On a Class of Law Invariant Convex Risk Measures*

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Abstract

We consider the class of law invariant convex risk measures with robust representation $\rho_{h,p}(X) = \sup_f \int_0^1 [AV@R_s(X)f(s) - f^p(s)h(s)]ds$, where $1 \leq p < \infty$ and h is a positive and strictly decreasing function. The supremum is taken over the set of all Radon Nikodym derivatives corresponding to the set of all probability measures on $(0,1]$ which are absolutely continuous with respect to the Lebesgue measure. We provide necessary and sufficient conditions for the position X such that $\rho_{h,p}(X)$ is real-valued and the supremum is attained. Using variational methods, an explicit formula for the maximizer is given. We exhibit two examples of such risk measures and compare them to the average value at risk.

Key words: Law Invariant Convex Risk Measures, Robust Representation, Variational Methods

1 Introduction

Risk assessment is a fundamental activity for both regulators and agents in financial markets. A formal and axiomatic characterization of *coherent risk measures* has been initiated by Artzner, Delbaen, Eber and Heath [1, 2]. Since then, risk measures have been generalized in several directions. Föllmer and Schied [13] as well as Frittelli and Rosazza Gianin [15] introduced the concept of *convex risk measures*, which naturally appear in pricing and hedging problems in incomplete markets and serve as building blocks for the

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variational preferences [20]. Moreover, there exist characterization and representation results for risk measures which satisfy additional properties such as law invariance, comonotonicity, additivity for independent random variables, first and second order stochastic monotonicity, et cetera. For instance, *law invariant coherent and convex risk measures* have been investigated by Kusuoka [19], Frittelli and Rosazza Gianin [16], Kunze [18] and Jouini et al. [17] and have a robust representation of the form

$$\rho(X) = \sup_{\mu} \left\{ \int AV@R_u(X)\mu(du) - \beta(\mu) \right\}. \tag{1}$$

The risk of a position, which is here modeled as a random variable X, is understood as the minimal amount of money which has to be added to the position to make it acceptable and can therefore be seen as a capital requirement. We first discuss some special cases of this robust representation. The simplest case reduces to the *average value at risk* $AV@R_u$ with risk aversion coefficient $u \in (0, 1]$ corresponding to the penalty function β, whose domain is concentrated on the Dirac measure $\mu = \delta_u$. Another subclass of (1) are the *distortion risk measures* of the form $\rho(X) = \int AV@R_u(X)\mu(du)$. There, the penalty function β is concentrated on the probability measure μ, which corresponds to an average over different average value at risks weighted according to the measure μ. Distortion risk measures are widely used in practice and have in addition an intuitive representation of the form $\rho(X) = -\int_{\mathbb{R}} xd(\psi \circ F_X)(x)$, where ψ is a concave distortion function which is in one-to-one relation with the measure μ, and $\int_{\mathbb{R}} xd(\psi \circ F_X)(x)$ is the expectation of the distorted distribution function $\psi \circ F_X$. Any distortion risk measure $\rho_\mu(X) = \int AV@R_u(X)\mu(du)$ is positive homogeneous, that is, the capital requirement $\rho_\mu(\lambda X)$ for the position λX is $\lambda\rho_\mu(X)$ for any $\lambda > 0$.

For general risk measures of the form (1) the optimal weighting measure μ_X (if it exists) for which $\rho(X) = \int AV@R_u(X)\mu_X(du) - \alpha(\mu_X)$ depends on X. In this paper, we study a class of convex risk measures for which the weighting measures μ_X are absolutely continuous with respect to the Lebesgue measure and can be computed by use of variational methods. More precisely, we consider the subclass of risk measures of the form (1) with robust representation

$$\rho_{h,p}(X) = \sup_{f} \left\{ \int AV@R_u(X)f(u)du - \int f^p(u)h(u)du \right\},$$

for the penalty function $\beta(f) = \int f^p(u)h(u)du$. The supremum is taken over all probability densities on $(0, 1]$. The class is parameterized by a constant $1 < p < \infty$ and a nondecreasing function $h : (0, 1] \to \mathbb{R}_+$. We provide growth conditions on h at the origin such that $\rho_{h,p}(X)$ is real-valued. For instance, in case that $h(s) = s^{-\alpha}$ for some $\alpha > 1$ it follows that the respective risk measure is real-valued on the vector space of all integrable random variables. Moreover, we derive conditions on the function h such that the supremum is attained at some density f_X. It turns out, that for increasing potential losses of X the respective weighting density f_X typically concentrates more and more at zero. The sensitivity of the weighting measure f_X in terms of the size of the potential losses of X can be controlled by the constant p and the function $h : (0, 1] \to (0, \infty)$.

The risk measures $\rho_{h,p}$ are convex, second order stochastically monotone (Corollary 4.59 in [14]) and law invariant. As a consequence of Theorem 3.2 below, the class is in general not strictly convex, but its penalty functions are strictly convex, which is useful for uniqueness considerations. Any risk measure $\rho_{h,p}$ can be viewed as a distortion risk measure whose distortion function depends on the size of the evaluated position. Under adequate technical conditions, these weighting measures are given in closed form solutions. This makes the class analytically tracktable and extends the well-known examples of law invariant convex risk measures such as the entropic risk measure, the optimized certainty equivalents [3] and some parametric families of risk measures [6].

While the classical risk measure theory is developed for bounded random variables, Filipović and Svindland [12] and Cheridito and Li [5] studied extensions of convex risk measures from bounded to unbounded random variables. By definition, the risk measures $\rho_{h,p}$ are well-defined for unbounded positions, for which the negative part is integrable. For technical simplifications the main results are however stated for integrable random variables.

The paper is organized as follows. In Section 2, we provide necessary and sufficient conditions such that $\rho_{h,p}(X)$ is real-valued. The conditions immediately provide vector spaces on which $\rho_{h,p}$ is real-valued. From the computational point of view, we are interested in the existence and the shape of the maximizer of (2) because it allows for explicit computation of $\rho_{h,p}$. In Section 3, we give a simple sufficient condition, which guarantees the existence of such a maximizer, provided that $\rho_{h,p}(X)$ is real-valued. Once the existence of a maximizer is assured, we compute it by use of variational methods and provide a closed form solution. The class of risk measures (2) is fairly broad and includes, for instance, all $\rho_{h,p}$ for the two-parameter family $h(u) = u^{-\alpha}(1-u)^{\eta}$, $\alpha > 0$, $\eta \geq 0$. This subclass is treated in detail in Subsection 2.2, and the cases $p = 1$ and $p = 2$ are computed and illustrated in Section 4 for families of normal random variables.

2 Finiteness of the Risk Measure $\rho_{h,p}$

In this section, we give necessary and sufficient condition on the position X such that $\rho_{h,p}(X) \in \mathbb{R}$. We then determine vector spaces on which $\rho_{h,p}$ is real-valued and illustrate the results with some examples. We first introduce the formal setting for our results. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space. We identify random variables which coincide almost surely. Inequalities between random variables are understood in the almost sure sense. $L^0(\Omega)$ consists of the set of all real-valued random variables on $(\Omega, \mathcal{F})$. For $1 \leq p \leq \infty$, we denote by $L^p(\Omega)$ the set of random variables $X \in L^0(\Omega)$ with finite L^p-norm $||X||_p := \mathbb{E}[|X|^p]^{1/p}$ for $1 \leq p < \infty$ and $||X||_\infty := \text{ess.inf}\{m \in \mathbb{R} \mid m \geq |X|\}$. We will also deal with the probability space $((0,1], \mathcal{B}(0,1], \lambda)$, where $\mathcal{B}(0,1]$ denotes the Borel sigma-algebra on the interval $(0,1]$ and λ is the Lebesgue measure. $\mathcal{M}_1(0,1]$ denotes the set of all probability measures on $\mathcal{B}(0,1]$. Throughout, $h : (0,1] \to \mathbb{R}_+$ is a strictly decreasing function. By $L^p(\lambda)$ and $L^p(h)$ we denote the vector spaces of all functions on $(0,1]$ with finite L^p-norm with respect to the measures λ and $h(x)\lambda(dx)$, respectively.

