CONDITIONALLY ATOMLESS EXTENSIONS OF SIGMA ALGEBRAS

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Abstract. We give two equivalent definitions of sigma algebras that are atomless conditionally to a smaller sigma algebra.

1. Notation

In this paper [1] we will work with a probability space equipped with three sigma algebras $(\Omega, \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2, \mathbb{P})$. The sigma algebra $\mathcal{F}_0$ is supposed to be trivial $\mathcal{F}_0 = \{\emptyset, \Omega\}$ whereas the sigma algebra $\mathcal{F}_2$ is supposed to express innovations with respect to $\mathcal{F}_1$. Since we do not put topological properties on the set Ω we will make precise definitions later that do not use conditional probability kernels. But essentially we could say that we suppose that conditionally on $\mathcal{F}_1$ the probability $\mathbb{P}$ is atomless on $\mathcal{F}_2$. We will show that such an hypothesis implies that there is an atomless sigma algebra $\mathcal{B} \subset \mathcal{F}_2$ that is independent of $\mathcal{F}_1$. In some (under topological hypotheses on $\Omega, \mathcal{F}_1, \mathcal{F}_2$) cases the conditional expectation with respect to $\mathcal{F}_1$ is given by integration with respect to a kernel. We will use the notation K for such a kernel. More precisely: the mapping $K\colon \Omega \times \mathcal{F}_2 \to \mathbb{R}_+$ satisfies

(1) For almost every $\omega \in \Omega$, the mapping $K(\omega, .) : \mathcal{F}_2 \to [0,1]$ is a probability. It is no restriction to suppose that this property holds for every $\omega \in \Omega$.
(2) For each $A \in \mathcal{F}_2$, the mapping $K(., A)\colon \Omega \to [0,1]$ is $\mathcal{F}_1$ measurable.
(3) For each $\xi \in L^1(\Omega, \mathcal{F}_2, \mathbb{P})$ we have that almost surely

$$\mathbb{E}[\xi \mid \mathcal{F}_1](\omega) = \int \xi(\tau)\, K(\omega, d\tau).$$

The existence of such a kernel is not always easy to verify. Sometimes it is part of the model that is studied. Applying the property above and integrating with respect to $\mathbb{P}$ gives

$$\mathbb{P}[A] = \int_\Omega \mathbb{P}[d\omega] K(\omega, A).$$

Or for general $\xi \in L^1$:

$$\int_\Omega \mathbb{P}[dx]\xi(x) = \int_\Omega \mathbb{P}[d\omega] \int_\Omega \xi(\tau)\, K(\omega, d\tau).$$

Part of the results were developed and used in my paper on commonotonicity, see [1]

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[1] This paper is to be seen as an exercise in measure theory. It will not be sent to a math. journal

2. Atomless Extension of Sigma Algebras

Definition 1. We say that $\mathcal{F}_2$ is atomless conditionally to $\mathcal{F}_1$ if the following holds. For every $A \in \mathcal{F}_2$ there exists a set $B \subset A$, $B \in \mathcal{F}_2$, such that $0 < \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]$ on the set $\{\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1] > 0\}$.

In case the conditional expectation could be calculated with a – under extra topological conditions – regular probability kernel, say $K(\omega, A)$, then the above definition is a measure theoretic way of saying that the probability measure $K(\omega, .)$ is atomless for almost every $\omega \in \Omega$. This equivalence will be the topic of the next section. x

Theorem 1. *$\mathcal{F}_2$ is atomless conditionally to $\mathcal{F}_1$ if for every $A \in \mathcal{F}_2$, $\mathbb{P}[A] > 0$, there is $B \subset A$ such that*

$$\mathbb{P}\left[0 < \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\right] > 0.$$

Proof The proof is a standard exhaustion argument. For completeness we give a proof. Let $\mathcal{D}$ be the collection of $\mathcal{F}_1-$measurable sets:

$$\mathcal{D} = \{\{0 < \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\} \mid B \subset A\}$$

We show that there is a biggest set in $\mathcal{D}$ and this set must then be equal to $\{\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1] > 0\}$. To show that there is a biggest set in $\mathcal{D}$ it is sufficient to show that $\mathcal{D}$ is stable for countable unions. Let D_n be a sequence in $\mathcal{D}$ and suppose that for each n we have a set $B_n \subset A$ such that $D_n = \{0 < \mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\}$. Now take

$$B = \cup_n \left(B_n \cap \left(D_n \setminus \left(\cup_{k \le n-1} D_k\right)\right)\right).$$

