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A Note of Option Pricing for Constant Elasticity of Variance Model

by
Freddy Delbaen
Department of Mathematics, Eidegenössische Technische Hochschule Zürich
CH-8092 Zurich, Switzerland
E-mail: delbaen@math.ethz.ch
and
Hiroshi Shirakawa
Department of Industrial Engineering and Management, Tokyo Institute of Technology
2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152, Japan
E-mail: sirakawa@me.titech.ac.jp

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Freddy Delbaen
Department of Mathematics, Eidgenössische Technische Hochschule Zürich
CH-8092 Zürich, Switzerland
E-mail : delbaen@math.ethz.ch
and
Hiroshi Shirakawa†
Department of Industrial Engineering and Management, Tokyo Institute of Technology
2-12-1 Oh-Okayama, Meguro-ku, Tokyo 152, Japan
E-mail : sirakawa@me.titech.ac.jp

Abstract: We study the arbitrage free option pricing problem for constant elasticity of variance (CEV) model. To treat the stochastic aspect of the CEV model, we direct attention to the relationship between the CEV model and squared Bessel processes. Then we show the existence of a unique equivalent martingale measure and derive the Cox's arbitrage free option pricing formula through the properties of squared Bessel processes. Finally we show that the CEV model admits arbitrage opportunities when it is conditioned to be strictly positive.

Keywords: Constant Elasticity of Variance Model, Squared Bessel Process, Option Pricing, Equivalent Martingale Measure, Arbitrage.

1 Introduction

We consider a security market model of two assets, bond and stock. Let $(\Omega, (\mathcal{F}_t)_{0\leq t}, P)$ be a filtered probability space satisfying the usual condition and let W_t be a Wiener process under P. We suppose that $\mathcal{F}_t$ is generated by $\{W_u; 0 \leq u \leq t\}$, the bond price B_t increases with constant riskless return $r > 0$ and stock price process S_t has independent increments with constant elasticity of variance. That is :

$$dB_t = rB_t dt, \quad B_0 = 1, \tag{1.1}$$

$$dS_t = \mu S_t dt + \sigma S_t^\rho dW_t, \quad S_0 = s \geq 0, \tag{1.2}$$

where ρ means the elasticity of variance. This model was first considered by Cox [1] where it was called the constant elasticity of variance model (hereafter abbreviated as the CEV model).

We require the following conditions for the parameters in (1.2).

$$\text{(C.1)} \qquad 0 < \rho < 1.$$

Under Condition (C.1), the point 0 is an attainable state. As soon as the process S reaches zero. we keep it equal to zero. The reader can verify that the process so defined, still satisfies the stochastic differential equation (1.2). In doing so the point 0 becomes the absorbing state for the stock price process S_t. A straightforward analysis shows that when we want the equation (1.2) to be satisfied, this is the only way to treat the point 0.

The object of this paper is to study the existence of a unique equivalent martingale measure of the CEV model and to derive arbitrage free option pricing formula through the probabilistic analysis. Cox

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[1] has already investigated the transition probability of the stock price process for the CEV model and obtained the arbitrage free call option pricing formula through the risk neutral evaluation [2]. However it was not shown whether there is a unique equivalent martingale measure for the CEV model. The risk neutral method is a convenient approach to derive the arbitrage free option prices, but it requires only the local arbitrage free property and that is not equivalent to the arbitrage free property [4, 5]. Here we direct attention to the relationship between the CEV model and squared Bessel process which enables us to study the CEV model through the basic properties of squared Bessel process. Then we prove the existence of a unique equivalent martingale measure and derive the law of stock price process for the CEV model. Furthermore we show that the CEV model admits arbitrage opportunities when the stock price is conditioned to be strictly positive. This is the advantage of our approach since the standard risk neutral argument is useless to study the existence of arbitrage for the conditioned the CEV model. The analysis also shows that one cannot discard the possibility that a CEV process will hit 0.

The paper is organized as follows. In Section 2, we show the relationship between the CEV model and squared Bessel process. Then we prove the existence of a unique equivalent martingale measure. In Section 3, we study the law of the risk neutral stock price process and derive the arbitrage free option pricing formula. Finally in Section 4, we show the existence of arbitrage opportunities for the CEV model when the stock price is conditioned to be strictly positive.

2 Weak Solution by Squared Bessel Process

First we shall represent the weak solution of the stochastic differential equation (1.2) by a squared Bessel process with time and state changes. We denote the δ-dimensional squared Bessel process ($\delta \in \mathbf{R}$) by $X_t^{(\delta)}$. It follows the stochastic differential equation

$$dX_t^{(\delta)} = 2\sqrt{|X_t^{(\delta)}|}dW_t + \delta dt, \tag{2.1}$$

starting with $X_0^{(\delta)} = s^{\frac{2}{2-\delta}}$. Let ζ be the first passage time of 0 for the process $X_t^{(\delta)}$, i.e.,

$$\zeta = \inf\{t > 0\,;X_t^{(\delta)} = 0\}. \tag{2.2}$$

For the parameters $\nu > 0$ and $\delta < 2$, we consider a deterministic time change defined by

$$\tau_t^{(\delta,\nu)} = \frac{\sigma^2}{2\nu(2-\delta)}\left(1 - \exp\left\{-\frac{2\nu t}{2-\delta}\right\}\right). \tag{2.3}$$