Further, $F_X(x) := \mathbb{P}[X \le x]$ denotes the cumulative distribution function of $X \in L^0(\Omega)$ with respective (upper) quantile function $q_X(s) := \inf\{x \mid F_X(x) > s\}$. Note that X on $(\Omega, \mathcal{F}, \mathbb{P})$ and q_X on $((0,1], \mathcal{B}(0,1], \lambda)$ have the same law. For $X \in L^1(\Omega)$, the average value at risk with risk aversion coefficient $u \in (0,1]$ is defined as

$$AV@R_u(X) := -\frac{1}{u}\int_0^u q_X(s)ds,$$

so that the function $u \mapsto AV@R_u(X)$ is decreasing and $AV@R_1(X) = \mathbb{E}[-X]$.

The risk measures studied in this paper have a robust representation of the form

$$\rho_{h,p}(X) = \sup_{f \in \mathcal{M}_a(0,1] \cap L^p(h)} \int_0^1 \left[AV@R_u(X)f(u) - f^p(u)h(u)\right] du - C, \quad X \in L^1(\Omega), \tag{2}$$

where $1 \le p < \infty$ and $\mathcal{M}_a(0,1]$ denotes the set of all probability measures $\mu \in \mathcal{M}_1(0,1]$ which are absolutely continuous with respect to the Lebesgue measure λ. Absolutely continuous measures $\mu \in \mathcal{M}_a(0,1]$ are identified with their Radon-Nikodym densities $d\mu/d\lambda$. Finally, $C \in \mathbb{R}$ is a normalizing constant guaranteeing $\rho_{h,p}(0) = 0$.

Theorem 2.1 *Let $X \in L^1(\Omega)$, $h : (0,1] \to (0,\infty)$ strictly decreasing and $1 < p < \infty$ with conjugate $q = p/(p-1)$. Then, $\rho_{h,p}(X) \in \mathbb{R}$ if and only if*

$$\int_0^{1-\varepsilon} h(u)^{1-q} \left|AV@R_u(X)\right|^q du < \infty, \quad \text{for some } 0 < \varepsilon < 1. \tag{3}$$

Proof. *Step 1.* Since $\rho_{h,p}(X+m) = \rho_{h,p}(X) - m$, $AV@R_u(X+m) = AV@R_u(X) - m$ for all $m \in \mathbb{R}$ and $AV@R_u(X) \ge \mathbb{E}[-X]$, we may assume w.l.o.g. that $AV@R_u(X) \ge 0$ for all $u \in (0,1]$. The vector space $E := L^p(h) \cap L^1(\lambda)$ endowed with the norm

$$|||X||| = ||X||_{L^p(h)} + ||X||_{L^1(\lambda)}, \quad X \in E,$$

is a Banach space. The mapping $i : E \to L^p(h) \times L^1(\lambda)$, $f \mapsto (f,f)$ shows that E is isometric isomorph to a closed subspace of $L^p(h) \times L^1(\lambda)$. The dual space of $L^p(h) \times L^1(\lambda)$ is isometric isomorph to $L^q(h) \times L^\infty(\lambda)$. Hence, the elements of the dual space of $i(E)$ are $(k_1, k_2) \in L^q(h) \times L^\infty(\lambda)$ where (k_1, k_2) and $(\tilde{k}_1, \tilde{k}_2)$ are identified if

$$\int_0^1 [k_1(u)h(u) + k_2(u)]f(u)du = \int_0^1 [\tilde{k}_1(u)h(u) + \tilde{k}_2(u)]f(u)du, \quad \text{for every } f \in E.$$

This shows that the dual E^* of E is a quotient space with elements $k(u) = k_1(u)h(u) + k_2(u)$, $k_1 \in L^q(h)$, $k_2 \in L^\infty(\lambda)$. Any $k \in E^*$ defines a linear mapping on E given by $\int_0^1 k(u)f(u)du$, $f \in E$. The dual norm of $k \in E^*$ is

$$\begin{aligned} |||k|||_* &= \inf_{k = k_1 \cdot h + k_2} \left(||k_1||_{L^q(h)} + ||k_2||_{L^\infty(\lambda)}\right) \\ &= \inf_{k_2 \in L^\infty(\lambda)} \left\{ \left(\int_0^1 \left(\frac{|k(u) - k_2(u)|}{h(u)} \right)^q h(u)du \right)^{1/q} + ||k_2||_{L^\infty(\lambda)} \right\}. \end{aligned}$$

In the case that $k \geq 0$ the dual norm simplifies to

$$|||k|||_* = \inf_{s>0} \left\{ \left(\int_0^1 h(u)^{1-q} (k(u) - s)_+^q du \right)^{1/q} + s \right\}. \tag{4}$$

Step 2. We show that (3) holds if and only if the mapping $A : f(u) \mapsto AV@R_u(X)f(u)$ is a bounded function from E to $L^1(\lambda)$.

Indeed, suppose that (3) holds for some $0 < \varepsilon < 1$. Then, there is $K \in \mathbb{R}$ such that $\int_0^{1-\varepsilon} h(u)^{1-q} AV@R_u(X)^q du \leq K^q$ and $||AV@R_u(X)||_{L^\infty(1-\varepsilon,1)} \leq K$. Hölder's inequality implies the boundednes of A as

$$\begin{aligned} ||Af||_{L^1(\lambda)} &= \int_0^{1-\varepsilon} |f(u)| \, AV@R_u(X) du + \int_{1-\varepsilon}^1 |f(u)| \, AV@R_u(X) du \\ &\leq \int_0^{1-\varepsilon} |f(u)| \frac{1}{h(u)} AV@R_u(X) h(u) du + K||f||_{L^1(\lambda)} \\ &\leq \left(\int_0^{1-\varepsilon} |f(u)|^p \, h(u) du \right)^{1/p} \left(\int_0^{1-\varepsilon} \left(\frac{1}{h(u)} AV@R_u(X) \right)^q h(u) du \right)^{1/q} + K||f||_{L^1(\lambda)} \\ &\leq K||f||_{L^p(h)} + K||f||_{L^1(\lambda)} = K|||f|||. \end{aligned}$$

Conversely, suppose that $A : E \to L^1(\lambda)$ is bounded. Hence, its adjoint $A^* : L^\infty(\lambda) \to E^*$ is bounded. By definition of the adjoint operator A^*,

$$\int_0^1 g(u)[Af(u)]du = \int_0^1 [A^* g(u)] f(u) du, \quad \text{for all } f \in E, \, g \in L^\infty(\lambda),$$

it follows

$$A^* g(u) = g(u) AV@R_u(X), \quad g \in L^\infty(\lambda).$$

Since A^* is bounded, there is $K \in \mathbb{R}_+$ such that

$$|||A^* g|||_* \leq K||g||_{L^\infty(\lambda)}, \quad \text{for all } g \in L^\infty(\lambda). \tag{5}$$

In particular, for $g = 1$ we have $A^* g \geq 0$ so that by (4) and (5) it follows

$$\inf_{s>0} \left\{ \left(\int_0^1 h(u)^{1-q} \left(AV@R_u(X) - s \right)_+^q du \right)^{1/q} + s \right\} \leq K,$$

and therefore (3).

Step 3. We finally show that $\rho_{h,p}(X) \in \mathbb{R}$ if and only if (3) holds.