It is easy to check that $\{0 < \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\} = \cup_n D_n$ and therefore $\cup_n D_n \in \mathcal{D}$. Let $D = \{0 < \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\}$ be a maximum in $\mathcal{D}$. Suppose now that $\mathbb{P}[\{\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1] > 0\} \setminus D] > 0$. This implies that $\mathbb{P}[A \setminus D] > 0$. According to the hypothesis of the theorem, there will be a set $B' \subset (A \setminus D)$ with $D' \subset \{0 < \mathbb{E}[\mathbf{1}_{B'} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_{A \setminus D} \mid \mathcal{F}_1]\}$ and non-negligible. Since $D \cup D' \in \mathcal{D}$ and $D \cap D' = \emptyset$, the element D is not a maximum, a contradiction.

The main result of this section is the following

Theorem 2. *$\mathcal{F}_2$ is atomless conditionally to $\mathcal{F}_1$ if and only if there exists an atomless sigma algebra $\mathcal{B} \subset \mathcal{F}_2$ that is independent of $\mathcal{F}_1$.*

The “if" part is easy but requires some continuity argument. Because $\mathcal{B}$ is atomless, there is a $\mathcal{B}$-measurable, $[0, 1]$ uniformly distributed random variable U. The sets $B_t = \{U \le t\}, 0 \le t \le 1$ form an increasing family of sets with $\mathbb{P}[B_t] = t$. Let $A \in \mathcal{F}_2$ and let $F = \{0 < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\}$. We may suppose that $\mathbb{P}[F] > 0$ since otherwise there is nothing to prove. We will show that there is $t \in]0, 1[$ with $\mathbb{P}\left[0 < \mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\right] > 0$. According to the previous theorem, $\mathcal{F}_2$ is conditionally atomless with respect to $\mathcal{F}_1$. Obviously for $0 \le s \le t \le 1$ we have, by independence of $\mathcal{B}$ and $\mathcal{F}_1$:

$$\|\mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1] - \mathbb{E}[\mathbf{1}_{A \cap B_s} \mid \mathcal{F}_1]\|_\infty \le \|\mathbb{E}[\mathbf{1}_{B_t \setminus B_s} \mid \mathcal{F}_1]\|_\infty = t - s.$$

It follows that there is a set of measure 1, say Ω', such that for all $s \le t$, rational,

$$|\mathbb{E}[\mathbf{1}_{A \cap B_t} \mid \mathcal{F}_1] - \mathbb{E}[\mathbf{1}_{A \cap B_s} \mid \mathcal{F}_1]| \le t - s$$

on Ω'. For $\omega \in \Omega'$ we can extend the function

$$\{q \in [0, 1] \mid q \text{ rational }\} \to \mathbb{E}[\mathbf{1}_{A \cap B_q} \mid \mathcal{F}_1](\omega)$$

to a continuous function on $[0,1]$. The resulting continuous extension then represents $(\mathbb{E}[\mathbf{1}_{A\cap B_t} \mid \mathcal{F}_1])_t$. For $t=0$ we have zero and for $t=1$ we find $\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]$. Because for $\omega\in\Omega'$, the trajectories are continuous, a simple application of Fubini's theorem shows that the real valued function

$$t \to \mathbb{P}\left[0 < \mathbb{E}[\mathbf{1}_{A\cap B_t} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\right]$$

becomes strictly positive for some t. For completeness let us now give the details of the application of Fubini's theorem. Suppose on the contrary that for all $t\in[0,1]$ we have

$$\mathbb{P}\left[0 < \mathbb{E}[\mathbf{1}_{A\cap B_t} \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\right] = 0.$$

