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Risk Measures with the CxLS property

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Abstract In the present contribution we characterise law determined convex risk measures that have convex level sets at the level of distributions. By relaxing the assumptions in Weber [41], we show that these risk measures can be identified with a class of generalised shortfall risk measures. As a direct consequence, we are able to extend the results in Ziegel [42] and Bellini and Bignozzi [7] on convex elicitable risk measures and confirm that expectiles are the only elicitable coherent risk measures. Further, we provide a simple characterisation of robustness for convex risk measures in terms of a weak notion of mixture continuity.

Keywords Decision Theory · Elicitability · Convex level sets · Mixture continuity · Robustness

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1 Introduction

The main goal of this paper is to investigate and characterise risk measures that have "convex level sets at the level of distributions" (CxLS), in the sense that

$$\rho(F) = \rho(G) = \gamma \Rightarrow \rho(\lambda F + (1-\lambda)G) = \gamma, \text{ for each } \lambda \in (0,1),$$

where F and G are probability distributions and the risk measure ρ is considered as a mapping from the space $\mathcal{M}_{1,c}(\mathbb{R})$ of all compactly supported probability measures to the real line $\mathbb{R}$. The financial interpretation of this property is that any mixture of two equally risky positions remains with the same risk. In the axiomatic theory of risk measures, it is customary to impose convexity or quasiconvexity requirements with respect to the pointwise sum of the risks, viewed as random variables on a common state space Ω, in order to model an incentive to diversification. On the contrary, the CxLS property arises naturally as a necessary condition for elicitability, that is defined as the property of being the minimiser of a suitable expected loss. More formally, we say that a law determined risk measure ρ is elicitable if there exists a loss function $L : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\rho(F) = \arg\min_x \int L(x,y)\,\mathrm{d}F(y).$$

It has been suggested by several authors (see e.g. [26], [42], [7]) that elicitability is relevant in connection with backtesting and with forecast comparison of risk measure estimates. The basic idea is that if the estimate of a risk measure is accurate then the ex-post realised loss $\hat{L}$, defined as

$$\hat{L} = \frac{1}{N}\sum_{n=1}^{N} L(\hat{\rho}_n, y_n),$$

where $\hat{\rho}_n$ is the estimated risk measure at time n and y_n is the corresponding financial position, should be small. Indeed, the quantity $\hat{L}$ can be used for comparing risk measure forecasts from different models, for testing a computation model, and even for pooling different models (see e.g. [9], [28], [8] and the references therein).

It is very important not to confuse the forecast comparison process with backtesting. Backtesting refers to a statistical test of the null hypothesis that the stipulated model for the financial position is correct and that the risk measure in question has been estimated accurately up to random fluctuations. Therefore, if the backtest rejects a model or risk measure estimate then the model or the estimation procedure are so wrong that the fluctuations observed in the finite sample cannot only stem from random fluctuations with a small probability. However, if the null hypothesis is not rejected, then either could be true, the model and estimation procedure are correct, or, there is not enough evidence in the finite sample to show that they are incorrect. Forecast comparison is concerned with assessing which one of two or more competing risk measure estimates is the closest to the truth, that is, the risk measure with the smallest ex-post realised loss $\hat{L}$ is considered most accurate. This reasoning is justified by the fact that loss functions are *order-sensitive* or *accuracy-rewarding* (under weak continuity assumptions on ρ, see [36], [34], [40]). The ex-post realised loss $\hat{L}$ can also serve for a statistical test of the hypothesis that a newly proposed model and risk measure estimation procedure is actually a significant improvement with respect to an existing or a standard procedure.

The case of Expected Shortfall (ES) has been the subject of interesting discussions. As it has been pointed out in [41] and [26], ES does not satisfy the CxLS property, and hence is not elicitable. This does not mean that it is not possible to do backtesting by means of a statistic derived from ES; for example a bootstrap approach is always possible (see e.g. [35], [31], [2]). However, the lack of elicitability of ES means that it is not possible to assess the accuracy of its estimation by means of a realised loss functional implying for example that it is an open question how to provide a meaningful ranking of different forecasts with respect to predictive performance; to find alternative approaches is an active research field.