Indeed, suppose that (3) is satisfied. Due to Step 2, the mapping A is bounded, whence there is $K \in \mathbb{R}$ such that for all $f \in \mathcal{M}_a(0,1] \cap L^p(h)$

$$\begin{aligned} \int_0^1 [AV@R_u(X) f(u) - f^p(u) h(u)] du &\leq K|||f||| - ||f||_{L^p(h)}^p \\ &= K \left(||f||_{L^p(h)} + ||f||_{L^1(\lambda)} \right) - ||f||_{L^p(h)}^p. \end{aligned} \tag{6}$$

Choose a maximizing sequence $f_k \in \mathcal{M}_a(0,1] \cap L^p(h)$ for the right hand side of (2). The sequence (f_k) is bounded in $L^p(h)$. Indeed, if not, it follows $||f_k||_{L^p(h)} \to \infty$ for $k \to \infty$. Then, $K||f_k||_{L^p(h)} - ||f_k||^p_{L^p(h)}$ tends to minus infinity, which in view of (6) contradicts the fact that f_k is a maximizing sequence for (2). Hence, $||f_k||_{L^p(h)}$ is bounded and (6) yields $\rho_{h,p}(X) \in \mathbb{R}$.

Conversely, let us assume that (3) does not hold. Due to Step 2, A is not bounded. Hence, there exists a sequence $f_k \in E$ with $|||f_k||| \to 0$ and $\int_0^1 AV@R_u(X) f_k(u) du \to \infty$ for $k \to \infty$. We can assume that $f_k \geq 0$, otherwise we replace f_k by $|f_k|$. There is a subsequence (f_k), still denoted by (f_k), such that $||f_k||_{L^1(\lambda)} \leq 2^{-k}$ and $||f_k||_{L^p(h)} \leq 2^{-k}$. For the normalized sequence

$$F_k := \frac{\sum_{j=1}^k f_j}{||\sum_{j=1}^k f_j||_{L^1(\lambda)}} \in \mathcal{M}_a(0,1],$$

it follows

$$\sup_{k\in\mathbb{N}} \left\{ \int_0^1 F_k^p(u) h(u) du \right\} \leq \sup_{k\in\mathbb{N}} \left\{ \frac{1}{||f_1||^p_{L^1(\lambda)}} \left(\sum_{j=1}^k ||f_j||_{L^p(h)} \right)^p \right\} < \infty, \tag{7}$$

showing that $F_k \in L^p(h)$. Moreover,

$$\int_0^1 AV@R_u(X) F_k(u) du = \frac{1}{||\sum_{j=1}^k f_j||_{L^1(\lambda)}} \sum_{j=1}^k \int_0^1 AV@R_u(X) f_j(u) du \to \infty \quad \text{as } k \to \infty. \tag{8}$$

Hence, (2), (7) and (8) imply

$$\rho_{h,p}(X) \geq \liminf_{k\to\infty} \left(\int_0^1 \left[AV@R_u(X) F_k(u) - F_k^p(u) h(u) \right] du - C \right) = \infty.$$

This shows $\rho_{h,p}(X) = \infty$ ans the proof is completed. □

Remark 2.1 *The set* $\{X \in L^1(\Omega) : X$ *satisfies* $(3)\}$ *is a convex cone. Moreover, in case that* $\int_0^1 h(u)^{1-q} du < \infty$, *the condition (3) is equivalent to* $\int_0^1 h(u)^{1-q} |AV@R_u(X)|^q du < \infty$, *which for instance is satisfied if* h *is strictly bounded away from zero.*

2.1 Sufficient conditions for (3)

While (3) is a condition in terms of $AV@R_u(X)$, we present here sufficient conditions for (3) involving only the quantile function q_X. These sufficient conditions will be applied in Subsection 2.2 below to derive vector spaces on which $\rho_{h,p}$ is real-valued defined. The first condition reads as follows.

Proposition 2.2 *Let $X \in L^1(\Omega)$, $h : (0,1] \to (0,\infty)$ strictly decreasing and $1 < p < \infty$ with conjugate $q = p/(p-1)$. A sufficient condition for (3) is*

$$\int_0^{1-\varepsilon} |q_X(u)| \left(\int_u^{1-\varepsilon} h(s)^{1-q} s^{-q} ds \right)^{1/q} du < \infty \quad \text{for some } 0 < \varepsilon < 1. \tag{9}$$

Proof. As shown in the proof of Theorem 2.1 condition (3) is equivalent to the boundedness of the mapping $A : f(u) \mapsto AV@R_u(X)f(u)$ from E to $L^1(\lambda)$. We therefore have to show that there exist constants $c_1, c_2 \in \mathbb{R}_+$ such that

$$\int_0^1 |AV@R_u(X)f(u)| \; du \leq c_1 ||f||_{L^p(h)} + c_2 ||f||_{L^1(\lambda)}, \quad \text{for all } f \in E.$$

To this end, we decompose the integral in

$$\int_0^1 |AV@R_u(X)f(u)| \; du = \int_0^{1-\varepsilon} |AV@R_u(X)f(u)| + \int_{1-\varepsilon}^1 |AV@R_u(X)f(u)| \, .$$

Fubini's theorem and Hölder's inequality imply

$$\begin{aligned} \int_0^{1-\varepsilon} |AV@R_u(X)f(u)| \, du &\leq \int_0^{1-\varepsilon} \left(\frac{1}{s} \int_0^s |q_X(u)| \, du \right) |f(s)| \, ds \\ \leq \int_0^{1-\varepsilon} |q_X(u)| \left(\int_u^{1-\varepsilon} \frac{|f(s)|}{s} ds \right) du &= \int_0^{1-\varepsilon} |q_X(u)| \left(\int_u^{1-\varepsilon} \frac{|f(s)|}{h(s)s} h(s) ds \right) du \\ \leq \int_0^{1-\varepsilon} |q_X(u)| \left(\int_0^1 h(s) \, |f(s)|^p \, ds \right)^{1/p} &\left(\int_u^{1-\varepsilon} h(s)^{1-q} s^{-q} ds \right)^{1/q} du \\ = ||f||_{L^p(h)} \int_0^{1-\varepsilon} |q_X(u)| \left(\int_u^{1-\varepsilon} h(s)^{1-q} s^{-q} ds \right)^{1/q} &du = c_1 ||f||_{L^p(h)} \end{aligned}$$

and

$$\int_{1-\varepsilon}^1 |f(u)| \, |AV@R_u(X)| \, du \leq ||f||_{L^1(\lambda)} ||AV@R_u(X)||_{L^\infty(1-\varepsilon,1)} = c_2 ||f||_{L^1(\lambda)}.$$

□

Based on the Muckenhoupt-Wheeden inverse Hölder theory, we provide another sufficient condition for (3), stated in the following proposition.

Proposition 2.3 *Let $X \in L^1(\Omega)$, $h : (0,1] \to (0,\infty)$ strictly decreasing and $1 < p < \infty$ with conjugate $q = (p-1)/p$. Suppose*

$$\sup_{0<u\leq 1-\varepsilon} \frac{1}{uh(u)} \int_0^u h(s) ds < \infty, \quad \text{for some } 0 < \varepsilon < 1. \tag{10}$$

Then, there is $K \in \mathbb{R}$ such that

$$\int_0^{1-\varepsilon} h(u)^{1-q} \, |AV@R_u(X)|^q \, du \leq K \int_0^{1-\varepsilon} h(u)^{1-q} \, |q_X(u)|^q \, du, \tag{11}$$

i.e., $\int_0^{1-\varepsilon} h(u)^{1-q} \, |q_X(u)|^q \, du < \infty$ is a sufficient condition for (3).

