\eta}{2}$$

which is implied by

$$\frac{\|z_0^*\|}{\gamma - 1} < \frac{\eta}{2}$$

(a consequence of the inequality that defined γ).

By Goldstine's theorem B_{E^*} is weak*, i.e. $\sigma(E^{***}, E^{**})$, dense in $B_{E^{***}}$. Therefore by the Mackey-Arens theorem it is also Mackey, i.e $\tau(E^{***}, E^{**})$, dense. We can find a sequence $x_n^* \in (\lambda + 1)B_{E^*}$ such that

$$\text{for all } p \leq n\text{: } |x_n^*(x_p) - x^{***}(x_p)| < 1/2^n \tag{7}$$

$$|(x_n^* - x^{***})(x_0^{**})| < 1/2^n \tag{8}$$

$$|(x_n^* - x^{***})(u_0)| < 1/2^n \tag{9}$$

$$\sup_{d\in D} |x_n^*(d) - x^{***}(d)| < 1/2^n. \tag{10}$$

We then have:

$$\lim_{n\to+\infty} x_n^*(x_p) = \langle x^*, x_p\rangle \tag{11}$$

for every $p \in \mathbb{N}$,

$$\lim_{n\to+\infty} x_n^*(x_0^{**}) = \langle x^{***}, x_0^{**}\rangle \tag{12}$$

$$\lim_{n\to+\infty} x_n^*(u_0) = \langle x^*, u_0\rangle \tag{13}$$

and

$$\lim_{n\to+\infty} x_n^*(d) = \langle x^*, d\rangle \tag{14}$$

uniformly on $d \in D$.

The inequalities above have some consequences which follow from (11) and (5):

$$\text{for every } p \geq 1, \text{ there is } n_p \text{ such that } \beta > x_n^*(x_p) \text{ for } n \geq n_p, \tag{15}$$

and without loss of generality we may assume that:

$$x_0^{**}(x_n^*) > \alpha, x_n^*(d) + x_n^*(u_0) > \alpha, \forall n \in \mathbb{N} \text{ and } d \in D \tag{16}$$

by (4),(12) and the uniform convergence on the weakly compact set D, (14).

Note that given any pointwise–cluster point x_0^* on E^* for the $\sigma(E^*, E)$-topology of the bounded sequence $\{x_n^*\}_{n\geq 1}$, we have that

$$x_0^{**}(x_0^*) \leq \beta, \tag{17}$$

because $x_0^{**} \in \overline{\{x_p : p \geq 1\}}^{\sigma(E^{**},E^*)}$ and for all $p \geq 1, x_n^*(x_p) \leq \beta$ by (15).

We now have all ingredients to find non-attaining linear functionals. We will use *undetermined function procedure* of Pryce [18] and James [9], as it was improved by Galan and Simons in [8], see also [2]. In fact, we are going to use

Pryce's arguments (see [8] Lemma 9,c; Proposition 10.14 and Theorem 10.15 in [2]) on $\mathbb{R}^A$ with the uniformly bounded sequence $\{\sqrt{r}x_n^* : n = 1, 2, \cdots\}$. Then we know there is a subsequence

$$\{\sqrt{r}x_{n_k}^* : k = 1, 2, \cdots\}$$

such that, for all $h_0 \in \mathrm{co}_\sigma \left\{\sqrt{r}x_n^* : n \geq 1\right\}$, we have:

$$\sup_{a\in\sqrt{r}A}\left(h_0 - \limsup_{k\geq 1}\sqrt{r}x_{n_k}^* - z_0^*\right)(a) = \tag{18}$$

$$= \sup_{a\in\sqrt{r}A}\left(h_0 - \liminf_{k\geq 1}\sqrt{r}x_{n_k}^* - z_0^*\right)(a).$$

Let us now observe that for x_0^{**} we have by (16) that

$$\forall h_0 \in \mathrm{co}_\sigma \left\{\sqrt{r}x_{n_k}^* : k \geq 1\right\}, \qquad \sqrt{r}x_0^{**}(h_0) > r\alpha. \tag{19}$$