The process $Y_t^{(\delta,\nu)}$ is defined as

$$Y_t^{(\delta,\nu)} = \exp\{\nu t\}\left(X^{(\delta)}_{\tau_t^{(\delta,\nu)}\wedge\zeta}\right)^{1-\frac{1}{2}\delta}. \tag{2.4}$$

From Itô's lemma, we can "easily" check that $Y_t^{(\delta,\nu)}$ follows

$$dY_t^{(\delta,\nu)} = \begin{cases} \nu Y_t^{(\delta,\nu)}dt + \sigma(Y_t^{(\delta,\nu)})^{\frac{1-\delta}{2-\delta}}dW_t^{(\delta,\nu)}, & \text{if } \tau_t^{(\delta,\nu)} \le \zeta, \\ 0, & \text{if } \tau_t^{(\delta,\nu)} > \zeta, \end{cases} \tag{2.5}$$

where $W_t^{(\delta,\nu)}$ is the Wiener process defined by

$$W_t^{(\delta,\nu)} = \int_0^{\tau_t^{(\delta,\nu)}} \frac{2-\delta}{\sqrt{\sigma^2 - 2\mu(2-\delta)v}}dW_v. \tag{2.6}$$

Let $\delta_\rho = \frac{1-2\rho}{1-\rho}$ which means $\frac{1-\delta_\rho}{2-\delta_\rho} = \rho$. Since $\rho \in (0,1)$, we have $\delta_\rho \in (-\infty, 1)$ and hence

$$\frac{2-\delta}{2\nu}\log\left(\frac{\sigma^2}{\sigma^2 - 2\nu(2-\delta)\zeta}\right) \text{ on the set } \left\{\zeta < \frac{\sigma^2}{2\nu(2-\delta)}\right\}$$

also plays a role as first passage time of the point 0 for the process $Y_t^{(\delta_\rho,\mu)}$.

Remark 2.1 *Before continuing the analysis of the relation between the Bessel square processes and the CEV model, we make the remark that for $\delta < 1$ and for $\tau_t^{(\delta,\nu)} > \zeta$, the process Y still satisfies, in a trivial way, the first equation of 2.5. The solution of the stochastic differential equation 1.2 (except for the substitution of μ by ν is therefore given by the transformation 2.4. It follows that the process S for $0 \le t < +\infty$ only uses the part of the squared Bessel process X up to time $\frac{\sigma^2}{2-\delta} = \sigma^2(1-\rho)$. Also it follows that there are two kinds of trajectories for the process S. The first kind consists of the trajectories absorbed at 0. The second kind consists of the trajectories that (at least for $\mu > 0$) will converge to $+\infty$ when $t \to +\infty$. (Just for the information of the reader we add that for $\mu \le 0$ the above analysis does not apply and that all trajectories will be absorbed by 0.) The passage time through zero is given by*

$$\zeta^S = \frac{1}{2\mu(1-\rho)}\log\left(\frac{\sigma^2(1-\rho)}{\sigma^2(1-\rho) - 2\mu\zeta}\right) \text{ on the set } \left\{\zeta < \frac{\sigma^2(1-\rho)}{2\mu}\right\}$$

and by

$$\zeta^S = +\infty \text{ on the set } \left\{\zeta \ge \frac{\sigma^2(1-\rho)}{2\mu}\right\}.$$

We can summarize the previous discussion in the following:

Theorem 2.2

$$\{S_t; 0 \le t\} \stackrel{law}{=} \{Y_t^{(\delta_\rho,\mu)}; 0 \le t\} \tag{2.7}$$

where $\stackrel{law}{=}$ means equivalence in law under the original probability measure P. □

Let us consider the risk neutral evaluation when B_t is used as a numéraire. Define the process U_t by

$$U_t = \frac{S_t}{B_t} = \exp\{-rt\}S_t. \tag{2.8}$$

From Itô's lemma, U_t follows

$$dU_t = \sigma\exp\{-(1-\rho)rt\}U_t^\rho d\tilde{W}_t \tag{2.9}$$

where

$$\tilde{W}_t = W_t + \int_0^t \theta S_v^{1-\rho}dv \tag{2.10}$$

and $\theta = \frac{\mu - r}{\sigma}$. Using the process $\tilde{W}_t$, the stock price process follows

$$dS_t = rS_tdt + \sigma S_t^\rho d\tilde{W}_t. \tag{2.11}$$

By Cameron-Martin-Maruyama-Girsanov's theorem [9, p.191], we shall consider the unique equivalent measure change candidate for $P|_{\mathcal{F}_{T\wedge\zeta^S}}$ defined by the Radon-Nikodym derivative

$$\eta_T = \exp\left\{-\theta\int_0^T S_t^{1-\rho}dW_t - \frac{\theta^2}{2}\int_0^T S_t^{2(1-\rho)}dt\right\} = \exp\left\{-\theta\int_0^{T\wedge\zeta^S} S_t^{1-\rho}dW_t - \frac{\theta^2}{2}\int_0^{T\wedge\zeta^S} S_t^{2(1-\rho)}dt\right\}, \tag{2.12}$$

under which $\{U_t; 0 \le t \le T\}$ would be martingale.