Proof. We first introduce a filtration $(\mathcal{F}_u)_{0\le u\le 1-\varepsilon}$ on the probability space $((0,1-\varepsilon],\mathcal{B}(0,1-\varepsilon],\tilde{\lambda})$, $\tilde{\lambda}:=\lambda/(1-\varepsilon)$, where the time is supposed to run backwards from $1-\varepsilon$ to 0. For $0<u\le 1-\varepsilon$, the sigma-field $\mathcal{F}_u$ is generated by the atom $(0,u)$ and the Borel sigma-field on $[u,1-\varepsilon]$. At time zero we set $\mathcal{F}_0=\mathcal{B}(0,1-\varepsilon]$. Let Y_0 be an $\mathcal{F}_0$-measurable, $\tilde{\lambda}$-integrable function and define the martingale

$$Y_u(s) := \mathbb{E}_{\tilde{\lambda}}[Y_0(s) \mid \mathcal{F}_u], \quad 0\le u\le 1-\varepsilon,$$

with respective maximum function

$$Y^*(s) = \sup_{0\le u\le 1-\varepsilon} |Y_u(s)| \, .$$

Let $\nu \ll \tilde{\lambda}$ denote a probability measure on $\mathcal{B}(0,1-\varepsilon]$ with respective density process $Z_u := \mathbb{E}_{\tilde{\lambda}}\left[\frac{d\nu}{d\tilde{\lambda}} \mid \mathcal{F}_u\right]$. If $\sup_{0<u\le 1-\varepsilon} Z_{u-}/Z_u<\infty$, then Proposition 1' in [11] (see also [4]) stats that

$$\sup_{0<u\le 1-\varepsilon} \mathbb{E}_{\tilde{\lambda}}\left[\left(\frac{Z_u}{Z_0}\right)^{\frac{1}{q-1}} \mid \mathcal{F}_u\right] < \infty \quad \text{implies} \quad \mathbb{E}_\nu\left[(Y^*)^q\right] \le K\mathbb{E}_\nu\left[|Y_0|^q\right] \tag{12}$$

for some constant $K\in\mathbb{R}_+$. We will apply (12) on the $\tilde{\lambda}$-integrable function $Y_0=q_X(u)$ and the density $Z_0 = c\,h(u)^{1-q}$, where $c\in\mathbb{R}_+$ is a normalizing constant guaranteeing $\int_0^{1-\varepsilon} c\,h(u)^{1-q}du=1$. Then

$$Z_u(v) = \begin{cases} \frac{c}{u}\int_0^u h(s)^{1-q}ds & \text{if } v\in[0,u) \\ c\,h(v)^{1-q} & \text{if } v\in[u,1-\varepsilon] \end{cases}$$

and therefore

$$\frac{Z_u}{Z_0}(v) = \begin{cases} \frac{1}{uh(v)^{1-q}}\int_0^u h(s)^{1-q}ds & \text{if } v\in[0,u) \\ 1 & \text{if } v\in[u,1-\varepsilon] \end{cases} .$$

This shows

$$\left(\frac{Z_u}{Z_0}(v)\right)^{\frac{1}{q-1}} = \begin{cases} h(v)\left(\frac{1}{u}\int_0^u h(s)^{1-q}ds\right)^{\frac{1}{q-1}} & \text{if } v\in[0,u) \\ 1 & \text{if } v\in[u,1-\varepsilon] \end{cases} . \tag{13}$$

Since $u\mapsto Z_u(v)$ is left-continuous, it follows $\sup_{0<u\le 1-\varepsilon} Z_{u-}/Z_u \le 1$. The inequality $h(s)^{1-q}\le h(u)^{1-q}$ for all $0<s\le u$ and (13) imply

$$\begin{aligned}\mathbb{E}_{\tilde{\lambda}}\left[\left(\frac{Z_u}{Z_0}\right)^{\frac{1}{q-1}} \mid \mathcal{F}_u\right] &\le \max\left\{\left(\frac{1}{u}\int_0^u h(v)dv\right)\left(\frac{1}{u}\int_0^u h(s)^{1-q}ds\right)^{\frac{1}{q-1}}, 1\right\} \\ &\le \max\left\{\frac{1}{uh(u)}\int_0^u h(v)dv, 1\right\}\end{aligned}$$

which by (10) is bounded. Hence, for $Z_0 = c\,h(u)^{1-q}$ the left hand side of (12) is verified. For $Y_0 := q_X$ with corresponding martingale

$$Y_u(v) = \begin{cases} \frac{1}{u}\int_0^u q_X(s)ds & \text{on } [0,u) \\ q_X(v) & \text{on } [u, 1-\varepsilon] \end{cases},$$

it follows

$$Y^*(v) \geq \left| \frac{1}{v} \int_0^v q_X(s)ds \right| = |AV@R_v(X)|,$$

and (12) yields

$$\mathbb{E}_{\tilde{\lambda}}\left[c\,h(s)^{1-q}\,|AV@R_s(X)|^q\right] \leq \mathbb{E}_{\tilde{\lambda}}\left[c\,h(s)^{1-q} Y^*(s)^q\right] \leq K\mathbb{E}_{\tilde{\lambda}}\left[c\,h(s)^{1-q}\,|q_X(s)|^q\right].$$

This shows

$$\int_0^{1-\varepsilon} h(u)^{1-q}\,|AV@R_u(X)|^q\,du \leq K\int_0^{1-\varepsilon} h(u)^{1-q}\,|q_X(u)|^q\,du.$$

□

2.2 The example $h(u) = u^{-\alpha}\,(1-u)^\eta$

In the following we consider the example $h(u) = u^{-\alpha}\,(1-u)^\eta$, where $\alpha > 0$, $\eta \geq 0$. By use of Proposition 2.2 and Proposition 2.3, we derive vector spaces on which $\rho_{h,p}$ is real-valued. Troughout, we assume that $0 < p < \infty$ with conjugate $q = p/(p-1)$.

Corollary 2.4 *Suppose $h(u) = u^{-\alpha}\,(1-u)^\eta$, $\alpha > 1$, $\eta \geq 0$, and $X \in L^1(\Omega)$. Then, the condition (3) holds.*

Proof. We show that (9) holds and conclude by Proposition 2.2. There is a constant $K > 0$ such that $0 < K \leq (1-s)^\eta \leq 1$ on $(0, 1-\epsilon)$. Since $\alpha(q-1) - q > -1$, it follows

$$\int_u^{1-\varepsilon} h(s)^{1-q} s^{-q} ds = \int_u^{1-\varepsilon} \left(s^{-\alpha}\,(1-s)^\eta\right)^{1-q} s^{-q} ds \leq K^{1-q} \int_u^{1-\varepsilon} s^{\alpha(q-1)-q} ds.$$

Hence $\sup_{u\in(0,1-\varepsilon)} \int_u^{1-\varepsilon} h(s)^{1-q} s^{-q} ds < \infty$ and condition (9) follows. □

Corollary 2.5 *Suppose $h(u) = u^{-1}\,(1-u)^\eta$, $\eta \geq 0$, and let X belong to the Orlicz space*

$$\left\{ Z : \mathbb{E}\left[|Z|\,(\log(1+|Z|))^{1/q}\right] < \infty \right\}. \tag{14}$$

Then, the condition (3) holds.

Proof. We show that (9) holds and conclude by Proposition 2.2. Note that

$$\begin{aligned} \int_u^{1-\varepsilon} h(s)^{1-q} s^{-q} ds &= \int_u^{1-\varepsilon} \left(s^{-1}\left(1-s\right)^{\eta}\right)^{1-q} s^{-q} ds \leq K \int_u^{1-\varepsilon} s^{(q-1)-q} ds \\ &= K \int_u^{1-\varepsilon} s^{-1} ds \leq K \log\left(u^{-1}\right), \end{aligned} \tag{15}$$

for some constant $K > 0$. Fix $\tilde{K} > K$ and define the convex function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Psi(x) := \exp\left(\frac{1}{\tilde{K}} x^q\right)$, with convex conjugate

$$\Psi^*(x) := \sup_{y \in \mathbb{R}_+} \left\{xy - \Psi(y)\right\}, \quad x \in \mathbb{R}_+. \tag{16}$$

For any $x \in \mathbb{R}_+$, the supremum in (16) is attained at $y_*(x)$ and differentiation of (16) yields

$$0 = x - \left.\frac{d}{dy}\right|_{y=y_*(x)} \Psi(y) = x - \exp\left(\frac{y_*(x)^q}{\tilde{K}}\right) \frac{q}{\tilde{K}} y_*(x)^{q-1}.$$

Hence, there exists a constant $c > 0$ such that $y_*(x) \leq c(1 + \log(1+x)^{1/q})$ for all $x \in \mathbb{R}_+$. Since X belongs to (14) and $\Psi^*(x) \leq x y_*(x)$ for all $x \in \mathbb{R}_+$, it follows

$$\mathbb{E}\left[\Psi^*(|X|)\right] \leq \mathbb{E}\left[|X|\, y_*(|X|)\right] \leq c\left(\mathbb{E}\left[|X|\right] + \mathbb{E}\left[|X| \log(1+|X|)^{1/q}\right]\right) < \infty. \tag{17}$$