Let us fix a $\sigma(E^*, E)$-cluster point x_0^* of the bounded sequence $\left\{x_{n_k}^* : k \geq 1\right\}$; then it follows that for all $a \in A$,

$$\limsup_{k\geq 1}\sqrt{r}x_{n_k}^*(a) \geq \sqrt{r}x_0^*(a) \geq \liminf_{k\geq 1}\sqrt{r}x_{n_k}^*(a)$$

and thus, for all $a \in A$,

$$\begin{aligned}&\left(h_0(\sqrt{r}a) - \liminf_{k\geq 1}\sqrt{r}x_{n_k}^*(\sqrt{r}a) - z_0^*(\sqrt{r}a)\right)\\ &\geq \left(h_0 - \sqrt{r}x_0^* - z_0^*\right)(\sqrt{r}a)\\ &\geq \left(h_0(\sqrt{r}a) - \limsup_{k\geq 1}\sqrt{r}x_{n_k}^*(\sqrt{r}a) - z_0^*(\sqrt{r}a)\right).\end{aligned}$$

Therefore, in view of (18) we deduce that for all $h_0 \in \mathrm{co}_\sigma \left\{\sqrt{r}x_{n_k}^* : k \geq 1\right\}$

$$\begin{aligned}&\sup_{\sqrt{r}A}\left(h_0 - \limsup_{k\geq 1}\sqrt{r}x_{n_k}^* - z_0^*\right)\\ &= \sup_{\sqrt{r}A}\left(h_0 - \liminf_{k\geq 1}\sqrt{r}x_{n_k}^* - z_0^*\right)\\ &= \sup_{\sqrt{r}A}\left(h_0 - \sqrt{r}x_0^* - z_0^*\right).\end{aligned}$$

Let us observe that for $h_0 \in \mathrm{co}_\sigma \left\{\sqrt{r}x_{n_k}^* : k \geq 1\right\}$ we have:

$$\begin{aligned}&\sup_{\sqrt{r}A}\left(h_0 - \sqrt{r}x_0^* - z_0^*\right)\\ &= \sup_{\overline{\sqrt{r}A}^{w^*}}\left(h_0 - \sqrt{r}x_0^* - z_0^*\right)\\ &\geq \langle\sqrt{r}x_0^{**}, h_0 - \sqrt{r}x_0^* - z_0^*\rangle\end{aligned}$$

and

$$\langle\sqrt{r}x_0^{**}, h_0 - \sqrt{r}x_0^* - z_0^*\rangle > r\alpha - r\beta - \sqrt{r}x_0^{**}(z_0^*) > 0,$$

by (17), (19) and (6), as it is needed to apply [8], Corollary 8, to get a sequence $\{g_i^*\}_{i\geq 1}$ with $g_i^* \in \mathrm{co}_\sigma\{\sqrt{r}x^*_{n_k} : k \geq i\}$ and $g_0^* \in \mathrm{co}_\sigma\,\{g_i^* : i \geq 1\}$ such that for all $\tilde{g} \in \ell_\infty(A)$, with

$$\liminf_{i\geq 1} g_i^* \leq \tilde{g} \leq \limsup_{i\geq 1} g_i^* \text{ on } A$$

we have that

$$(g_0^* - \tilde{g} - z_0^*) \text{ does not attain its supremum on } \sqrt{r}A.$$

Thus it does not attain its supremum on A whenever $\tilde{g} \in E^*$, which does happen by $\sigma(E^*, E)$- compactnes.

We now take care of the fact that

$$|(g_0^* - \tilde{g})(a)| \leq 2\sqrt{r} < \eta$$

since we assumed $A \subset B_E$, and thus

$$(z_0^* - (g_0^* - \tilde{g}))(A) > 0$$

Let us observe that $\tilde{g}$ does coincide with $\sqrt{r}x^{***}$ on D and that g_i^* are σ-convex combinations of linear forms of the sequence $\{\sqrt{r}x_n^*\}$. By uniform convergence on D we may assume that

$$x^{***}(d) + \frac{\epsilon}{2\sqrt{r}} > x_n^*(d) > x^{***}(d) - \frac{\epsilon}{2\sqrt{r}}$$

for all $d \in D$ and every $n \in \mathbb{N}$, and finally

$$\sqrt{r}x^{***}(d) + \epsilon > g_i^*(d) > \sqrt{r}x^{***}(d) - \epsilon$$

for all $d \in D$ and every $i \in \mathbb{N}$, therefore

$$+\epsilon > (g_0^* - \tilde{g})(d) > -\epsilon$$

for every $d \in D$ and the proof should be over. In particular, for every $\sigma(E^*, E)$–cluster point of the sequence $\{g_i^*\}_{i\geq 1}$, let us say $\tilde{g}^*$, we have that $g_0^* - \tilde{g}^* - z_0^* \in E^*$, and it does not attain its supremum on A. Thus $(z_0^* + \tilde{g}^* - g_0^*) \in B_{p_D}(z_0^*, \epsilon)$, it does not attain its infimum on A but

$$\inf(z_0^* + \tilde{g}^* - g_0^*)(A) > 0,$$

so the proof is over. □

The proof of Theorem 3, and its consequences Theorem 4 and Corollary 5, immediately follow as a consequence of the former result.