Theorem 2.3 *For all $T < \infty$, $E[\eta_T] = 1$ and hence there exists a unique P- equivalent martingale measure $\tilde{P}$ on $\mathcal{F}_T$ defined by*

$$\tilde{P}[A] = E[1_A\eta_T] \quad \textit{for} \quad A \in \mathcal{F}_T. \tag{2.13}$$

Proof. There are (at least) two ways to prove the theorem. One way is to analyse the Novikov condition for the local martingale

$$\eta_u = \exp\left\{-\theta \int_0^{u\wedge\zeta^S} S_t^{1-\rho} dW_t - \frac{\theta^2}{2}\int_0^{u\wedge\zeta^S} S_t^{2(1-\rho)} dt\right\}.$$

This is not completely possible and only works for $(1-\rho)\mu T \leq 1$. A repeated application of the Markov property allows to extend the equality $E[\eta_T] = 1$ also fortimes T that are bigger than $frac1(1-\rho)\mu$. The calculations are not difficult but are tedious. We will proceed in another way using a more qualitative approach. We first consider two stochastic differential equations. The first equation is the original equation 1.2:

$$dS_t = \mu S_t\, dt + \sigma S_t^\rho dW_t.$$

The second equation is the equation obtained when replacing μ by r, i.e.

$$dS_t = rS_t\, dt + \sigma S_t^\rho dW_t.$$

The law of the first process, a measure on the space $C[0,T]$ of continuous functions on the interval $[0,T]$, is denoted by P, the law of the second process is denoted by $\tilde{P}$. The coordinate process is denoted by S, it generates a filtration denoted by $\mathcal{F}_t$. This notation is not really misleading, it is inspired by transporting the processes defined on Ω to the canonical space $C[0,T]$.

Since we supposed that the process S generates the filtration (up to time ζ^S) we can find a $P-$Brownian motion W so that on the space $C[0,T]$ we have $dS_t = \mu S_t\, dt + \sigma S_t^\rho dW_t$. In the same way we can find a $\tilde{P}-$ Brownian motion $\tilde{W}$ so that $dS_t = rS_t\, dt + \sigma S_t^\rho d\tilde{W}_t$. The passage from P to $\tilde{P}$ is not difficult. If we define the stopping times

$$\tau_n = \inf\left\{u \mid \int_0^u S_t^{2(1-\rho)}\, dt \geq n\right\},$$

we have that on the $\sigma-$algebra $\mathcal{F}_{\tau_n}$ both measures, P and $\tilde{P}$, are equivalent. Furthermore the density of $\tilde{P}$ with respect to P is given by the random variable

$$\eta_{\tau_n} = \exp\left\{-\theta \int_0^{\tau_n} S_t^{1-\rho} dW_t - \frac{\theta^2}{2}\int_0^{\tau_n} S_t^{2(1-\rho)} dt\right\}.$$

Since $P-$almost surely we have $\int_0^T S_t^{2(1-\rho)}\, dt < \infty$, we get that for $n \to +\infty$, necessarily $\tau_n \to +\infty$. This implies that $P \ll \tilde{P}$.

Conversely the same reasoning applies to the measure $\tilde{P}$ and using that $\tilde{P}-$almost surely we have $\int_0^T S_t^{2(1-\rho)}\, dt < \infty$, we get that $\tilde{P}$ is absolutely continuous with respect to P.

As a result, we find that both measures on $C[0,T]$ are equivalent. Transporting back to Ω and using the fact that also on Ω, the filtration is generated by S, we can conclude that on Ω, the density process

$$\eta_u = \exp\left\{-\theta \int_0^{u\wedge\zeta^S} S_t^{1-\rho} dW_t - \frac{\theta^2}{2}\int_0^{u\wedge\zeta^S} S_t^{2(1-\rho)} dt\right\}.$$

is not only a local martingale but is a strictly positive martingale. Therefore necessarily $E[\eta_T] = 1$ and $\eta_T > 0$. □

3 Arbitrage Free Option Pricing

By Theorem 2.3, there always exists an P-equivalent measure $\tilde{P}$ on $\mathcal{F}_T$ so that $\tilde{W}_t$ becomes a Wiener process under the measure $\tilde{P}$. By construction, the discounted price process U_t is a local martingale under the P- equivalent measure.

From the reasoning in section 2, we immediately deduce the following

Corollary 3.1

$$\{S_t; 0 \le t\} \text{ under the measure } \tilde{P} \overset{law}{=} \{Y_t^{(\delta_\rho, r)}; 0 \le t\} \text{ under the measure } P \tag{3.1}$$

□

Before giving explicit formulas we first need to state some important results for the squared Bessel process.

Lemma 3.2 *For any $\delta \in [0, \infty)$, we have*

$$X_t^{(\delta)} \overset{law}{=} t \cdot V^{(\delta, \frac{x}{t})}, \quad x \ge 0, \quad t > 0, \tag{3.2}$$

where $X_0^{(\delta)} = x$ and $V^{(a,b)}$ means the noncentral χ^2 random variable with a (≥ 0) degrees of freedom and noncentrality parameter $b \ge 0$. That is, the density of $V^{(a,b)}$ is given by

$$f(v; a, b) = \frac{1}{2^{\frac{a}{2}}} \exp\left\{-\frac{1}{2}(b + v)\right\} v^{\frac{a}{2}-1} \sum_{n=0}^{\infty} \left(\frac{b}{4}\right)^n \frac{v^n}{n!\ \Gamma(\frac{1}{2}a + n)}. \tag{3.3}$$