Then on the product space $[0,1]\times\Omega'$ we find that the (clearly measurable) set

$$\{(t,\omega) \mid 0 < \mathbb{E}[\mathbf{1}_{A\cap B_t} \mid \mathcal{F}_1](\omega) < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1](\omega)\}$$

has $m\times\mathbb{P}$ measure zero (m denotes Lebesgue measure). By Fubini's theorem we have that for almost all $\omega\in\Omega'$, the set

$$\{t \mid 0 < \mathbb{E}[\mathbf{1}_{A\cap B_t} \mid \mathcal{F}_1](\omega) < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1](\omega)\}$$

must have Lebesgue measure zero. However, for $\omega\in\Omega'$, this contradicts the continuity of the mapping

$$t \to \mathbb{E}[\mathbf{1}_{A\cap B_t} \mid \mathcal{F}_1](\omega).$$

The proof of the "only if" part is broken down in several steps. We will without further notice, always suppose that $\mathcal{F}_2$ is atomless conditionally to $\mathcal{F}_1$.

Lemma 1. *Suppose $A\in\mathcal{F}_1$ and $C\subset A$ is such that $\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]>0$ on A. Then we can construct a decreasing sequence of sets $(B_n)_{n\geq 0}$, $B_n\subset C$, such that $0<\mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1]\leq 2^{-n}$ on A.*

Proof The statement is obviously true for $n=0$ since we can take $B_0=C$. We now proceed by induction and suppose the statement holds for n. So the set $B_n\subset A$ satisfies $0<\mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1]\leq 2^{-n}$ on A. Clearly $A\subset\{\mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1]>0\}$. By assumption there is a set $D\subset B_n$ such that on $A\subset\{\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]>0\}$ we have

$$0<\mathbb{E}[\mathbf{1}_D \mid \mathcal{F}_1]<\mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1].$$

We now take

$$B_{n+1}=\left(D\cap\left\{\mathbb{E}[\mathbf{1}_D \mid \mathcal{F}_1]\leq\frac{1}{2}\mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1]\right\}\right)\cup\left((B_n\setminus D)\cap\left\{\mathbb{E}[\mathbf{1}_D \mid \mathcal{F}_1]>\frac{1}{2}\mathbb{E}[\mathbf{1}_{B_n} \mid \mathcal{F}_1]\right\}\right).$$

The set B_{n+1} satisfies the requirements.

Lemma 2. *Let $C\in\mathcal{F}_2$ and let $h\colon\Omega\to[0,1]$ be $\mathcal{F}_1$ measurable. Then there is a set $B\subset C$ such that $\mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1]=h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]$.*

Proof Let $B_0=\emptyset$. Inductively we define for $n\geq 1$, classes $\mathcal{B}_n$ and sets $B_n\in\mathcal{B}_n$. For $n\geq 1$ let

$$\mathcal{B}_n=\{B_{n-1}\subset B\subset C \mid B\in\mathcal{F}_2,\ \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1]\leq h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]\}.$$

Let $\beta_n=\sup\{\mathbb{P}[B] \mid B\in\mathcal{B}_n\}$ and take $B_n\in\mathcal{B}_n$ such that $\mathbb{P}[B_n]\geq(1-2^{-n})\beta_n$. Clearly B_n is non-decreasing and let $B_\infty=\cup_n B_n$. Obviously

$$\mathbb{P}[B_\infty]\geq\limsup\beta_n\geq\liminf\beta_n\geq\lim\mathbb{P}[B_n]=\mathbb{P}[B_\infty].$$

We claim that $\mathbb{E}[\mathbf{1}_{B_\infty} \mid \mathcal{F}_1] = h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]$. Obviously we already have that $\mathbb{E}[\mathbf{1}_{B_\infty} \mid \mathcal{F}_1] \le h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]$. If $\mathbb{P}\left[\mathbb{E}[\mathbf{1}_{B_\infty} \mid \mathcal{F}_1] < h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]\right] > 0$ then $\mathbb{P}[B] < \mathbb{P}[C]$ and there must be $m \ge 1$ such that $\mathbb{P}\left[\mathbb{E}[\mathbf{1}_{B_\infty} \mid \mathcal{F}_1] < h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1] - 2^{-m}\right] > 0$. The previous lemma allows to find $D \subset C \setminus B_\infty$, $\mathbb{P}[D] = \eta > 0$, such that $0 < \mathbb{E}[\mathbf{1}_D \mid \mathcal{F}_1] \le 2^{-m}$ on the set $\{\mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] < h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1] - 2^{-m}\}$ and zero elsewhere. The set $D \cup B_\infty$ is in all classes $\mathcal{B}_n$ and for n big enough:

$$\beta_n \ge \mathbb{P}[D \cup B_\infty] \ge \mathbb{P}[B_n] + \eta \ge (1 - 2^{-n})\beta_n + \eta \ge \beta_n + \eta - 2^{-n} > \beta_n,$$

yielding a contradiction. So we must have $\mathbb{E}[\mathbf{1}_{B_\infty} \mid \mathcal{F}_1] = h\,\mathbb{E}[\mathbf{1}_C \mid \mathcal{F}_1]$.