3. MACKEY TOPOLOGY AND RISK MEASURES

For the duality $(\mathbb{L}^1, \mathbb{L}^\infty)$ the Mackey topology plays a central role. It is the topology of uniform convergence on weakly compact sets of $\mathbb{L}^1$. The Dunford-Pettis Theorem says that relatively weakly compact sets in $\mathbb{L}^1$ are precisely the sets that are uniformly integrable. From this it follows that the Mackey topology on bounded sets of $\mathbb{L}^\infty$ is the topology of convergence in probability. Because of the de la Vallée-Poussin Theorem, $\tau(\mathbb{L}^\infty, \mathbb{L}^1)$ is also the initial topology of the imbeddings $\mathbb{L}^\infty \longrightarrow \mathbb{L}^\Phi$, where $\mathbb{L}^\Phi$ is the Orlicz space

$$\{\xi \in L^0 : \exists \lambda > 0 \text{ with } \mathbb{E}[\Phi(\lambda|\xi|)] < \infty\}$$

equiped with the Luxemburg norm:

$$\|\xi\|_\Phi = \inf\{\lambda > 0 : \mathbb{E}\left[\Phi\left(\frac{|\xi|}{\lambda}\right)\right] \leq 1\}$$

and $\Phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a convex function with $\Phi(0) = 0$ and

$$\lim_{x\to+\infty} \frac{\Phi(x)}{x} = \infty.$$

$\mathbb{L}^\Phi$ is not always the closure of $\mathbb{L}^\infty$. The closure of $\mathbb{L}^\infty$ is called the Orlicz heart and it is equal to:

$$\mathbb{L}^{(\Phi)} = \{\xi \in \mathbb{L}^0 : \text{ for all } \lambda > 0 \text{ we have } \mathbb{E}[\Phi(\lambda|\xi|)] < \infty\}.$$

Clearly we have $\mathbb{L}^{(\Phi)} \subset \mathbb{L}^\Phi$. Both spaces are equal if and only if Φ satisfies the Δ_2 condition. The Legendre transform of Φ is again a convex function with the same properties. It defines Orlicz spaces $\mathbb{L}^{(\Psi)}$ and $\mathbb{L}^\Psi$. The spaces $\mathbb{L}^\Phi, \mathbb{L}^\Psi$ form a dual pair with the coupling between $\xi \in \mathbb{L}^\Phi, \eta \in \mathbb{L}^\Psi$ defined as $\mathbb{E}[\xi\eta]$. That $\xi\eta \in \mathbb{L}^1$ follows from $xy \leq \Phi(x) + \Psi(y)$ for $x, y \geq 0$. The Banach space $\mathbb{L}^{(\Phi)}$ has $\mathbb{L}^\Psi$ as its topological dual.

Before stating the relations with risk measures or monetary utility functions, let us recall some definitions.

Definition 7. *A function $u\colon \mathbb{L}^\infty \to \mathbb{R}$ is called a monetary utility function if*

(i) *if $\xi \geq 0$ then $u(\xi) \geq 0$, $u(0) = 0$*
(ii) *u is concave*
(iii) *for $\xi \in \mathbb{L}^\infty$ and $a \in \mathbb{R}$: $u(\xi + a) = u(\xi) + a$*
(iv) *u has the Fatou property if for a sequence $\xi_n \downarrow \xi_0$: $u(\xi_n) \to u(\xi_0)$*
(v) *u has the Lebesgue property if for uniformly bounded sequences $\xi_n \to \xi_0$ in probability we have $u(\xi_n) \to u(\xi_0)$*
(vi) *u is called coherent if it is positively homogeneous, i.e. for $0 \leq \lambda \in \mathbb{R}$, $u(\lambda\xi) = \lambda u(\xi)$*

The Eberlein-Smulian or Banach-Dieudonné theorem implies that for monetary utility functions, the Fatou property is the same as the upper semi continuity for the weak* topology on $\mathbb{L}^\infty$. The Lebesgue property is the same as continuity on bounded sets for the Mackey topology.