Proof. Consider the Laplace transform of $V^{(a,b)}$,

$$\begin{aligned}
&E[\exp\{-\lambda V^{(a,b)}\}] \\
&= \int_{v\ge 0} \frac{1}{2^{\frac{a}{2}}} \exp\left\{-\frac{1}{2}(b + (1+2\lambda)v)\right\} v^{\frac{a}{2}-1} \sum_{n=0}^{\infty} \left(\frac{b}{4}\right)^n \frac{v^n}{n!\ \Gamma(\frac{1}{2}a+n)} dv \\
&= \frac{\exp\left\{-\frac{\lambda}{1+2\lambda} b\right\}}{(1+2\lambda)^{\frac{\delta}{2}}} \int_{v\ge 0} \frac{1}{2^{\frac{a}{2}}} \exp\left\{-\frac{1}{2}\left(\frac{b}{1+2\lambda} + v\right)\right\} v^{\frac{a}{2}-1} \sum_{n=0}^{\infty} \left(\frac{b}{4(1+2\lambda)}\right)^n \frac{v^n}{n!\ \Gamma(\frac{1}{2}a+n)} dv \\
&= \frac{\exp\{-\frac{\lambda}{1+2\lambda} b\}}{(1+2\lambda)^{\frac{b}{2}}}.
\end{aligned}$$

On the other hand, the Laplace transform of $X_t^{(\delta)}$ is given by (see [10, p.411]),

$$E[\exp\{\lambda X_t^{(\delta)}\}] = \frac{\exp\{-\frac{\lambda t}{1+2\lambda t} \frac{x}{t}\}}{(1+2\lambda t)^{\frac{\delta}{2}}} = E[\exp\{-\lambda t V^{(\delta, \frac{x}{t})}\}].$$

Since the Laplace transforms for both random variables are equal, we have (3.2). □

Yor [11, (2.c)] derived the following relationship between the squared Bessel processes $X_t^{(\delta)}$ and $X_t^{(4-\delta)}$.

Lemma 3.3 *Let $\delta \in (-\infty, 2)$ and ϕ be a function such that*

$$\lim_{x\downarrow 0} E\left[\phi\left(X_t^{(4-\delta)}\right)\left(X_t^{(4-\delta)}\right)^{\frac{\delta}{2}-1} \middle| X_0^{(4-\delta)} = x\right] < \infty. \tag{3.4}$$

Then for any and $x > 0$,

$$E[\phi(X_t^{(\delta)}) 1_{\{\zeta > t\}} | X_0^{(\delta)} = x] = x^{1-\frac{\delta}{2}} \cdot E\left[\phi\left(X_t^{(4-\delta)}\right)\left(X_t^{(4-\delta)}\right)^{\frac{\delta}{2}-1} \middle| X_0^{(4-\delta)} = x\right]. \quad \square \tag{3.5}$$

The martingale property of U_t under $\tilde{P}$ follows from Corollary 3.1 and Lemma 3.3. That is, let $\phi(x) = x^{1-\frac{\delta_\rho}{2}}$, then

$$\begin{aligned}
\tilde{E}[U_t | U_0 = u] &= \tilde{E}\left[\left(\tilde{X}^{(\delta_\rho)}_{\tau_t^{(\delta_\rho, r)} \wedge \tilde{\zeta}}\right)^{1-\frac{\delta_\rho}{2}} \middle| \tilde{X}_0^{(\delta_\rho)} = u^{\frac{2}{2-\delta_\rho}}\right] \\
&= \tilde{E}\left[\left(\tilde{X}^{(\delta_\rho)}_{\tau_t^{(\delta_\rho, r)}}\right)^{1-\frac{\delta_\rho}{2}} 1_{\{\tilde{\zeta} \ge t\}} \middle| \tilde{X}_0^{(\delta_\rho)} = u^{\frac{2}{2-\delta_\rho}}\right] \\
&= \left(u^{\frac{2}{2-\delta_\rho}}\right)^{1-\frac{\delta_\rho}{2}} \tilde{E}\left[\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_t^{(\delta_\rho, r)}}\right)^{1-\frac{\delta_\rho}{2}} \left(\tilde{X}^{(4-\delta_\rho)}_{\tau_t^{(\delta_\rho, r)}}\right)^{\frac{\delta_\rho}{2}-1} \middle| \tilde{X}_0^{(\delta_\rho)} = u^{\frac{2}{2-\delta_\rho}}\right] \\
&= u.
\end{aligned} \tag{3.6}$$

Remark 3.4 *The martingale property of the process U can also be proved by using more structural methods. However since we need the density functions later on, we did not insist on such a presentation.*

Furthermore by Theorem 2.2 and Lemma 3.2, 3.3, we can derive the probability distribution of S_T under the original measure P (see also Cox [1]).

Theorem 3.5

$$P[S_T \le x | S_0 = s] = 1 - \sum_{n=1}^{\infty} g(n+\lambda, z) G(n, w) \tag{3.7}$$

where

$$\begin{cases} \lambda &= \frac{1}{2(1-\rho)} \\ z &= \frac{s^{2(1-\rho)}}{2\tau_T^{(\delta_\rho,\mu)}} = \frac{2\mu\lambda \exp\{\frac{\mu T}{\lambda}\} s^{\frac{1}{\lambda}}}{\sigma^2(\exp\{\frac{\mu T}{\lambda}\}-1)} \\ w &= \frac{(\exp\{-\mu T\}x)^{2(1-\rho)}}{2\tau_T^{(\delta_\rho,\mu)}} = \frac{2\mu\lambda x^{\frac{1}{\lambda}}}{\sigma^2(\exp\{\frac{\mu T}{\lambda}\}-1)} \\ g(u,v) &= \frac{v^{u-1}}{\Gamma(u)} \exp\{-v\} \\ G(u,v) &= \int_{w \ge v} g(u,w) dw. \end{cases} \tag{3.8}$$