Remark 1. The lemma above is a variant of Sierpiński's theorem, [3]. This theorem states that in an atomless probability space $(\Omega, \mathcal{E}, \mathbb{P})$, for every set $A \in \mathcal{E}$ and every $0 < t < 1$, there is a set $B \subset A$ with $\mathbb{P}[B] = t\mathbb{P}[A]$. The usual proof — presented in many probability courses — uses the Axiom of Choice (AC). A referee of [1] pointed out that for many people AC – or Zorn's lemma – is an extra assumption. To prove Sierpiński's theorem we only need the Axiom of Countable Dependent Choice, which is a countable form of the axiom of choice. In analysis this is the axiom that is usually needed and used. The proof above follows the approach given by Lorenc and Witula, [2].

Lemma 3. *There is an increasing family of sets $(B_t)_{t\in[0,1]}$ such that $\mathbb{E}[\mathbf{1}_{B_t} \mid \mathcal{F}_1] = t$. The sigma algebra $\mathcal{B}$, generated by the family $(B_t)_t$ is independent of $\mathcal{F}_1$. The system $(B_t)_t$ can also be described as $B_t = \{U \le t\}$ where U is a random variable that is independent of $\mathcal{F}_1$ and uniformly distributed on* $[0, 1]$.

Proof The proof is a repeated use of the previous lemma where we take $h = 1/2$. We start with $B_0 = \emptyset, B_1 = \Omega$. Suppose that for the diadic numbers $k2^{-n}, k = 0, \dots 2^n$ the sets are already defined. Then we consider the set $B_{(k+1)2^{-n}} \setminus B_{k2^{-n}}$ and apply the previous lemma with $h = 1/2$. We get a set $D \subset B_{(k+1)2^{-n}} \setminus B_{k2^{-n}}$ with $\mathbb{E}[\mathbf{1}_D \mid \mathcal{F}_1] = 2^{-(n+1)}$. We then define $B_{(2k+1)2^{-(n+1)}} = B_{k2^{-n}} \cup D$. For non-diadic numbers t we find a sequence of diadic numbers d_n such that $d_n \uparrow t$. Then we define $B_t = \cup_n B_{d_n}$. This completes the construction. Since the system $(B_t)_t$ is trivially stable for intersection, the relation $\mathbb{E}[\mathbf{1}_{B_t} \mid \mathcal{F}_1] = t$ shows that the sigma algebra $\mathcal{B}$ generated by $(B_t)_t$, is independent of $\mathcal{F}_1$. The construction of U is standard. At level n we put $U_n = \sum_{k=1,\dots 2^n} k2^{-n}\mathbf{1}_{B_{k2^{-n}} \setminus B_{(k-1)2^{-n}}}$. U_n then decreases to a random variable U that satisfies the needed properties.

Remark 2. After the first version was made available, I got the remark that the paper [4] of Shen, J., Shen, Y., Wang, B., and Wang, R. contains similar concepts and results.[2] In their notation they work with a measurable space $(\Omega, \mathcal{A})$ on which they have a finite number of probability measures $\mathbb{Q}_1, \dots, \mathbb{Q}_n$.[3] They introduce

Definition 2. The set $(\mathbb{Q}_1, \dots, \mathbb{Q}_n)$ is conditionally atomless if there exists a dominating measure $\mathbb{Q}$ (i..e $\mathbb{Q}_k \ll \mathbb{Q}$ for each $k \le n$) as well as a continuously distributed random variable X (for the measure $\mathbb{Q}$) such that the vector of Radon-Nikodym derivatives $\left(\frac{d\mathbb{Q}_k}{d\mathbb{Q}}\right)_k$ is independent of X.

[2] I thank Ruodu Wang for pointing out these relations and for the subsequent discussions we had on the topic.