The Fenchel transform of u is a convex lower semi continuous mapping from $\mathbb{L}^1$ to $\mathbb{R}_+$ and it easy to see that it is $+\infty$ outside the set of densities of probability measures $\mathbb{Q} \ll \mathbb{P}$. It is usually denoted by $c(\mathbb{Q}) = \sup\{\mathbb{E}_\mathbb{Q}[-\xi] \mid u(\xi) \geq 0\}$. If u is coherent then c is the (convex) indicator of a closed convex set of probabilities (also called scenarios) $\mathcal{S} \subset \mathbb{L}^1$. Consequently, see e.g. [6], we have

$$u(\xi) = \inf\{\mathbb{E}_\mathbb{Q}[\xi] + c(\mathbb{Q}) \mid \mathbb{Q} \ll \mathbb{P}\},$$

or in the coherent case

$$u(\xi) = \inf\{\mathbb{E}_{\mathbb{Q}}[\xi] \mid \mathbb{Q} \in \mathcal{S}\}.$$

These expressions already suggest that James's theorem will play a role. As proved in [6] we have for coherent Fatou monetary utility functions u that weak compactness of $\mathcal{S}$ is equivalent to the property that for every $\xi \in \mathbb{L}^\infty$, there is $\mathbb{Q} \in \mathcal{S}$ with $u(\xi) = \mathbb{E}_{\mathbb{Q}}[\xi]$. For concave functions this was generalised by Jouini-Schachermayer-Touzi theorem, [11] where it is shown (using a modification of the proof of James's theorem) that if for every $\xi \in \mathbb{L}^\infty$ there is $\mathbb{Q}$ with $u(\xi) = \mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q})$ then the sublevel sets $\{\mathbb{Q} \mid c(\mathbb{Q} \leq \alpha\}$ are weakly compact for every (or just for one) $\alpha > 0$. In [6] it is shown that one can reduce the concave case to the coherent case. In [6] and [11] it is also shown that this property is equivalent to the Lebesgue property. The results on the Fenchel transform then state that if the sublevel sets are weakly compact, the function u is continuous for the Mackey topology. This result is due to Moreau and for a proof one can consult [19] and the references there. We will see below that Mackey continuity on the bounded sets implies continuity on the whole space.

In case the monetary utility function u is not necessarily Fatou, we must work with the dual of $\mathbb{L}^\infty$, the space, $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$ or $\mathbf{ba}$, of finitely additive measures absolutely continuous to $\mathbb{P}$. The scenario-set (in the case of coherent functions) is then denoted by $\mathcal{S}^{\mathbf{ba}}$. It is defined in the same way:

$$\mathcal{S}^{\mathbf{ba}} = \{\mu \mid \text{ for all } \xi \text{ with } u(\xi) \geq 0 \text{ we have } \mu(\xi) \geq 0\}.$$

The Fatou property just means that $\mathcal{S}^{\mathbf{ba}}$ is the $\sigma(\mathbf{ba}, \mathbb{L}^\infty)$ closure of $\mathcal{S}$. The Lebesgue property means that $\mathcal{S} = \mathcal{S}^{\mathbf{ba}}$. For concave monetary utility functions we define for finitely additive probability measures:

$$c^{\mathbf{ba}}(\mu) = \sup\{\mu(-\xi) \mid u(\xi) \geq 0\}.$$

The Fatou property means that the epigraph $\mathcal{G}$ of c,

$$\mathcal{G} = \{(\mathbb{Q}, \alpha) \mid \alpha \geq c(\mathbb{Q}), \alpha \in \mathbb{R}, \mathbb{Q} \ll \mathbb{P}\},$$

is $\sigma(\mathbf{ba}, \mathbb{L}^\infty)$ dense in the epigraph $\mathcal{G}^{\mathbf{ba}}$ of $c^{\mathbf{ba}}$, see [6] for details.

For two monetary concave utility functions u_1 and u_2 we can optimise in the sense that we can define a new monetary utility function, called the convolution. It is defined as

$$u_1 \Box u_2(\xi) = \sup\{u_1(\eta_1) + u_2(\eta_2) \mid \eta_1 + \eta_2 = \xi\}.$$

One can show (see [6] for an analysis also in the case of concave monetary utility functions) that for coherent functions we have that $u_1 \Box u_2$ is a coherent utility function if and only if $\mathcal{S}_1^{\mathbf{ba}} \cap \mathcal{S}_2^{\mathbf{ba}} \neq \emptyset$. The scenario-set of the convolution is then $\mathcal{S}_1^{\mathbf{ba}} \cap \mathcal{S}_2^{\mathbf{ba}}$. Of course this implies that if u_1 has the Lebesgue property and if $u_1 \Box u_2$ is defined (meaning it is not identically $+\infty$), then the convolution has the Lebesgue property.