In Decision Theory, the CxLS property is usually known as Betweenness; it is one of the possible relaxations of the independence axiom of the Von Neumann-Morgenstern theory (see for example [17] and [14]). In the seminal paper of Weber [41], the author proved that, under additional conditions that we discuss in detail in Section 3, a monetary risk measure with upper and lower convex level sets at the level of distributions belongs to the class of shortfall risk measures introduced by [23] as follows:

$$\rho(F) = \inf \left\{ m \in \mathbb{R} \mid \int \ell(x-m)\, \mathrm{d}F(x) \leq 0 \right\},$$

for a non decreasing and non constant loss function $\ell\colon \mathbb{R} \to \mathbb{R}$ with 0 in its range. In comparison with Weber's theorem, we limit ourselves to the more restricted case of convex risk measures. On the contrary, we relax Weber's additional conditions in order to completely characterise convex law determined risk measures with the CxLS property. Our results show that in the case of convex risk measures, Weber's condition is equivalent to requiring the weak compactness property (see Proposition 2.4 and the remark thereafter). We see in Theorem 3.9 that convex law determined risk measures with CxLS correspond to generalised shortfalls, in which the loss function can also assume the value $+\infty$. As a consequence, we show that the only elicitable coherent law determined risk measures are expectiles, that can be defined as the shortfall risk measures associated to the loss function

$$\ell_\alpha(x) = \alpha x^+ - (1-\alpha)x^-,$$

for $\alpha \geq \frac{1}{2}$. For more information on expectiles we refer to [37], [19], [10].

A similar result has been obtained in Ziegel [42], by means of a constructive argument that does not rely on Weber's theorem. Indeed, in [42] it is shown that the Kusuoka representation of a coherent risk measure with the CxLS property has a very peculiar structure, which also characterises expectiles. Bellini and Bignozzi [7] proved an analogous characterisation result in the non-convex case, by adding several additional requirements on the loss function $L(x, y)$ in order to guarantee Weber's continuity requirements.

As a byproduct of the analysis in this paper, we provide a simple characterisation of robustness for convex risk measures. Robustness of a risk measure is frequently identified with its continuity properties with respect to some metric on probability measures. Cont et al. [15] noticed that Hampel's classical notion of robustness (see [27] and [29]), corresponding to continuity with respect to weak convergence, is a strong requirement that is essentially only satisfied by risk measures that do not encorporate tail risk. For instance, for Value at Risk (VaR) at level α, one obtains continuity with respect to weak convergence for all α where the quantile function is continuous. For other important risk measures such as ES and the mean, this property does not hold. Stahl et al. [39] propose to define robustness as continuity with respect to the Wasserstein metric, which is satisfied by both VaR and ES. We refer to [21] and [22] for a short digression on the different notions of robustness available in the literature. Here, we follow the approach in Krätschmer et al. [32] and identify the notion of robustness for a law determined risk measure with its continuity with respect to ψ-weak convergence, that is

$$F_n \stackrel{\psi}{\to} F \;\text{ if } F_n \to F \text{ weakly and } \int \psi \, \mathrm{d}F_n \to \int \psi \, \mathrm{d}F,$$

where $\psi\colon \mathbb{R} \to [0, +\infty)$ is a continuous gauge function satisfying $\psi \geq 1$ outside some compact set and $\lim_{x\to\infty} \psi(x) = +\infty$. We show in Proposition 2.5 that, for a convex law determined risk measure ρ, robustness is equivalent to a weak form of mixture continuity:

$$\lim_{\lambda\to 0^+} \rho(\lambda\delta_x + (1-\lambda)\delta_y) = \rho(\delta_y), \text{ for each } x, y \in \mathbb{R}.$$

For ease of reference, in this paper we will follow the sign convention used in [18], so the notion of a convex risk measure will be replaced by that of a concave utility function, which is the same up to the sign.