Proof. From Theorem 2.2 and Lemma 3.3,

$$\begin{aligned} & P[S_T \ge x | S_0 = s] \\ &= P\left[Y_T^{(\delta_\rho,\mu)} \ge x \,\middle|\, Y_0^{(\delta_\rho,\mu)} = s \right] \\ &= P\left[X^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,\mu)} \wedge \zeta} \ge (\exp\{-\mu T\}x)^{\frac{2}{2-\delta_\rho}} \,\middle|\, X_0^{(\delta_\rho)} = s^{\frac{2}{2-\delta_\rho}} \right] \\ &= E\left[1\left\{ X^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,\mu)}} \ge (\exp\{-\mu T\}x)^{\frac{2}{2-\delta_\rho}} \right\} 1\{\zeta \ge T\} \,\middle|\, X_0^{(\delta_\rho)} = s^{\frac{2}{2-\delta_\rho}} \right] \\ &= s \cdot E\left[1\left\{ X^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,\mu)}} \ge (\exp\{-\mu T\}x)^{\frac{2}{2-\delta_\rho}} \right\} \left(X^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,\mu)}} \right)^{\frac{\delta_\rho}{2}-1} \,\middle|\, X_0^{(4-\delta_\rho)} = s^{\frac{2}{2-\delta_\rho}} \right] \end{aligned} \tag{3.9}$$

And from Lemma 3.2,

$$\begin{aligned} & E\left[\left(X^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,\mu)}} \right)^{\frac{\delta_\rho}{2}-1} 1\left\{ X^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,\mu)}} \ge (\exp\{-\mu T\}x)^{\frac{2}{2-\delta_\rho}} \right\} \,\middle|\, X_0^{(4-\delta_\rho)} = s^{\frac{2}{2-\delta_\rho}} \right] \\ &= \left(\tau_T^{(\delta_\rho,\mu)} \right)^{\frac{\delta_\rho}{2}-1} E\left[\left(V^{(4-\delta_\rho,2z)} \right)^{\frac{\delta_\rho}{2}-1} 1\{V^{(4-\delta_\rho,2z)} \ge 2w\} \right] \\ &= \frac{\exp\{-z\}}{\left(2\tau_T^{(\delta_\rho,\mu)} z \right)^{1-\frac{\delta_\rho}{2}}} \sum_{n=0}^{\infty} \frac{z^{n+1-\frac{\delta_\rho}{2}}}{\Gamma(n+2-\frac{\delta_\rho}{2})} \int_{v \ge w} \frac{v^n \exp\{-v\}}{n!} dv \\ &= \frac{1}{s} \sum_{n=1}^{\infty} g(n+\lambda, z) G(n, w). \end{aligned} \tag{3.10}$$

Thus from (3.9) and (3.10), (3.7) is obtained. □

As the special case of Theorem 3.5, we can derive the probability that $X_t^{(\delta)}$ hits the point 0 for the dimension $\delta \in (-\infty, 2)$.

Corollary 3.6 *For the squared Bessel process $X_t^{(\delta)}$ with dimension $\delta \in (-\infty, 2)$, we have*

$$\begin{aligned} & P[X_u^{(\delta)} \text{ hits the point 0 during } 0 \le u \le t | X_0^{(\delta)} = x] \\ &= 1 - \left(\frac{x}{2t}\right)^{1-\frac{\delta}{2}} \sum_{n=1}^{\infty} \frac{\left(\frac{x}{2t}\right)^{n-1}}{\Gamma(n+1-\frac{\delta}{2})} \exp\left\{-\frac{x}{2t}\right\}, \quad x \ge 0. \end{aligned} \tag{3.11}$$

Proof. Let $s=0$ $(w=0)$ and $S=x^{1-\frac{\delta_\rho}{2}}$ in (3.7). Then $G(n,w)=1$ and hence

$$P[S_T=0|S_0=x^{1-\frac{\delta_\rho}{2}}]=1-\sum_{n=1}^{\infty}g(n+\lambda,z)=P[\zeta\le\tau_T^{(\delta_\rho,\mu)}|X_0^{(\delta)}=x] \tag{3.12}$$

Substituting $\tau_T^{(\delta_\rho,\mu)}=t$, $z=\frac{x}{2t}$ and $\delta_\rho=\delta$ in (3.12), we get (3.11). □

Remark 3.7 *It is well known that when $\delta\ge 2$, the point 0 is polar and hence the probability that $X_t^{(\delta)}$ hits the point 0 is 0 (see [10, p.415]).*

Here note that the martingale measure $\tilde{P}$ is not unique on $\mathcal{F}_T$. However it is unique on $\mathcal{F}_\tau$. Hence if the payoff of the option depends only on the stock price process S_t, we can derive the unique option price to duplicate the terminal payoff [7, 8]. That is, the unique no arbitrage price of the option is given by its discounted expected value under the equivalent martingale measure $\tilde{P}$. Hereafter we shall consider the arbitrage free pricing for the option whose payoff C_T depends only on the stock price at the maturity S_T. That is, C_T is given by $C(S_T)$ for some function $C(\cdot)$. Then we can derive an arbitrage free option pricing formula.