Condition (9) finally follows from (15), (17) and the inequality $xy \leq \Psi(x) + \Psi^*(y)$ for all $x, y \in \mathbb{R}_+$. Indeed,

$$\begin{aligned} & \int_0^{1-\varepsilon} |q_X(u)| \left(\int_u^{1-\varepsilon} h(s)^{1-q} s^{-q} ds\right)^{1/q} du \\ \leq & \int_0^{1-\varepsilon} |q_X(u)| \left(K \log(u^{-1})\right)^{1/q} du \\ \leq & \int_0^{1-\varepsilon} \Psi^*\left(|q_X(u)|\right) du + \int_0^{1-\varepsilon} \exp\left(\frac{K}{\tilde{K}} \log(u^{-1})\right) du \\ \leq & \ \mathbb{E}\left[\Psi^*\left(|X|\right)\right] + \int_0^{1-\varepsilon} u^{-K/\tilde{K}} du < \infty, \end{aligned}$$

because $K/\tilde{K} < 1$ and $\int_0^1 \Psi^*(|q_X(u)|) du = \mathbb{E}\left[\Psi^*(|X|)\right]$ as the random variables X on $(\Omega, \mathcal{F}, \mathbb{P})$ and q_X on $((0,1], \mathcal{B}(0,1], \lambda)$ have the same law. $\square$

Corollary 2.6 *Suppose $h(u) = u^{-\alpha}(1-u)^{\eta}$, $0 < \alpha < 1$, $\eta \geq 0$ and $X \in L^{\gamma}(\Omega)$ with $\gamma = \frac{q}{1+\alpha(q-1)}$. Then, the condition (3) holds.*

Proof. It is straightforward to check that $h(u)$ satisfies condition (10) and Proposition 2.3 can be applied. Since $\int_0^{1-\varepsilon} |q_X(u)|^{\gamma} du < \infty$, there is $\tilde{K} > 0$ such that $u\,|q_X(u)|^{\gamma} \leq \tilde{K}$ for all $u \in (0, 1-\varepsilon)$. Moreover, $0 < K \leq (1-s)^\eta \leq 1$ on $(0, 1-\epsilon)$ for some constant $K > 0$. Hence

$$\begin{aligned}
\int_0^{1-\varepsilon} h(u)^{1-q} |q_X(u)|^q \, du &\leq K \int_0^{1-\varepsilon} u^{-\alpha(1-q)} |q_X(u)|^{q-\gamma} |q_X(u)|^\gamma \, du \\
&\leq K \int_0^{1-\varepsilon} u^{-\alpha(1-q)} \left(\frac{\tilde{K}}{u}\right)^{\frac{q-\gamma}{\gamma}} |q_X(u)|^\gamma \, du \\
&= K \tilde{K}^{\frac{q-\gamma}{\gamma}} \int_0^{1-\varepsilon} |q_X(u)|^\gamma \, du < \infty,
\end{aligned}$$

because $-\alpha(1-q) - \frac{q-\gamma}{\gamma} = 0$. $\square$

In the following table we summarize the results of the Corollaries 2.4–2.6 and provide admissible vector spaces on which $\rho_{h,p}$ is real-valued defined.

$h(s) = s^{-\alpha}(1-s)^\eta$	admissible vector space
$0 < \alpha < 1$	$L^{q/(1+\alpha(q-1))}(\Omega)$
$\alpha = 1$	$\{Z : \mathbb{E}[\lvert Z\rvert(\log(1+\lvert Z\rvert))^{1/q}] < \infty\}$
$\alpha > 1$	$L^1(\Omega)$

In the case $\eta = 0$, $h(u)$ reduces to $u^{-\alpha}$ which is considered in Section 4.

3 Existence and Characterization of the Maximizer in (2)

For computational aspects of the risk measure $\rho_{h,p}$ it is important to know, for which $X \in L^1(\Omega)$ the supremum in (2) is attained. In the second part of this section, we compute this maximizer $f_X \in \mathcal{M}_a(0,1] \cap L^p(h)$ (if it exists) by use of variational methods.

Theorem 3.1 *Let $h : (0,1] \to (0,\infty)$ be a strictly decreasing function and $1 < p < \infty$. For all $X \in L^1(\Omega)$ satisfying (3) the supremum in (2) is attained by a unique maximizer $f_X \in \mathcal{M}_a(0,1] \cap L^p(h)$ if and only if*

$$\int_0^1 h(u)^{1-q} du < \infty. \tag{18}$$

Proof. *Step 1.* The identity mapping $id : (L^p(h) \cap L^1(\lambda), ||\cdot||_{L^p(h)}) \to L^1(\lambda)$ is continuous if and only if (18) is satisfied.

Indeed, in case that (18) holds, Hölder's inequality implies

$$\int_0^1 |f(u)| \, du = \int_0^1 |f(u)| \frac{1}{h(u)} h(u) du \leq \left(\int_0^1 |f(u)|^p \, h(u) du\right)^{1/p} \left(\int_0^1 h(u)^{1-q} du\right)^{1/q},$$

showing that id is continuous. On the other hand, if id is continuous, it follows that its adjoint $id^* : L^\infty(\lambda) \to L^q(h)$ is continuous. Recall that the Hahn-Banach theorem implies that any linear continuous mapping from $(L^p(h) \cap L^1(\lambda), ||\cdot||_{L^p(h)})$ to $\mathbb{R}$ has a linear, continuous extension to $L^p(h)$, showing that the dual space of $(L^p(h) \cap L^1(\lambda), ||\cdot||_{L^p(h)})$ can be identified with $L^q(h)$. The adjoint id^* satisfies

$$\int_0^1 [id^*(f)(u)]g(u)h(u)du = \int_0^1 f(u)[id(g)(u)]du = \int_0^1 f(u)g(u)du,$$

for all $f \in L^\infty(\lambda)$ and $g \in L^p(h) \cap L^1(\lambda)$ showing that $id^*(f)(u) = f(u)/h(u)$. Since id^* is continuous there exists a constant $K > 0$ such that

$$K||g||_{L^\infty(\lambda)} \geq ||id^*(g)||_{L^q(h)} = \left(\int_0^1 |g(u)|^q\, h(u)^{-q} h(u)du\right)^{1/q} \quad \text{for all } g \in L^\infty(\lambda),$$

from which we deduce (18).

Step 2. Let us now assume that (18) holds and fix $X \in L^1(\Omega)$ for which (3) holds. The goal is to show the existence of a maximizer for the right hand side of (2). Indeed, let $f_k \in \mathcal{M}_a(0,1] \cap L^p(h)$ be a maximizing sequence for (2). Following the arguments given in the proof of Theorem 2.1 we derive from (6) that the sequence (f_k) is bounded in the reflexive Banach space $L^p(h)$. Hence, there exists a subsequence such that $f_k \rightharpoonup f_X$ weakly in $L^p(h)$ for some $f_X \in L^p(h)$. According to Mazur's lemma, there exists $\tilde{f}_k$ in the convex hull $\operatorname{conv}\{f_l \mid l \geq k\}$ such that $\tilde{f}_k \to f_X$ strongly in $L^p(h)$. The concavity of the optimization problem (2) implies that $\tilde{f}_k$ remains a maximizing sequence. Due to Step 1, the identity $id : (L^p(h) \cap L^1(\lambda), ||\cdot||_{L^p(h)}) \to L^1(\lambda)$ is continuous so that $\tilde{f}_k \to f_X$ in $L^1(\lambda)$ and $f_X \in \mathcal{M}_a(0,1]$. Hence, $\tilde{f}_k \to f_X$ in $|||\cdot|||$ and the continuity of the mapping $A : f(u) \mapsto AV@R_u(X)f(u)$ from E to $L^1(\lambda)$ (due to Step 2 in the proof of Theorem 2.1) implies that f_X is a maximizer for (2). The uniqueness of the maximizer f_X follows because $f \mapsto \int_0^1 [AV@R_u(X)f(u) - f^p(u)h(u)]\, du$ is strictly concave.