[3] Their paper also considers an infinite number of measures but to clarify the relation between their paper and my approach, I only consider a finite number of measures.

They then prove the following

Proposition 1. *Are equivalent*

(1) $(\mathbb{Q}_1,\dots,\mathbb{Q}_n)$ *is conditionally atomless*
(2) *in the definition we can take* $\mathbb{Q}=\frac{1}{n}(\mathbb{Q}_1+\dots+\mathbb{Q}_n)$
(3) X *can be taken as uniformly distributed over* $[0,1]$.

There are several differences with my approach. There is the technical difference that they suppose the existence of a continuously distributed random variable X. In doing so they avoid the technical points between the more conceptual definition using conditional expectations and the construction of a suitable sigma-algebra with a uniformly distributed random variable. A further difference is that they use a dominating measure that later can be taken as the mean of $(\mathbb{Q}_1,\dots,\mathbb{Q}_n)$. Of course their result together with the results here show that the definition of $(\mathbb{Q}_1,\dots,\mathbb{Q}_n)$ being conditionally atomless, is equivalent to the statement that for the measure $\mathbb{Q}_0=\frac{1}{n}(\mathbb{Q}_1+\dots+\mathbb{Q}_n)$, the sigma algebra $\mathcal{A}$ is conditionally atomless with respect to the sigma-algebra generated by the Radon-Nikodym derivatives $\left(\frac{d\mathbb{Q}_k}{d\mathbb{Q}_0}\right)_k$. In [4] it is also shown that one can take any strictly positive convex combination of the measures $(\mathbb{Q}_1,\dots,\mathbb{Q}_n)$. Below we will show that this sigma-algebra in some sense has a minimal property, a result that clarifies the relation between the two approaches. Before doing so, let us recall two easy results from introductory probability theory.

Result 1. For a given probability space $(\Omega,\mathcal{A},\mathbb{Q})$ let us denote $\mathcal{N}=\{N\in\mathcal{A}\mid\mathbb{Q}[N]=0\}$. Suppose that a sub sigma-algebra $\mathcal{F}\subset\mathcal{A}$ is given and that $\mathcal{G}$, $\mathcal{F}\subset\mathcal{G}$, is another sub sigma-algebra which is included in the sigma-algebra generated by $\mathcal{F}$ and $\mathcal{N}$. Then for each $\xi\in L^1(\Omega,\mathcal{A},\mathbb{Q})$

$$\mathbb{E}_\mathbb{Q}[\xi\mid\mathcal{F}]=\mathbb{E}_\mathbb{Q}[\xi\mid\mathcal{G}]\quad\text{a.s.}$$

Result 2. With the notation in the previous exercise let $F\colon\Omega\to\mathbb{R}^n$ and $F'\colon\Omega\to\mathbb{R}^n$ be two vectors that are equal a.s. . Let $\mathcal{F}$ be generated by F and $\mathcal{G}$ be generated by F'. Then $\mathcal{F}$ and $\mathcal{G}$ are equal up to sets in $\mathcal{N}$. More precisely $\mathcal{G}$ is included in the sigma-algebra generated by $\mathcal{F}$ and $\mathcal{N}$ (and of course conversely), i.e. $\sigma(\mathcal{F},\mathcal{N})=\sigma(\mathcal{G},\mathcal{N})$.

Proposition 2. *Let* $\mathbb{Q}_1,\dots,\mathbb{Q}_n$ *be probability measures on a measurable space* $(\Omega,\mathcal{A})$. *Let* $\mathbb{Q}_0$ *denote a convex combination of these measures* $\mathbb{Q}_0=\sum_k\lambda_k\mathbb{Q}_k$ *where each* $\lambda_k>0$. *Let* f_k *denote an* $\mathcal{A}$ *measurable version* $\frac{d\mathbb{Q}_k}{\mathbb{Q}_0}$. *Let* $\mathbb{Q}$ *be another dominating measure with* g_k *an* $\mathcal{A}$ *measurable version of* $\frac{d\mathbb{Q}_k}{d\mathbb{Q}}$. *Let* $\mathcal{N}=\{N\in\mathcal{A}\mid\mathbb{Q}_0[N]=0\}$. *Let* $\mathcal{F}$ *be generated by* $f_k,k=1\dots n$ *and let* $\mathcal{G}$ *be generated by* $g_k,k=1\dots n$. *Then* $\mathcal{F}\subset\sigma(\mathcal{G},\mathcal{N})$