Another approach to characterise the Lebesgue property is in the work of Cheredito and Li [3], if a monetary utility function u, satisfies the Lebesgue property then there is an Orlicz space, $\mathbb{L}^\Phi$, such that u can be extended to $\mathbb{L}^{(\Phi)}$ and

$$u : \mathbb{L}^{(\Phi)} \longrightarrow \mathbb{R}$$

is continuous. If we replace Φ by $\Xi(x) = \Phi(x^2)$, then $\mathbb{L}^\Xi$ injects into $\mathbb{L}^{(\Phi)}$ and hence we can extend u to an Orlicz space. We can rephrase this as: if u is continous on closed balls for the Mackey topology, then it is continuous on $\mathbb{L}^\infty$ for the Mackey topology. A first aim here is to show that this fact is true in full generality as Theorem 8 proves. It is stated for convex functions which might be more familiar.

Theorem 8. *Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex proper and lower semicontinuous function with Fenchel conjugate $f^* < \infty$ everywhere on E^*. If $f^*_{\rceil nB_{E^*}}$ is Mackey continuous for every $n \in \mathbb{N}$, then $f^* : E^* \longrightarrow \mathbb{R}$ is Mackey continuous.*

Proof. Let us remark that f^* is already lower semicontinuous for the Mackey topology. To prove the continuity at $x^* \in E^*$ we may replace f^* by a new function:

$$g(z^*) := f^*(x^* + z^*) - f^*(x^*)$$

Now $g(0) = 0$ and we must show that g is continuous at 0. But g is also a conjugate function or Fenchel transform of a convex lower semicontinuous proper function $\tilde{g} : E \longrightarrow \mathbb{R} \cup \{+\infty\}$. Therefore without loss of generality we may and do suppose $f^*(0) = 0$ and prove the continuity at 0. Because f^* is lower semicontinuous we must only show that for every $\epsilon > 0$

$$\{x^* : |f^*(x^*)| \le \epsilon\}$$

is a Mackey neighbourhood of the origin. Take $x_0 \in E$ an element of the $\epsilon/2$-subgradient of f^* at 0. This is possible since f^* is lower semicontinuous. We have $f^*(x^*) \ge x^*(x_0) - \epsilon/2$ for all $x^* \in E^*$. The function $h^*(x^*) := f^*(x^*) - x^*(x_0)$ is again a Fenchel transform and $h^*_{\rceil nB_{E^*}}$ is $\tau(E^*, E)$-continuous for each n. Now we see that $h^* \ge -\epsilon/2$ and hence

$$\{x^* \in E^* : |h^*(x^*)| \le \epsilon/2\} = \{x^* \in E^* : h^*(x^*) \le \epsilon/2\}$$

is a convex and $\sigma(E^*, E)$-closed set V with $0 \in V$. For every $n \in \mathbb{N}$ there exists a weakly compact subset $K_n \subset E$ with

$$V \cap nB_{E^*} \supset K_n^\circ \cap nB_{E^*}.$$

This implies

$$V^\circ \subset K_n + n^{-1}B_E$$

for every $n \in \mathbb{N}$. The well known Lemma of Grothendieck (Lemma 13.32, p.591 in [7]) tells us that V° is weakly compact and $V = V^{\circ\circ}$ is a Mackey neighbourhood of the origin. Now

$$\{x^* \in E^* : |f^*(x^*)| \le \epsilon\} \supset V \cap \{x^* \in E^* : |x^*(x_0)| \le \epsilon/2\}$$

and is clearly a Mackey neighbourhood of 0.

$\square$

The Lebesgue property is a sequential property and it triggers the question whether we can improve the previous theorem by just using sequences. As it is well known, in general this is not possible. For completeness we provide some proofs in the next paragraphs.