Step 3. Finally, we assume that (18) does not hold, i.e., $\int_0^1 h(u)^{1-q}du = \infty$ and show that the right hand side of (2) does not have a maximizer for the random variable $X = 0$. Indeed, due to Step 1, the identity $id : (L^p(h) \cap L^1(\lambda), ||\cdot||_{L^p(h)}) \to L^1(\lambda)$ is not continuous. Hence, there is a sequence $g_k \in L^p(h)$ with $||g_k||_{L^p(h)} \leq 2^{-k}$ and $\int_0^1 g_k(u)du \to \infty$ for $k \to \infty$. We can assume that g_k is positive and increasing (otherwise, we replace g_k by an increasing rearrangement of $|g_k|$). For the positive and increasing function

$$g := \sum_{k \geq 1} g_k \in L^p(h)$$

it follows $\int_0^1 g(u)du = \infty$. Since $\int_0^{1-\varepsilon} g(u)du < \infty$ for all $\varepsilon > 0$ and $\int_0^1 g(u)du = \infty$, there exists an increasing sequence $x_n \in (0,1)$ with $x_n \to 1$ such that for $f_k(u) := g(u)\mathbb{1}_{(x_k, x_{k+1}]}$,

$$\int_0^1 f_k(u)du = 1, \quad \text{for all } k \in \mathbb{N}.$$

The sequence (f_k) is bounded in $L^p(h)$, but it is not uniformly integrable. For $X = 0$, we derive for the right hand side of (2) that

$$\sup_{f\in\mathcal{M}_a(0,1]\cap L^p(h)} \int_0^1 [AV@R_u(0)f(u) - h(u)f^p(u)]\, du = \sup_{f\in\mathcal{M}_a(0,1]\cap L^p(h)} \int_0^1 -h(u)f^p(u)du$$
$$\geq \limsup_{k\to\infty} \int_0^1 -h(u)f_k^p(u)du = \limsup_{k\to\infty} \int_{x_k}^{x_{k+1}} -h(u)g^p(u)du = 0.$$

On the other hand, $-\int_0^1 f^p(u)h(u)du < 0$ for all $f \in \mathcal{M}_a(0,1] \cap L^p(h)$. Hence, for $X = 0$ the right hand side of (2) does not have a maximizer. □

Remark 3.1 *For $h(u) = u^{-\alpha}(1-u)^\eta$, $\alpha > 0$, $\eta \geq 0$, the condition (18) is satisfied if $0 \leq \eta < p-1$. Indeed, since $\eta < p-1 = 1/(q-1)$ where $q = p/(p-1)$, it follows*

$$\int_0^1 h(u)^{1-q}du = \int_0^1 \left(u^{-\alpha}(1-u)^\eta\right)^{1-q} du \leq \int_0^1 (1-u)^{\eta(1-q)}du < \infty.$$

In particular, for $h(u) = u^{-\alpha}$, $\alpha > 0$, the condition (18) is always satisfied.

Remark 3.2 *Theorem 3.1 is not valid for $p = 1$ as illustrated by the example in Subsection 4.1, below. The example shows that the supremum in*

$$\rho(X) = \sup_{f\in\mathcal{M}_a(0,1]\cap L^1(h)} \int_0^1 [AV@R_u(X)f(u) - u^{-\alpha}f(u)]\, du,$$

is not attained in $\mathcal{M}_a(0,1]\cap L^1(h)$, even though the condition (18) is satisfied for all $\alpha > 0$ by Remark 3.1.

Using variational methods we now compute the maximizer f_X for (2).

Theorem 3.2 *Let $1 < p < \infty$ and $h : (0,1] \to (0,\infty)$ be a strictly decreasing function for which (18) holds. For $X \in L^1(\Omega)$ satisfying (3) the unique maximizer f_X of (2) in $\mathcal{M}_a(0,1] \cap L^p(h)$ is given by*

$$f_X(u) = \left(\frac{AV@R_u(X) - \kappa}{ph(u)}\right)_+^{\frac{1}{p-1}}, \tag{19}$$

where κ is determined through $\int_0^1 f_X(u)du = 1$.

Proof. Suppose that $f_X \in \mathcal{M}_a(0,1] \cap L^p(h)$ is a maximizer for (2). For $f \in \mathcal{M}_a(0,1] \cap L^p(h)$, we consider the parameterized family $f_t := tf_X + (1-t)f$, $t \in \mathbb{R}$. By construction, $f_t \in \mathcal{M}_a(0,1] \cap L^p(h)$ for all $t \in [0,1]$ and since f_X is a maximizer it follows

$$\frac{d}{dt}\bigg|_{t=1} \left(\int_0^1 [AV@R_u(X)f_t(u) - f_t^p(u)h(u)]du\right) \geq 0.$$

Hence,

$$\int_0^1 \left(AV@R_u(X) - ph(u)f_X^{p-1}(u)\right)(f_X(u) - f(u))\,du \geq 0, \quad \text{for all } f \in \mathcal{M}_a(0,1] \cap L^p(h). \tag{20}$$

For fixed $u \in (0,1)$, the mapping

$$k \mapsto \left(\frac{AV@R_u(X) - k}{ph(u)}\right)_+^{\frac{1}{p-1}} \tag{21}$$

tends to zero and infinity as k tends to $+\infty$ and $-\infty$, respectively. Since $1/(p-1) = q-1$, (3) and (18) imply

$$\int_0^1 \left(\frac{AV@R_u(X)}{ph(u)}\right)_+^{\frac{1}{p-1}} \leq p^{1-q} \int_0^1 h(u)^{1-q} \left|AV@R_u(X)\right|^{q-1} du < \infty,$$

$$\int_0^1 \left(\frac{1}{ph(u)}\right)^{\frac{1}{p-1}} du = p^{1-q} \int_0^1 h(u)^{1-q}\,du < \infty.$$

Hence, the Lebesgue dominated and monotone convergence theorems yield

$$k \mapsto \int_0^1 \left(\frac{AV@R_u(X) - k}{ph(u)}\right)_+^{\frac{1}{p-1}} du$$

is a continuous function with limits 0 and $+\infty$ as k tends to $+\infty$ and $-\infty$, respectively. This shows the existence of $\kappa \in \mathbb{R}$ such that $\int_0^1 [(AV@R_u(X) - \kappa)/(ph(u))]_+^{1/(p-1)} du = 1$.

We finally show that

$$f_X := \left(\frac{AV@R_u(X) - \kappa}{ph(u)}\right)_+^{\frac{1}{p-1}} \tag{22}$$

is the unique solution f_X in $\mathcal{M}_a(0,1] \cap L^p(h)$ for the variational inequality (20) and is therefore the unique maximizer for (2), which in view of Theorem 3.1 exists. Indeed, in case that f_X is of the form (22), the inequality (20) becomes

$$\begin{aligned}
& \int_0^1 (AV@R_u(X) - (AV@R_u(X) - \kappa)_+)\left(\left(\frac{AV@R_u(X) - \kappa}{ph(u)}\right)_+^{\frac{1}{p-1}} - f(u)\right) du \\
= & \int_0^1 ((AV@R_u(X) - \kappa) - (AV@R_u(X) - \kappa)_+)\left(\left(\frac{AV@R_u(X) - \kappa}{ph(u)}\right)_+^{\frac{1}{p-1}} - f(u)\right) du \\
= & \int_0^1 (AV@R_u(X) - \kappa)_- f(u)\,du \geq 0, \quad \text{for all } f \in \mathcal{M}_a(0,1] \cap L^p(h).
\end{aligned}$$

Moreover, suppose there exists another $\tilde{f}_X \in \mathcal{M}_a(0,1] \cap L^p(h)$ such that $\tilde{f}_X \neq f_X$ and for which (20) is valid. Define

$$H(u) := (AV@R_u(X) - \kappa) - ph(u)(\tilde{f}_X(u))^{p-1}, \quad u \in (0,1],$$