Proof Clearly $\mathbb{Q}_0\ll\mathbb{Q}$ so let $h=\frac{d\mathbb{Q}_0}{d\mathbb{Q}}$. It is now immediate that $g_k=f_kh$ $\mathbb{Q}$ a.s. . To see this, observe that the values of f_k on $\{h=0\}$ do not matter. The functions g_k and h are $\mathcal{G}$ measurable since h can be taken as $h=\sum_k\lambda_kg_k$. Then we define $f'_k=\frac{g_k}{h}$ on $\{h>0\}$ and $f'_k=0$ on $\{h=0\}$. This choice shows that the f'_k are $\mathcal{G}$ measurable. It is immediate that $f_k=f'_k$ $\mathbb{Q}_0$ a.s. . The result now follows.

From the theorem it follows that the sigma-algebra augmented with the class $\mathcal{N}$ is the same for all strictly positive convex combinations. The theorem shows that in the definition of conditionally atomless with respect to $\mathcal{F}$, we can also add the null sets $\mathcal{N}$ to $\mathcal{F}$. To check that $\mathcal{A}$ is conditionally atomless with respect to a sigma-algebra $\mathcal{F}$ it is clear that the smaller

$\mathcal{F}$, the easier it is to satisfy the condition. In my opinion the above clarifies the relation between this paper and [4].

3. An Equivalent Definition

As already mentioned in the previous section, the definition of being conditionally atomless is related to a similar statement for the kernel K. We suppose that the conditional expectation with respect to $\mathcal{F}_1$ is given by the kernel K. We have the following

Theorem 3. *If $\mathcal{F}_2$ is conditionally atomless with respect to $\mathcal{F}_1$ then for almost every $\omega \in \Omega$ the probability measure $K(\omega, .)$ is atomless on $\mathcal{F}_2$. In case the sigma algebra $\mathcal{F}_2$ is generated by a countable family of sets, the converse holds, i.e. if for almost every $\omega \in \Omega$, the probability $K(\omega, .)$ is atomless on $\mathcal{F}_2$, then $\mathcal{F}_2$ is conditionally atomless with respect to $\mathcal{F}_1$.*

Proof We first suppose that $\mathcal{F}_2$ is atomless with respect to $\mathcal{F}_1$. According to the previous section there is an atomless sub sigma algebra $\mathcal{B} \subset \mathcal{F}_2$ that is independent of $\mathcal{F}_1$. There is also a random variable U which is independent of $\mathcal{F}_1$ and is uniformly distributed on $[0,1]$. Let $\mathcal{C}[0,1]$ be the space of real valued continuous functions on $[0,1]$, equipped with the sup norm. This space is separable and so we can take a (sup-norm) dense sequence $(g_n)_{n\geq 1}$ in $\mathcal{C}[0,1]$. For each $n \geq 1$ we have a.s. :

$$\mathbb{E}[g_n(U) \mid \mathcal{F}_1](\omega) = \mathbb{E}[g_n(U)] = \int_0^1 g_n(t)\, dt.$$

So we have a.s. , say on $\Omega_n, \mathbb{P}[\Omega_n] = 1$;

$$\int K(\omega, d\tau) g_n(U(\tau)) = \int_0^1 g_n(t)\, dt.$$

For $\omega \in \cap_{n\geq 1}\Omega_n$ we have by density of the sequence $(g_n)_n$, for all $g \in \mathcal{C}[0,1]$:

$$\int K(\omega, d\tau) g(U(\tau)) = \int_0^1 g(t)\, dt.$$

This proves that a.s. the random variable U is for $K(\omega, .)$ uniformly $[0,1]$ distributed. That can only happen when $K(\omega, .)$ is atomless on $\mathcal{F}_2$.