The paper is structured as follows: In Section 2 we discuss robustness issues, while in Section 3 we provide the characterisation of concave, law determined utilities with CxLS. Auxiliary results are moved to the Appendix.

2 Continuity properties of law determined concave monetary utility functions

2.1 Notations and preliminaries

In this subsection, we set our notation and review basic properties of concave utility functions. All results that are quoted without reference can be found in

[18].
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and denote $L^\infty := L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ the space of (almost surely) bounded random variables. The atomless assumption is not a very big restriction as it simply means that on Ω we can define a random variable with a continuous distribution function. Several statements are only valid for atomless spaces so we will use this as a standing assumption and will not repeat it. A utility function is any function $u\colon L^\infty \to \mathbb{R}$. We say that u is translation invariant if $u(\xi + h) = u(\xi) + h$, for each $\xi \in L^\infty$ and $h \in \mathbb{R}$; it is monotone if for any $\xi, \eta \in L^\infty$, $\xi \le \eta$ a.s. $\Rightarrow u(\xi) \le u(\eta)$; it is monetary, if u is translation invariant, monotone and satisfies $u(0) = 0$.
The properties of a monetary utility function can be recovered by means of its acceptance set

$$\mathcal{A} := \{\xi \in L^\infty \mid u(\xi) \ge 0\};$$

in particular, u is concave if and only if the acceptance set $\mathcal{A}$ is convex.
A utility function u is law determined (or law invariant) if

$$\mathrm{Law}(\xi) = \mathrm{Law}(\eta) \Rightarrow u(\xi) = u(\eta),$$

where $\mathrm{Law}(\cdot)$ denotes the probability law of a random variable. A law determined utility can be seen as a function on $\mathcal{M}_{1,c}(\mathbb{R})$, the set of probability measures on $\mathbb{R}$ with compact support. The monotonicity property implies that for each $F, G \in \mathcal{M}_{1,c}(\mathbb{R})$

$$F \le_{st} G \Rightarrow u(F) \le u(G),$$

where $\le_{st}$ denotes the usual stochastic order, also known as first order stochastic dominance; if u is concave then also

$$F \le_{cv} G \Rightarrow u(F) \le u(G),$$

where $\le_{cv}$ is the concave order, also known as second order stochastic dominance (see for example [6] and [13]). The set of distributions of acceptable positions will be denoted by

$$\mathcal{N} := \{\mathrm{Law}(\xi) \mid \xi \in \mathcal{A}\}.$$

A law determined functional $u\colon \mathcal{M}_{1,c} \to \mathbb{R}$ is said to be weakly continuous if $u(F_n) \to u(F)$ whenever $F_n \to F$ weakly; it is ψ-weakly continuous if

$$F_n \xrightarrow{\psi} F \Rightarrow u(F_n) \to u(F).$$

Clearly, since ψ-weak convergence implies weak convergence, it follows that weak continuity implies ψ-weak continuity and that a weakly closed set is ψ-weakly closed for each gauge function ψ. A utility function $u\colon L^\infty \to \mathbb{R}$ has the Fatou property if for each $\xi_n \in L^\infty$, with $\sup_n \|\xi_n\|_\infty < +\infty$, it holds that

$$\xi_n \xrightarrow{\mathbb{P}} \xi \Rightarrow u(\xi) \ge \limsup_{n \to +\infty} u(\xi_n). \tag{2.1}$$