$A_> := \{H > 0\}$, $A_0 := \{H = 0\}$ and $A_< := \{H < 0\}$. Note that $\lambda[A_>] > 0$ and $\lambda[A_<] > 0$. For the probability density

$$f(u) := c\left(2\tilde{f}_X(u)\mathbb{1}_{A_>} + \tilde{f}_X(u)\mathbb{1}_{A_0} + \frac{1}{2}\tilde{f}_X(u)\mathbb{1}_{A_<}\right), \quad u \in (0,1],$$

wheralle $c \in (0,1)$ is a normalizing constant guaranteeing $f \in \mathcal{M}_a(0,1] \cap L^p(h)$, we deduce

$$H(u)(\tilde{f}_X(u) - f(u)) = \begin{cases} < 0 & \text{on } A_> \cap \{\tilde{f}_X > 0\} \\ < 0 & \text{on } A_< \cap \{\tilde{f}_X > 0\} \\ = 0 & \text{else} \end{cases},$$

which, in view of $\lambda[(A_> \cup A_<) \cap \{\tilde{f}_X > 0\}] > 0$, is a contradiction that $\tilde{f}_X$ is the solution of the variational inequality (20). $\square$

4 Examples

In this section we consider the risk measure $\rho_{h,p}$ where $h(s) = h_\alpha(s) = s^{-\alpha}$ for some $\alpha > 0$. The goal is to illustrate how the weighting measure μ_X depends on the position X and how it concentrates more and more at zero with increasing potential losses of X. In case that $p = 1$ the optimal weighting measure μ_X is a Dirac measure, while in the second example $p = 2$ it follows from Theorem 3.2 that μ_X is absolutely continuous with respect to the Lebesgue measure with density f_X.

4.1 The case $p = 1$

We consider the example of the form (1) with penalty function $\beta(\mu) = \int_{(0,1]} h_\alpha(s)\mu(ds) - 1$. In this case,

$$\rho_\alpha(X) = \sup_{\mu \in \mathcal{M}_1(0,1]} \left(\int_{(0,1]} [AV@R_s(X) - h_\alpha(s)]\ \mu(ds) + 1\right). \tag{23}$$

We assume

$$\lim_{s \to 0} h_\alpha(s) - AV@R_s(X) = +\infty, \tag{24}$$

which for instance for normally distributed X is satisfied for all $\alpha > 0$. Since the mapping $s \mapsto AV@R_s(X) - h_\alpha(s)$ is differentiable, $s \mapsto AV@R_s(X)$, $s \mapsto h_\alpha(s)$ are decreasing and $AV@R_1(X) - h_\alpha(1) \in \mathbb{R}$, the supremum in (23) is attained for a Dirac measure $\mu = \delta_{s^*_\alpha} \in \mathcal{M}_1(0,1] \setminus \mathcal{M}_a(0,1]$ at the maximal point s^*_α of $s \mapsto AV@R_s(X) - h_\alpha(s)$. This maximizing point is implicitly given by the solution of

$$0 = \frac{d}{ds}\left(AV@R_s(X) - h_\alpha(s)\right) = -\frac{1}{s}\left(q_X(s) + AV@R_s(X)\right) + \alpha\frac{1}{s^{\alpha+1}}.$$

Thus we determine s^*_α, which is not unique in general, by the equation

$$\frac{\alpha}{(s^*_\alpha)^\alpha} = q_X(s^*_\alpha) + AV@R_{s^*_\alpha}(X).$$

For the risk measure ρ_α in (23) we deduce $\rho_\alpha(X) = AV@R_{s^*_\alpha}(X) - h_\alpha(s^*_\alpha) + 1$. In Table 1 we list the values of s^*_α, the average value at risk $AV@R_{0.01}(X)$ and the risk measure $\rho_\alpha(X)$, $X \sim \mathcal{N}(0, \sigma^2)$, for the cases $\alpha = 1, 1/2, 1/4$ and different standard deviations σ. Figure 1 shows the $AV@R_{0.01}(X)$ and the risk measures $\rho_1(X)$, $\rho_{1/2}(X)$, $\rho_{1/4}(X)$ against the standard deviation σ.

σ	$AV@R_{0.01}$	s^*_1	ρ_1	$s^*_{1/2}$	$\rho_{1/2}$	$s^*_{1/4}$	$\rho_{1/4}$
1	2.665	0.816	0.101	0.445	0.389	0.087	0.979
10	26.652	0.183	10.023	0.018	18.142	$1.2 \cdot 10^{-4}$	30.592
100	266.521	0.026	194.836	$3.7 \cdot 10^{-4}$	312.318	$4.1 \cdot 10^{-8}$	484.524

Table 1

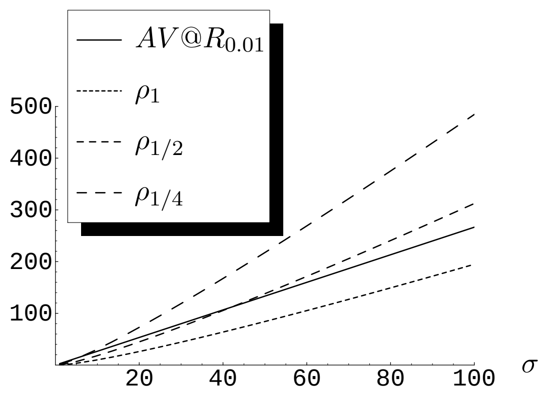

Figure 1: Comparison of the $AV@R_{0.01}(X)$ with the risk measures $\rho_1(X)$, $\rho_{1/2}(X)$, $\rho_{1/4}(X)$ for normally distributed X with mean zero and standard deviation σ. The plot shows the dependence on σ which varies in the range of the values in Table 1.

4.2 The case $p = 2$

In the following proposition we demonstrate an example of a typical risk measure of the form (2). We study the case $p = 2$ and $h(s) = s^{-\alpha}$ for $\alpha > 0$ which corresponds to the risk measure

$$\rho_\alpha(X) = \rho_{s^{-\alpha},2} = \sup_{f \in \mathcal{M}_a(0,1] \cap L^2(h)} \left\{ \int_0^1 [AV@R_s(X) f(s) - \frac{1}{s^\alpha} f^2(s)] ds - C \right\}.$$

Let us denote the primitive of an arbitrary function $f \in L^1(\lambda)$ by $f^\uparrow(s) := \int_0^s f(u) du$ and introduce the function $Q_X(s) := s^{\alpha-1} q_X^\uparrow(s)$.

Proposition 4.1 *Suppose that $X \in L^1(\Omega)$ satisfies condition (3). Let*

$$\varphi(s) := (\alpha+1)(2 + Q_X^{\uparrow}(s)) - sQ_X(s)$$

and define s_0 as the unique root $\varphi(s_0) = 0$ if it exists in $(0,1)$ or as 1 if $\varphi > 0$ on $(0,1)$. Then, for the risk measure ρ_α it follows $C = -(\alpha+1)/s_0^{\alpha+1}$ and

$$f_X(s) = \frac{1}{2}\left(\frac{(\alpha+1)s^\alpha}{s_0^{\alpha+1}}\left(2 + Q_X^{\uparrow}(s_0)\right) - Q_X(s)\right)\mathbb{1}_{\{s \le s_0\}}(s) \tag{25}$$

leading to

$$\rho_\alpha(X) = -\frac{\alpha+1}{s_0^{\alpha+1}}\left(Q_X^{\uparrow}(s_0) + \frac{3}{4}\left(Q_X^{\uparrow}(s_0)\right)^2\right) + \frac{3}{4}\int_0^{s_0}\frac{Q_X^2(s)}{s^\alpha}ds. \tag{26}$$