Theorem 3.8 *Let the initial stock price $S_0=s$ and $C_0(s)$ be the arbitrage free price for the option at time 0 with the terminal payoff $C(S_T)$. Then*

$$\begin{aligned} C_0(s) &= s\cdot\tilde{E}\left[\exp\{-rT\}C\left(\exp\{rT\}\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}}\right)\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{\frac{\delta_\rho}{2}-1}\Big|\tilde{X}_0=s^{2(1-\rho)}\right] \\ &\quad+\exp\{-rT\}C(0)\left(1-s\left(\frac{1}{2\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}}\sum_{n=1}^{\infty}\frac{\left(\frac{s^{2(1-\rho)}}{2\tau_T^{(\delta_\rho,r)}}\right)^{n-1}}{\Gamma(n+1-\frac{\delta_\rho}{2})}\exp\left\{-\frac{s^{2(1-\rho)}}{2\tau_T^{(\delta_\rho,r)}}\right\}\right). \end{aligned} \tag{3.13}$$

Proof. By the general theorem for the complete market asset pricing [7, 8], the unique arbitrage free price $C_0(S)$ is given by

$$\begin{aligned} C_0(s) &= \tilde{E}[\exp\{-rT\}C(S_T)|S_0=s] \\ &= \tilde{E}\left[\exp\{-rT\}C\left(\exp\{rT\}\left(\tilde{X}^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)\right)^{1-\frac{\delta_\rho}{2}}\Big|\tilde{X}_0^{(\delta_\rho)}=s^{\frac{2}{2-\delta_\rho}}\right] \\ &= \tilde{E}\left[\exp\{-rT\}C\left(\exp\{rT\}\left(\tilde{X}^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}}\right)1_{\{\zeta>\tau_T^{(\delta_\rho,r)}\}}\Big|\tilde{X}_0^{(\delta_\rho)}=s^{\frac{2}{2-\delta_\rho}}\right] \\ &\quad+\exp\{-rT\}C(0)\tilde{P}[\tilde{\zeta}\le\tau_T^{(\delta_\rho,r)}|\tilde{X}_0^{(\delta_\rho)}=s^{\frac{2}{2-\delta_\rho}}]. \end{aligned} \tag{3.14}$$

From Lemma 3.3,

$$\begin{aligned} &\tilde{E}\left[\exp\{-rT\}C\left(exp\{rT\}\left(\tilde{X}^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}}\right)1_{\{\tilde{\zeta}>\tau_T^{(\delta_\rho,r)}\}}\Big|\tilde{X}_0=s^{\frac{2}{2-\delta_\rho}}\right] \\ &= s\cdot\tilde{E}\left[\exp\{-rT\}C\left(exp\{rT\}\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}}\right)\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{\frac{\delta_\rho}{2}-1}\Big|\tilde{X}_0=s^{\frac{2}{2-\delta_\rho}}\right]. \end{aligned} \tag{3.15}$$

Also from Corollary 3.6,

$$\begin{aligned} &\tilde{P}[\tilde{\zeta}\le\tau_T^{(\delta_\rho,r)}|\tilde{X}_0=s^{\frac{2}{2-\delta_\rho}}] \\ &= 1-s\left(\frac{1}{2\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}}\sum_{n=1}^{\infty}\frac{\left(\frac{s^{\frac{2}{2-\delta_\rho}}}{2\tau_T^{(\delta_\rho,r)}}\right)^{n-1}}{\Gamma(n+1-\frac{\delta_\rho}{2})}\exp\left\{-\frac{s^{\frac{2}{2-\delta_\rho}}}{2\tau_T^{(\delta_\rho,r)}}\right\}. \end{aligned} \tag{3.16}$$

Then from (3.14) through (3.16) and $\frac{2}{2-\delta_\rho} = 2(1-\rho)$, we have (3.13). $\square$

Call Option Case : Consider the arbitrage free pricing for the call option $C_T = (S_T - K)^+$ with the nonnegative exercise price K. From Theorem 3.8, the arbitrage free price of C_T is given by

$$\begin{aligned}
&C_0(s) \\
&= s \cdot \tilde{E}\left[\exp\{-rT\}\left(exp\{rT\}\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}} - K\right)^+ \left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{\frac{\delta_\rho}{2}-1} \Big| \tilde{X}_0 = s^{\frac{2}{2-\delta_\rho}}\right] \\
&= s \cdot \tilde{P}\left[\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}} \geq (\exp\{-rT\}K)^{\frac{2}{2-\delta_\rho}} \Big| \tilde{X}^{(4-\delta_\rho)}_0 = S^{\frac{2}{2-\delta_\rho}}\right] \qquad (3.17)\\
&\quad - \exp\{-rT\}Ks \cdot \tilde{E}\left[\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{\frac{\delta_\rho}{2}-1} 1_{\{\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}} \geq (\exp\{-rT\}K)^{\frac{2}{2-\delta_\rho}}\}} \Big| \tilde{X}^{(4-\delta_\rho)}_0 = s^{\frac{2}{2-\delta_\rho}}\right].
\end{aligned}$$