Let us warm up for the next result and suppose that the Mackey topology of the dual space $\tau(E^*, E)$ is metrizable, then there is a countable neighbourhood base of the origin given by

$$\{K_n^\circ = \{x^* \in E^* \mid \forall x \in K_n : x^*(x) \leq 1\}; n = 1, 2, \cdots\},$$

where every K_n is a weakly compact and absolutely convex subset of E. Because it is a neighbourhood base we have:

$$\forall F \subset E \text{ convex and weakly compact, there is } n \in \mathbb{N} \text{ such that } F^\circ \supseteq K_n^\circ$$

This implies that $F \subset K_n$. Let us now take

$$K := \sum_{n=1}^{\infty} 2^{-n} \frac{K_n}{m_n}$$

where $m_n = \sup\{\|x\| : x \in K_n\}$. K is a weakly compact subset of E and for every weakly compact subset F in E there is $m \in \mathbb{N}$ such that $F \subset mK$. This implies that K is absorbing, hence a neighbourhood of the origin by Baire's Category theorem, which implies that E is reflexive.

Let us go a little bit further and suppose now that for every positive integer n we have that $\tau(E^*, E)_{\downharpoonright nB_{E^*}}$ is metrizable, or just locally countable at 0. As in the warm up there exists $K_{k,n}$ absolutely convex and weakly compact subsets of E such that

$$\{K_{k,n}^\circ \cap nB_{E^*}.k = 1, 2, \cdots\}$$

forms a neighbourhood base of the origin in nB_{E^*}. For every F weakly compact subset of E and every positive integer n there is $l \in \mathbb{N}$ such that

$$F^\circ \cap nB_{E^*} \supset K_{l,n}^\circ \cap nB_{E^*}$$

This implies that

$$F \subset (K_{l,n}^\circ \cap nB_{E^*})^\circ = \text{co}(K_{l,n}, n^{-1}B_E) \subset K_{l,n} + n^{-1}B_E$$

Let us now define:

$$K := \sum_{k,n=1}^{\infty} 2^{-k-n} \frac{K_{k,n}}{m_{k,n}}$$

where $m_{k,n} = \sup\{\|x\| : x \in K_{k,n}\}$. K is a weakly compact subset of E and satisfies:

(20)
$$\forall F \text{ weakly compact subset of } E \text{ and } \forall n \in \mathbb{N}\ \exists m \text{ such that } F \subset mK + \frac{1}{n}B_E.$$

Banach spaces E that verify the former condition, with some K weakly compact subset of it, are called *strongly weakly compactly generated*, SWCG in short. This class of Banach spaces was introduced and studied by Schlüchtermann and Wheeler [21] to better understand the Mackey topology of a dual Banach space, indeed they started proving that a Banach space E is SWCG if and only if the dual unit ball B_{E^*} with the relative Mackey topology $\tau(E^*, E)$ is (completely) metrisable. It is also shown in [21] that SWCG spaces are weakly sequentially complete, which proves immediately that c_0 or $C([0,1])$ are not SWCG. In $\mathbb{L}^1$ the Dunford-Pettis theorem tells us that $\mathbb{L}^1$ is SWCG, see, for instance, p. 615 in[7]. Other spaces that are SWCG are: $\mathbb{H}^1$, the Hardy space of holomorphic functions where a subset R of $\mathbb{H}^1$ is relatively weakly compact if, and only if, for every $\epsilon > 0$ there is $m > 0$ such that $R \subset mB_{\mathbb{H}^\infty} + \epsilon B_{\mathbb{H}^1}$, [5]. The Orlicz space $\mathbb{L}^\Phi$ where Φ satisfies $\lim_{x\to\infty} \frac{\Phi(x)}{x\Phi'(x)} = 1$ is SWCG too. Or more generally, when the conjugate Young function Ψ satisfies for all (or equivalently for one) $\alpha > 1$: $\lim_{x\to+\infty} \frac{\Psi(\alpha x)}{\Psi(x)} = +\infty$. In this case we have that $K \subset \mathbb{L}^\Phi$ is relatively weakly compact if, and only if,$\{\Phi(|\xi|) : \xi \in K\}$ is uniformly integrable. This applies for instance, to $\mathbb{L}^\Phi$ with $\Phi(x) = (x+1)\log(x+1) - x$, that is the $L\log L$-space.