A monetary concave utility function $u\colon L^\infty \to \mathbb{R}$ with the Fatou property has the following dual representation:

$$u(\xi) = \inf\left\{\mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q}) \mid \mathbb{Q} \in \mathbf{P}\right\}, \tag{2.2}$$

where $\mathbf{P} = \{\mathbb{Q} \mid \mathbb{Q} \ll \mathbb{P}\}$ is the set of probability measures that are absolutely continuous with respect to $\mathbb{P}$ and the penalty function $c\colon \mathbf{P} \to [0, +\infty]$ is convex and lower semicontinuous. We will often identify $\mathbf{P}$ with a subset of L^1_+ via the Radon-Nikodym derivative $\mathrm{d}\mathbb{Q}/\,\mathrm{d}\mathbb{P}$. If the monetary concave utility function u is law determined, then $u(\xi) \leq \mathbb{E}_{\mathbb{P}}[\xi]$ (see [24, Corollary 4.65]), which implies that $c(\mathbb{P}) = 0$. Further, for law determined monetary concave utility function u the Fatou property is always satisfied, see [30] and [18, Section 5.1]. The following Kusuoka representation holds (see [33], [25]):

$$u(\xi) = \inf\left\{\int u_\alpha(\xi)\nu(\,\mathrm{d}\alpha) + \overline{c}(\nu) \mid \nu \in \mathcal{M}_1[0,1]\right\}, \tag{2.3}$$

where $\mathcal{M}_1[0,1]$ is the set of all probability measures on $[0,1]$, the function $\overline{c}\colon \mathcal{M}_1[0,1] \to [0,+\infty]$ is convex and lower semicontinuous, u_α represents the Tail Value-at-Risk (TVaR) at level $\alpha \in (0,1]$ defined by

$$u_\alpha(\xi) = \frac{1}{\alpha}\int_0^\alpha q_x(\xi)\,\mathrm{d}x,$$

where q_x denotes a quantile function at level x and $u_0(\xi) = \operatorname{ess\,inf}(\xi)$.
We say that a monetary concave utility u has the weak compactness (WC) property if the penalty function $c\colon \mathbf{P} \to [0,+\infty]$ in the dual representation (2.2) has lower level sets $\mathcal{S}_m := \{\mathbb{Q} \in \mathbf{P} \mid c(\mathbb{Q}) \leq m\}$ that are compact in the weak topology $\sigma(L^1, L^\infty)$. The WC property is equivalent to the so called Lebesgue property:

$$\xi_n \xrightarrow{\mathbb{P}} \xi,\ \sup_n \|\xi_n\|_\infty < +\infty \Rightarrow u(\xi_n) \to u(\xi), \tag{2.4}$$

which is a stronger continuity requirement than the Fatou property. If u is law determined, then the WC property is equivalent to the following property of the penalty function $\overline{c}$ in the Kusuoka representation:

$$\nu(\{0\}) > 0 \Rightarrow \overline{c}(\nu) = +\infty.$$

In the coherent case the dual representation becomes

$$u(\xi) = \inf\left\{\mathbb{E}_{\mathbb{Q}}[\xi] \mid \mathbb{Q} \in \mathcal{S}\right\},$$

where $\mathcal{S} \subset \mathbf{P}$ is a convex set of probability measures, closed in the $\sigma(L^1, L^\infty)$ topology. The WC property is equivalent to the compactness of $\mathcal{S}$ in the $\sigma(L^1, L^\infty)$ topology, that by the Dunford-Pettis theorem is equivalent to the uniform integrability of $\mathcal{S}$. Recall that $\mathcal{S} \subset L^1$ is uniformly integrable if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \mathbb{P}(A) < \delta \Rightarrow \sup_{\phi \in \mathcal{S}} \int_A \phi\,\mathrm{d}\mathbb{P} < \varepsilon.$$

Uniform integrability is characterised by de la Vallée-Poussin's criterion: $\mathcal{S}$ is uniformly integrable if and only if there exists a function $\Phi\colon \mathbb{R}_+ \to \mathbb{R}$, increasing, convex, with $\Phi(0) = 0$ and $\Phi(x)/x \to +\infty$ for $x \to +\infty$, such that

$$\sup_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}\Big[\Phi\Big(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big)\Big] < +\infty.$$