We now prove the converse. Suppose that $\mathcal{F}_2$ is not conditionally atomless with respect to $\mathcal{F}_1$. In this case there is a set A with $\mathbb{P}[A] > 0$ such that for all $B \subset A$:

$$\mathbb{P}\left[0 < \mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] < \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]\right] = 0.$$

In order words, if $B \subset A$ then a.s. either $\mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] = 0$ or $\mathbb{E}[\mathbf{1}_B \mid \mathcal{F}_1] = \mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1]$. By definition of the kernel K, this means $K(\omega, B) = 0$ or $K(\omega, B) = K(\omega, A)$ a.s. . In other words for $B \subset A$:,

$$\mathbb{P}\left[\{\omega \mid K(\omega, B) = 0 \text{ or } K(\omega, B) = K(\omega, A)\}\right] = 1.$$

Since $\mathcal{F}_2$ is countably generated there is a countable Boolean algebra $\mathcal{A} \subset \mathcal{F}_2$ that generates $\mathcal{F}_2$. For each set $B \in \mathcal{A}$ we have that

$$\Omega_B = \left\{\omega \mid K(\omega, B \cap A)^2 = K(\omega, A)\, K(\omega, B \cap A)\right\},$$

has measure 1. The set $\Omega' = \cap_{B\in\mathcal{A}}\Omega_B$ still has probability 1. We claim that for each $\omega \in \Omega'$ and each $B \in \mathcal{F}_2$ we have that either $K(\omega, B \cap A) = 0$ or $= K(\omega, A)$. This means that for each $\omega \in \Omega'$ with $K(\omega, A) > 0$, the measure $K(\omega, .)$ has A as an atom, a contradiction to the hypothesis. To show the claim we use a monotone class argument. Let

$$\mathcal{M} = \left\{B \in \mathcal{F}_2 \mid \text{ for each } \omega \in \Omega' : K(\omega, B \cap A)^2 = K(\omega, A)\, K(\omega, B \cap A)\right\}.$$

Clearly $\mathcal{A} \subset \mathcal{M}$ and it is obvious that $\mathcal{M}$ is a monotone class, meaning that it is stable for increasing countable unions and for decreasing countable intersections. It is well known that this implies $\mathcal{M} = \mathcal{F}_2$, completing the proof of the theorem.

4. A Counterexample

We now give a counterexample when $\mathcal{F}_2$ is not countably generated. The basic ingredient is the interval $[0, 1]$ with its Borel sigma algebra $\mathcal{B}$ and the Lebesgue measure m. We define $S = [0, 1]$ and $\Omega = [0, 1] \times (S \times [0, 1])$. The sigma algebra $\mathcal{F}_1$ is generated by the first coordinate and the Borel sigma algebra, $\mathcal{B}$, on $[0, 1]$. On $S \times [0, 1]$ we put the sigma algebra defined as follows:

$$\mathcal{A} = \{B \mid \text{ there is a countable set } D \text{ and for } s \in D : B_s \in \mathcal{B} \text{ otherwise } B_s = \emptyset \text{ or } B_s = [0, 1]\}.$$

The sigma algebra $\mathcal{F}_2$ is the product sigma algebra $\mathcal{B} \otimes \mathcal{A}$. For each $x \in [0, 1]$ we define the kernel $K(x, C)$ as follows. We first define the transition probability $k(x, B)$ for $B \in \mathcal{A}$:

$$k(x, B) = \sum_{s \in S} \mathbf{1}_{\{x\}}(s)\, m(B_s).$$

Then we define the kernel (defined on Ω) as $K(\omega, .) = \delta_x \otimes k(x, .)$, where $\omega = (x, s, y)$ and where δ_x is the Dirac measure concentrated on the point x. The probability measure on Ω is constructed with m and the transition kernel k:

$$E \in \mathcal{B}, B \in \mathcal{A} \quad \mathbb{P}[E \times B] = \int_E m(dx)\, m(B_x) = m\left(E \cap \{x \mid B_x = [0, 1]\}\right).$$

For each $x \in [0, 1]$ the kernel K is atomless.
For $A \in \mathcal{F}_2$ we find putting $A_x = \{(s, z) \mid (x, s, z) \in A\}$ and $A_{x,x} = \{y \mid (x, x, y) \in A\}$:

$$\mathbb{E}[\mathbf{1}_A \mid \mathcal{F}_1](x) = k(x, A_x) = m(A_{x,x}),$$

which is almost surely 0 or 1. This makes it impossible that $\mathcal{F}_2$ is conditionally atomless with respect to $\mathcal{F}_1$.

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