Proof. We have $\varphi(0) = 2\alpha + 2$ and $\varphi'(s) = s^{\alpha-1}(q_X^{\uparrow}(s) - sq_X(s))$. For $s > 0$ the second factor of φ' is negative since $(q_X^{\uparrow}(s) - sq_X(s))' = -sq_X'(s) < 0$ and $(q_X^{\uparrow}(s) - sq_X(s))|_{s=0} = q_X^{\uparrow}(0) = 0$. Therefore φ is strictly decreasing and has at most one root in $(0,1]$. By Theorem 3.2 the maximizer is given by

$$f_X(s) = \left(\frac{AV@R_s(X) - \kappa}{2h(s)}\right)\mathbb{1}_{\{s \le \tilde{s}\}}(s)$$

where $\tilde{s}$ is the unique (since $h(u) \ge 0$ and $AV@R_s(X)$ is decreasing) root of f_X if it exists in $(0,1]$ or $\tilde{s} = 1$ otherwise. The parameter κ is adjusted in dependence of $\tilde{s}$ by the condition $f_X^{\uparrow}(\tilde{s}) = 1$:

$$\begin{aligned}
f_X^{\uparrow}(\tilde{s}) &= \int_0^{\tilde{s}} \frac{s^\alpha(AV@R_s(X) - \kappa)}{2}du = \int_0^{\tilde{s}}\frac{s^\alpha(-q_X^{\uparrow}(s)/s - \kappa)}{2}du \\
&= -\frac{\kappa\tilde{s}^{\alpha+1}}{2(\alpha+1)} - \frac{Q_X^{\uparrow}(\tilde{s})}{2} = 1 \quad \Leftrightarrow \quad \kappa = -\frac{\alpha+1}{\tilde{s}^{\alpha+1}}(2 + Q_X^{\uparrow}(\tilde{s})) \\
&\Rightarrow \; f_X(s) = \frac{1}{2}\left(\frac{(\alpha+1)s^\alpha}{\tilde{s}^{\alpha+1}}\left(2 + Q_X^{\uparrow}(\tilde{s})\right) - Q_X(s)\right)\mathbb{1}_{\{s \le \tilde{s}\}}(s).
\end{aligned}$$

From $f_X(\tilde{s}) = \frac{1}{2\tilde{s}}\varphi(\tilde{s})$ we deduce $\tilde{s} = s_0$. Since $\rho_\alpha(X+m) = \rho_\alpha(X) - m$ for $m \in \mathbb{R}$, we may assume w.l.o.g. that the expected value of the random variable X equals 0. Then we have

$$\int_0^{s_0} AV@R_s(0)f_0(s)ds = 0, \quad Q_0(s) = 0 \quad \text{and} \quad Q_0^{\uparrow}(s) = 0,$$

and the condition $\rho_\alpha(0) = 0$ yields

$$C = -\int_0^{s_0}\frac{1}{s^\alpha}f_0^2(s)ds = -\frac{\alpha+1}{s_0^{\alpha+1}}.$$

The penalty function thus reads

$$\beta(f_X) = \int_0^{s_0} \frac{1}{s^\alpha} f_X^2(s)ds + C = -\frac{\alpha+1}{4s_0^{\alpha+1}}\left(Q_X^{\uparrow}(s_0)\right)^2 + \frac{1}{4}\int_0^{s_0} \frac{Q_X^2(s)}{s^\alpha}ds.$$

Furthermore

$$\begin{aligned}\int_0^{s_0} AV@R_s(X)f_X(s)ds &= -\int_0^{s_0} \frac{q_X^{\uparrow}(s)}{2s}\left(\frac{(\alpha+1)s^\alpha}{s_0^{\alpha+1}}\left(2+Q_X^{\uparrow}(s_0)\right) - Q_X(s)\right)ds \\ &= -\frac{(\alpha+1)Q_X^{\uparrow}(s_0)}{2s_0^{\alpha+1}}\left(2+Q_X^{\uparrow}(s_0)\right) + \frac{1}{2}\int_0^{s_0}\frac{Q_X^2(s)}{s^\alpha}ds.\end{aligned}$$

This leads to the following expression for the risk measure ρ_α:

$$\begin{aligned}\rho_\alpha(X) &= \int_0^{s_0} AV@R_s(X)f_X(s)ds + \beta(f_X) \\ &= -\frac{\alpha+1}{s_0^{\alpha+1}}\left(Q_X^{\uparrow}(s_0) + \frac{3}{4}\left(Q_X^{\uparrow}(s_0)\right)^2\right) + \frac{3}{4}\int_0^{s_0}\frac{Q_X^2(s)}{s^\alpha}ds.\end{aligned}$$

□

Remark 4.1 *For practical computation it may be useful to simplify the expression for f_X in the case that $s_0 < 1$ by using the identity $f_X(s_0) = 0$, which leads to $\kappa = s_0^{-\alpha}Q(s_0)$, so that*

$$f_X(s) = \frac{1}{2}\left(\left(\frac{s}{s_0}\right)^\alpha Q_X(s_0) - Q_X(s)\right)\mathbb{1}_{\{s\le\tilde{s}\}}(s).$$

In what follows we apply Proposition 4.1 to normal distributions. Normally distributed random variables X with mean m and standard deviation σ do not provide explicit expressions for the quantile. However, X satisfies condition (3) for all $\alpha > 0$. The function f_X for mean zero and different values of σ and α is shown in Figure 2. Again, the maximizer concentrates at zero as the potential losses increase, which for the normal distribution is characterized by increasing σ.

In Table 2 we list the values of s_0 (denoted by $s_{0,\alpha}$ corresponding to the different values of α), the average value at risk $AV@R_{0.01}(X)$ and the risk measure $\rho_\alpha(X)$ given in (26) for the cases $\alpha = 1, 1/2, 1/4$ and normally distributed X with mean zero and different standard deviations σ.

σ	$AV@R_{0.01}$	$s_{0,1}$	ρ_1	$s_{0,1/2}$	$\rho_{1/2}$	$s_{0,1/4}$	$\rho_{1/4}$
1	2.665	1	0.625	1	0.804	1	0.951
10	26.652	0.910	10.970	0.712	14.675	0.561	17.409
100	266.521	0.368	166.986	0.204	210.621	0.122	242.472

Table 2

Compared to the first example ($p = 1$) the risk adjusted value calculated by the method in the second example ($p = 2$) is less sensitive to variations both of α and the standard deviation σ. The graphs of $AV@R_{0.01}(X)$ and $\rho_\alpha(X)$ as functions in dependence of the standard deviation σ look qualitatively the same as the corresponding graphs in Figure 1.

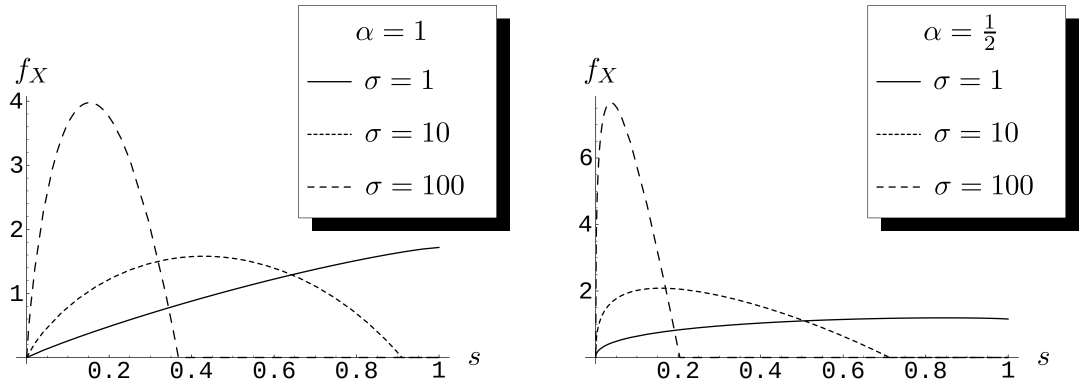

Figure 2: Function f_X given in (25) resulting from the quantile q_X of the normal distribution with mean zero and different standard deviations σ. The plots cover the cases $\alpha = 1$ and $\alpha = \frac{1}{2}$.

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