In the law determined coherent case, the Kusuoka representation becomes

$$u(\xi) = \inf\left\{\int \nu(\,\mathrm{d}\alpha) u_\alpha(\xi) \mid \nu \in \mathcal{S}\right\},$$

and the WC property is equivalent to $\nu(\{0\}) = 0$ for each $\nu \in \mathcal{S}$.
A (finite-valued) Young function is a convex function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ and $\lim_{x\to+\infty} \Phi(x) = +\infty$. A Young function is necessarily non decreasing, continuous and strictly increasing on $\{\Phi > 0\}$. The Orlicz space L^Φ is defined as

$$L^\Phi := \{X \mid \mathbb{E}[\Phi(c|X|)] < \infty \text{ for some } c > 0\};$$

the Orlicz heart is

$$H^\Phi := \{X \mid \mathbb{E}[\Phi(c|X|)] < \infty \text{ for every } c > 0\}.$$

The Luxemburg norm is defined as

$$\|X\|_\Phi := \inf\Big\{\lambda > 0 \mid \mathbb{E}[\Phi(|X/\lambda|)] \le 1\Big\}$$

and makes both L^Φ and H^Φ Banach spaces. Finally, we say that Φ satisfies a Δ_2 condition if $\Phi(2x) \le k\Phi(x)$, for some $k > 0$ and $x \ge x_0$; in this case $H^\Phi = L^\Phi$. For Orlicz space theory and applications to risk measures we refer to [38], [20], [11], [12] and the references therein.

2.2 Mixture continuity properties

In this subsection we study mixture continuity properties of law determined concave monetary utilities. We say that a utility u is mixture continuous if for each $F, G \in \mathcal{M}_{1,c}$, the function

$$\lambda \mapsto u(\lambda F + (1-\lambda)G)$$

is continuous on $[0,1]$. The next proposition shows that in the case $F = \delta_x$ and $G = \delta_y$ with $x < y$, continuity for $\lambda \in (0,1]$ is always satisfied.

Proposition 2.1. *Let $u\colon L^\infty \to \mathbb{R}$ be a concave law determined monetary utility, and let $x, y \in \mathbb{R}$, with $x < y$. Then the mapping*

$$\lambda \mapsto u(\lambda\delta_x + (1-\lambda)\delta_y)$$

is continuous at each $\lambda \in (0,1]$.

Proof. The TVaR at level α of the dyadic variables under consideration is given by

$$u_\alpha(\lambda\delta_x + (1-\lambda)\delta_y) = \begin{cases} \frac{\lambda}{\alpha}x + \frac{\alpha-\lambda}{\alpha}y & \text{if } 0 < \lambda < \alpha \\ x & \text{if } \alpha \le \lambda \le 1 \end{cases},$$

and is a piecewise linear, non increasing and convex function of λ.
It follows that for each $\nu \in \mathcal{M}_1[0,1]$, the function

$$g_\nu(\lambda) := \int u_\alpha(\lambda\delta_x + (1-\lambda)\delta_y)\nu(d\alpha)$$

is also non increasing and convex with $g_\nu(0) = y$ and $g_\nu(1) = x$.
By the Kusuoka representation (2.3), we have

$$u(\lambda\delta_x + (1-\lambda)\delta_y) = \inf\left\{g_\nu(\lambda) + \bar{c}(\nu) \mid \nu \in \mathcal{M}_1[0,1]\right\},$$

and by monotonicity and the Fatou property, the function $u(\lambda\delta_x + (1-\lambda)\delta_y)$ is left continuous on $(0,1]$ (at least for $x < y$). Then Lemma A.1 implies that $u(\lambda\delta_x + (1-\lambda)\delta_y)$ is continuous in λ for $\lambda \in (0,1)$. Finally, (2.1) gives

$$\limsup_{\lambda\to 1^-} u(\lambda\delta_x + (1-\lambda)\delta_y) \le u(\delta_x),$$

so from monotonicity it follows that

$$\lim_{\lambda\to 1^-} u(\lambda\delta_x + (1-\lambda)\delta_y) = u(\delta_x).$$