From Lemma 3.2,

$$\begin{aligned}
&\tilde{P}\left[\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}} \geq (\exp\{-rT\}K)^{\frac{2}{2-\delta_\rho}} \Big| \tilde{X}^{(4-\delta_\rho)}_0 = s^{\frac{2}{2-\delta_\rho}}\right] \\
&= P\left[V^{(4-\delta_\rho, 2z')} \geq 2w'\right] \\
&= \exp\{-z'\}\sum_{n=0}^{\infty}\frac{z'^n}{n!}\int_{v\geq w'}\frac{v^{n+1-\frac{\delta_\rho}{2}}\exp\{-v\}}{\Gamma(n+2-\frac{\delta_\rho}{2})}dv \\
&= \sum_{n=1}^{\infty} g(n,z')G(n+\lambda, w') \qquad (3.18)
\end{aligned}$$

where λ, $g(x,y)$, $G(x,y)$ are given by (3.8) and z', w' are defined by

$$\begin{cases} z' = \frac{s^{2(1-\rho)}}{2\tau_T^{(\delta_\rho,r)}} = \frac{2r\lambda\exp\{\frac{rT}{\lambda}\}s^{\frac{1}{\lambda}}}{\sigma^2(\exp\{\frac{rT}{\lambda}\}-1)} \\ w' = \frac{(\exp\{-rT\}K)^{2(1-\rho)}}{2\tau_T^{(\delta_\rho,r)}} = \frac{2r\lambda K^{\frac{1}{\lambda}}}{\sigma^2(\exp\{\frac{rT}{\lambda}\}-1)}. \end{cases} \qquad (3.19)$$

On the other hand from (3.10),

$$\begin{aligned}
&\tilde{E}\left[\left(\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{\frac{\delta_\rho}{2}-1} 1_{\{\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}} \geq (\exp\{-rT\}K)^{\frac{2}{2-\delta_\rho}}\}} \Big| \tilde{X}^{(4-\delta_\rho)}_0 = s^{\frac{2}{2-\delta_\rho}}\right] \\
&= \tfrac{1}{s}\textstyle\sum_{n=1}^{\infty} g(n+\lambda, z')G(n, w'). \qquad (3.20)
\end{aligned}$$

Thus from (3.18) through (3.20), we arrive at the following call option pricing formula which is derived by Cox [1, 3].

$$C_0(s) = s\sum_{n=1}^{\infty} g(n,z')G(n+\lambda,w') - \exp\{-rT\}K\sum_{n=1}^{\infty} g(n+\lambda,z')G(n,w'), \qquad (3.21)$$

where z', w', $g(x,y)$ and $G(x,y)$ are given by (3.8) and (3.19).

4 Arbitrage for Positive Conditional Process

From Theorem 2.2 and Corollary 3.6, we see that for a CEV model with $\rho < 1$, the process S_t is absorbed at the point 0 with positive probability. In this section, we study the existence of arbitrage opportunities for the CEV model when it is conditioned to the strictly positive region during the finite time interval

$[0,T]$. If we presume the risk neutralized stock price process with drift r, it is absorbed in 0 with strictly positive probability as is the original process. This means that the conventional risk neutral approach is useless to study this process. However our approach for the CEV model can be applied to show the existence of arbitrage opportunity for the positive conditional process.

Let us define ξ_T by

$$\xi_T = \frac{\exp\{-rT\}S_T}{s} \overset{\tilde{law}}{=} \frac{1}{s}\left(\tilde{X}^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}\wedge\tilde{\zeta}}\right)^{1-\frac{\delta_\rho}{2}}. \tag{4.1}$$

From (3.6), we have $\tilde{E}[\xi_T|\tilde{X}^{(\delta_\rho)}_0 = s^{2(1-\rho)}] = 1$. Hence we can use ξ_T as a Radon-Nikodym derivative for an absolutely continuous (not equivalent) measure change from $\tilde{P}$ to another distribution $\hat{P}$ on $\mathcal{F}_T$. Furthermore since $\xi_T > 0 \Leftrightarrow S_T > 0$, $\hat{P}$ is equivalent to the conditional distribution $\tilde{P}[\cdot|S_T > 0]$. This together with the equivalence of P and $\tilde{P}$ yields the equivalence of $P[\cdot|S_T > 0]$ and $\hat{P}$.

Next we shall consider the Markov process $\hat{S}_t$ defined by

$$d\hat{S}_t = (r\hat{S}_t + \sigma^2\hat{S}_t^{2\rho-1})dt + \sigma\hat{S}_t^{\rho}d\tilde{W}_t, \tag{4.2}$$

starting with $\hat{S}_0 = S$. Then we have the following lemma.

Lemma 4.1

$$\{S_t; 0 \le t \le T\} \text{ under } \hat{P} \overset{law}{=} \{\hat{S}_t; 0 \le t \le T\} \text{ under } \tilde{P}. \tag{4.3}$$

Proof. First we shall represent $\hat{S}_t$ by squared Bessel process with dimension $\delta' > 2$. For $\nu > 0$ and $\delta < 2$, let

$$Z^{(\delta,\nu)}_t = \exp\{\nu t\}\left(\tilde{X}^{(4-\delta)}_{\tau_t^{(\delta,\nu)}}\right)^{1-\frac{\delta}{2}},$$

starting with $\tilde{X}^{(4-\delta)}_0 = s^{\frac{2}{2-\delta}}$. From Itô's lemma,

$$dZ^{(\delta,\nu)}_t = \left(\nu Z^{(\delta,\nu)}_t + \sigma^2\left(Z^{(\delta,\nu)}_t\right)^{-\frac{\delta}{2-\delta}}\right)dt + \sigma\left(Z^{(\delta,\nu)}_t\right)^{\frac{1-\delta}{2-\delta}}d\tilde{W}^{(\delta,\nu)}_t$$

where $\tilde{W}^{(\delta,\nu)}_t$ is another Wiener process under $\tilde{P}$ given by (2.6) for $\tilde{W}_t$. Since $-\frac{\delta_\rho}{2-\delta_\rho} = 2\rho - 1$ and $\frac{1-\delta_\rho}{2-\delta_\rho} = \rho$, we have

$$\{\hat{S}_t; 0 \le t\} \overset{\tilde{law}}{=} \{\tilde{Z}^{(\delta_\rho,r)}_t; 0 \le t\}.$$