Our next result shows how sequential continuity and continuity do coincide for the Mackey topology in the dual space E^* whenever E is SWCG. Let us first show with an example that this is not always the case, even in separable Banach spaces. First we need to recall the following known fact:

Remark 9. *If (x_n^*) is $\tau(l^1, c_0)$-null, then it is also $\|.\|$ and $\sigma(l^1, l^\infty)$-null.*

Proof. Suppose we have $\|x_n^*\| \geq \epsilon > 0$ for a subsequence

$$n \in L = \{n_1 < n_2 < \cdots < n_j < \cdots\} \subset \mathbb{N}.$$

Then we can construct a further subsequence (N_k) $(N_k < N_{k+1})$ such that

$$\sum_{j=1}^{N_k} |x^*_{N_{k+1}}(j)| \leq 2^{-k}; \sum_{j \geq N_{k+1}+1} |x^*_{N_{k+1}}(j)| \leq 2^{-k}$$

and

$$\sum_{j=N_k+1}^{N_{k+1}} |x^*_{N_{k+1}}(j)| \geq \epsilon - 2^{-k+1}.$$

Let us now define

$$x_k(j) := \text{ sign } (x^*_{N_{k+1}}(j)) \text{ for } N_k + 1 \leq j \leq N_{k+1}$$

and equal 0 otherwise, Then $(x_k)_{k\geq 1}$ is a weakly null sequence in c_0. Because $(x^*_{N_k})_{k\geq 1}$ tends to zero in the Mackey topology we must have

$$\lim_{k\longrightarrow\infty} \sup_m \{|x^*_{N_k}(x_m)|\} = 0,$$

a contradiction to the fact that $|x^*_{N_{k+1}}(x_k)| \geq \epsilon - 2^{-k+1}$.

□

Example 10. *Take $x^{**} \in l^\infty \setminus c_0$. Then $x^{**} : l^1 \longrightarrow \mathbb{R}$ is sequentially Mackey continuous but not Mackey continuous.*

Indeed, as we have seen in the previous remark *if (x_n^*) is $\tau(l^1, c_0)$-null, then it is $\sigma(l^1, l^\infty)$-null too*. But x^{**} is not Mackey continuous and $x^{**}_{\rceil nB_{E^*}}$ is not Mackey continuous for some n, and then for all n. As a consequence $\tau(l^1, c_0)_{\rceil nB_{E^*}}$ is not metrizable and therefore c_0 does not satisfy (20). In other words for all K absolutely convex and weakly compact subset of c_0 there is F weakly compact and $\epsilon > 0$ such that for every $m \in \mathbb{N}$ we have

$$F \nsubseteq mK + \epsilon B_E.$$

Summarising, on the unit ball B_{l^1} the sequential continuity and the continuity for the Mackey topology $\tau(l^1, c_0)$ are clearly different.

Since SWCG Banach spaces are exactly the Banach spaces E with dual ball B_{E^*} metrizable in the Mackey topology $\tau(E^*, E)$ we arrive at the following consequence of Theorem 8

Corollary 11. *Let E be a SWCG Banach space. Let $f : E \longrightarrow \mathbb{R}_+$ be a convex proper and lower semicontinuous function with Fenchel conjugate $f^* < \infty$ everywhere on E^*. $f^* : E^* \longrightarrow \mathbb{R}$ is Mackey continuous if and only if f^* is sequentially Mackey continuous.*

In particular,we have another proof of the fact that a monetary utility function

$$u : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$$

is $\tau(\mathbb{L}^\infty, \mathbb{L}^1)$-continuous if, and only if, it verifies the Lebesgue property.

For our last result, a new characterization of coherent monetary utility functions with the Lebesgue property, we shall need the new ingredients of the one-sided James theorem:

Theorem 12. *Let $u_1 : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a Fatou coherent monetary utility function. Suppose that u_1 is not the function* ess.inf. *The following are equivalent:*

(i) *u_1 is a Lebesgue monetary utility function*
(ii) *$u_1 \Box u_2$ is Fatou for all Fatou coherent utility functions u_2*
(iii) *$u_1 \Box u_2$ is Lebesgue for all Fatou coherent utility function u_2*

Proof. We already remarked that (i) implies (iii) and it is obvious that (iii) implies (ii). To complete the equivalence we need to prove the implication $(ii) \Rightarrow (i)$.

Let us suppose that u_1 is Fatou but not Lebesgue, i.e $\mathcal{S}_1$ is closed convex but not weakly compact. Because u_1 is not the essential infimum, its scenario-set is not the set of all probability measures absolutely continuous with respect to $\mathbb{P}$. We will see that there is a Fatou coherent monetary utility function, u_2 with scenario-set $\mathcal{S}_2$, such that $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ but ${\mathcal{S}_1}^{\mathbf{ba}} \cap {\mathcal{S}_2}^{\mathbf{ba}} \neq \emptyset$, and thus $u_1 \Box u_2$ is a non Fatou well defined coherent monetary utility function. We start by selecting

$$\mathbb{Q}_0 \notin \mathcal{S}_1, \mathbb{Q}_0 \ll \mathbb{P}.$$

By the one sided James Theorem 1, we know there is $\xi_0 \in \mathbb{L}^\infty$ such that

$$\int \xi_0 d\mathbb{Q}_0 < \inf \left\{ \int \xi_0 d\mathbb{Q} : \mathbb{Q} \in \mathcal{S}_1 \right\} = u_1(\xi_0)$$

and the infimum is not attained. Now take

$$\mathcal{S}_2 = \{\mathbb{Q} : \int \xi_0 d\mathbb{Q} \leq u_1(\xi_0)\}$$

$\mathcal{S}_2$ is non empty since $\mathbb{Q}_0 \in \mathcal{S}_2$, $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ because the infimum $u_1(\xi_0)$ is not attained, but $\mathcal{S}_1^{\mathbf{ba}} \cap \mathcal{S}_2^{\mathbf{ba}} \neq \emptyset$ since the infimum will be attained on the bidual; i.e. there is $\mu_0 \in \mathbf{ba}$ with $\mu_0(\xi_0) = u_1(\xi_0)$ and $\mu_0 \in \mathcal{S}_1^{\mathbf{ba}} \cap \mathcal{S}_2^{\mathbf{ba}}$. Indeed,

$$\mathcal{S}_2^{\mathbf{ba}} = \{\mu \in \mathbf{ba} : \mu(\xi_0) \leq u_1(\xi_0)\},$$

which is the closure of $\mathcal{S}_2$ (use the bipolar theorem to see this).

□

Remark 13. *Let us remark that our application of the one-sided James theorem provide us with*

$$\int \xi_0 dQ_0 < \inf \left\{ \int \xi_0 dQ : Q \in \mathcal{S}_1 \right\} = u_1(\xi_0)$$

and the infimum is not attained with

$$\text{ess.inf}(\xi_0) < u_1(\xi_0),$$

otherwise $\mathcal{S}_2$ could not be defined. Let us remark that the usual James theorem would only give $\xi_0 \in \mathbb{L}^\infty$ with infimum not attained but $\mathcal{S}_2$ could be empty, for instance if

$$\text{ess.inf}(\xi_0) = u_1(\xi_0);$$

which could not be excluded.

Remark 14. *The essential infimum acts as a neutral element when taking inf convolutions. Indeed if u_1 is* ess.inf *then for all Fatou coherent utility function u_2 we have $u_1 \Box u_2 = u_2$. Let us remind that* ess.inf *is Fatou but not Lebesgue.*

For arbitrary utility functions the characterisation is not that easy. In fact we have the following

Theorem 15. *Suppose that u_1 and u_2 are two Fatou monetary utility functions. Suppose that $c_1^{\mathbf{ba}}$ is finite for all finitely additive probability measures and is $\sigma(\mathbf{ba}, \mathbb{L}^\infty)$ continuous. Then $u_1 \Box u_2$ has the Fatou property.*

Proof. The convolution $u_1 \Box u_2$ has the Fenchel transform $c_1^{\mathbf{ba}} + c_2^{\mathbf{ba}}$. We must show that the epigraph of $c_1 + c_2$ is weak* dense in the epigraph of $c_1^{\mathbf{ba}} + c_2^{\mathbf{ba}}$. Take a finitely additive probability measure μ. Because u_2 has the Fatou property there is a generalised sequence (net)

$$\{\mathbb{Q}_\alpha : \alpha \in (D, \succeq)\} \tag{21}$$

converging to μ and such that $\lim_{\alpha \in D} c_2(\mathbb{Q}_\alpha) = c_2^{\mathbf{ba}}(\mu)$. Because $c_1^{\mathbf{ba}}$ is continuous we also have for the *same* net (21) that $\lim_{\alpha \in D} c_1(\mathbb{Q}_\alpha) = c_1^{\mathbf{ba}}(\mu)$. Hence $\lim_{\alpha \in D}(c_1 + c_2)(\mathbb{Q}_\alpha) = (c_1^{\mathbf{ba}} + c_2^{\mathbf{ba}})(\mu)$. This is sufficient to show the density of the epigraphs. □

It is easy to find examples where the transform c is continuous and finite for each finitely additive probability measure. For instance we could define u using the function $c(\mu) = \sum_{n \geq 1} 2^{-n} \mu[A_n]^2$, where $(A_n)_n$ is a countable partition of Ω in nonempty sets. Such a utility function has the Fatou property but does not have the Lebesgue property.

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