This together with Lemma 3.3 yields

$$\begin{aligned}
&\hat{P}[S_T \le u|S_0 = s] \\
&= \hat{E}[1_{\{S_T\le u\}}|S_0 = s] \\
&= \frac{1}{S}\tilde{E}\left[1_{\left\{\tilde{X}^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\le(\exp\{-\mu T\}u)^{\frac{2}{2-\delta_\rho}}\right\}}\left(\tilde{X}^{(\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\right)^{1-\frac{\delta_\rho}{2}}1_{\left\{\tilde{\zeta}>\tau_T^{(\delta_\rho,r)}\right\}}\Big|\tilde{X}^{(\delta_\rho)}_0 = s^{\frac{2}{2-\delta_\rho}}\right] \\
&= \tilde{E}\left[1_{\left\{\tilde{X}^{(4-\delta_\rho)}_{\tau_T^{(\delta_\rho,r)}}\le(\exp\{-\mu T\}u)^{\frac{2}{2-\delta_\rho}}\right\}}\Big|\tilde{X}^{(4-\delta_\rho)}_0 = s^{\frac{2}{2-\delta_\rho}}\right] \\
&= \tilde{P}[\hat{S}_T \le u|\hat{S}_0 = s],
\end{aligned}$$

for any $u \ge 0$. Thus the transition probabilities for the Markov processes $(\{S_t; 0 \le t\}, \hat{P})$ and $(\{\hat{S}_t; 0 \le t\}, \tilde{P})$ coincide and hence they are equivalent in law. □

Now we can show the following result for the CEV model.

Theorem 4.2 *For the positive price process of the CEV model, i.e. : $\{S_t; 0 \le t \le T\}$ under the conditional probability measure $P[\cdot|S_T > 0]$, there always exists arbitrage opportunities.*

Proof. The probability measures $\hat{P}$ and $P[\cdot|S_T > 0]$ are equivalent. Furthermore from Lemma 4.1, $(\{S_t; 0 \le t \le T\}, \hat{P}) \overset{law}{=} (\{\hat{S}_t; 0 \le t \le T\}, \tilde{P})$. Then it is enough to show that there exists arbitrage opportunities for the process $\{\hat{S}_t; 0 \le t \le T\}$ under $\tilde{P}$. Define the process $\hat{X}_u$ by

$$\hat{X}_u = \left(\exp\{-rt_u^{(\delta_\rho,r)}\}\hat{S}_{t_u^{(\delta_\rho,r)}}\right)^{\frac{2}{2-\delta_\rho}} \tag{4.4}$$

where $t_u^{(\delta,\nu)}$ is given by the inverse transformation of 2.3. From Itô's lemma, $\hat{X}_u$ follows

$$d\hat{X}_u = 2\sqrt{\hat{X}_u}d\hat{W}_u + (4-\delta_\rho)du \tag{4.5}$$

where $\hat{W}_u$ is an other Wiener process under $\tilde{P}$ defined as :

$$\hat{W}_u = \int_0^{t_u^{(\delta,\nu)}} \frac{\sigma}{2-\delta_\rho} \exp\left\{-\frac{rv}{2-\delta_\rho}\right\} d\tilde{W}_v. \tag{4.6}$$

Thus $\hat{X}_u$ is a $(4-\delta_\rho)$-dimensional squared Bessel process. Since $4-\delta_\rho > 2$, we have the following result from Theorem 6 of Delbaen-Schachermayer [5] : $L_u = \hat{X}_u^{\frac{\delta_\rho}{2}-1}$ is a local martingale such that L_u^{-1} allows arbitrage with respect to the general admissible integrands under 0 interest rate. That is, there exists an admissible integrand φ_u such that

$$\hat{V}'_T = \int_0^{\tau_T^{(\delta_\rho,r)}} \varphi_u dL_u^{-1} \ge 0, \quad \tilde{P}\text{-a.s.}, \tag{4.7}$$

$$\tilde{P}[V'_T > 0] > 0. \tag{4.8}$$

Since $L_u^{-1} = \exp\{-rt_u^{(\delta_\rho,r)}\}\hat{S}_{t_u^{(\delta_\rho,r)}}$, we can easily show that

$$\hat{V}'_T = \exp\{-rT\}\hat{V}_T \tag{4.9}$$

where

$$\hat{V}_t = \int_0^t r(\hat{V}_u - \varphi_{\tau_u^{(\delta_\rho,r)}}\hat{S}_u)du + \varphi_{\tau_u^{(\delta_\rho,r)}}d\hat{S}_u, \quad 0 \le t \le T. \tag{4.10}$$

From (4.7) through (4.10), there exists an arbitrage opportunity for $\hat{S}_u$ under $\tilde{P}$. □

The intuitive explanation for the existence of arbitrage opportunity is as follows. If there exists an equivalent martingale measure $\check{P}$ for $\hat{S}_t$, it will have the same law as S_t under $\tilde{P}$ by Cameron-Martin-Maruyama-Girsanov's theorem. However this is impossible since $\tilde{P}[S_T = 0] > 0$ whereas $\check{P}[\hat{S}_T = 0] = 0$.

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