□

The simplest example in which mixture continuity for $\lambda \to 0^+$ fails is the coherent utility $u(\xi) = \operatorname{ess\,inf}(\xi)$. There are also many other examples, such as $u(\xi) = \gamma\mathbb{E}[\xi] + (1-\gamma)\operatorname{ess\,inf}(\xi)$, for $\gamma \in (0,1)$. In fact, we prove in Proposition 2.5 that for a law determined monetary concave utility the property of mixture continuity for $\lambda \to 0^+$ is equivalent to the WC property. We begin with a characterisation of the essential infimum.

Lemma 2.2. *Let $u\colon L^\infty \to \mathbb{R}$ be a concave law determined monetary utility. If there exists $a > 0$ and $k_n \uparrow 0$ such that $\forall\alpha \in (0,1)$ we have*

$$u(\alpha\delta_{k_n} + (1-\alpha)\delta_a) < 0,$$

then $u(\xi) = \operatorname{ess\,inf}(\xi)$.

Proof. We first show that for each k_n and for all $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$ there exists a $\mathbb{Q} \in \mathbf{P}$ with

$$c(\mathbb{Q}) \le -k_n \quad \text{and} \quad \mathbb{Q}(B^c) \le \frac{-k_n}{a}. \tag{2.5}$$

Let $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. Then from the hypothesis

$$u(k_n\mathbb{1}_B + a\mathbb{1}_{B^c}) < 0$$

and from the dual representation, there exists $\mathbb{Q} \in \mathbf{P}$ such that

$$k_n \mathbb{Q}(B) + a\mathbb{Q}(B^c) + c(\mathbb{Q}) < 0,$$

which yields

$$a\mathbb{Q}(B^c) + c(\mathbb{Q}) < -k_n \mathbb{Q}(B) \leq -k_n,$$

hence

$$c(\mathbb{Q}) \leq -k_n \quad \text{and} \quad \mathbb{Q}(B^c) \leq \frac{-k_n}{a}.$$

For any $\xi \in L^\infty$, $\xi \geq 0$, and $\beta > 0$, let

$$B := \{\xi \leq \operatorname{ess\,inf}(\xi) + \beta\}.$$

Clearly, $\mathbb{P}(B) > 0$. Taking $\mathbb{Q}$ as in (2.5), we have that

$$\begin{aligned} u(\xi) &\leq \mathbb{E}_{\mathbb{Q}}(\xi) + c(\mathbb{Q}) \leq (\operatorname{ess\,inf}(\xi) + \beta)\mathbb{Q}(B) + \|\xi\|_\infty \mathbb{Q}(B^c) + c(\mathbb{Q}) \\ &\leq \operatorname{ess\,inf}(\xi) + \beta + \frac{-k_n}{a}\|\xi\|_\infty - k_n. \end{aligned}$$

Since this inequality holds for each $\beta > 0$, letting $k_n \uparrow 0$ it follows that $u(\xi) = \operatorname{ess\,inf}(\xi)$. □

Lemma 2.3. *Let $u \colon L^\infty \to \mathbb{R}$ be a concave law determined monetary utility. If there exists $a > 0$, $k_n \to -\infty$ and $\alpha_n \in (0,1)$ such that*

$$u(\alpha_n \delta_{k_n} + (1-\alpha_n)\delta_a) \geq 0,$$

then u has the WC property.

Proof. We show that $\mathcal{S}_m = \{\, \mathrm{d}\mathbb{Q}/\,\mathrm{d}\mathbb{P} \mid c(\mathbb{Q}) \leq m\}$ is uniformly integrable. To this aim, we prove that $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\mathbb{P}(A) \leq \delta \Rightarrow \mathbb{Q}(a) \leq \varepsilon$, for each $\mathbb{Q} \in \mathcal{S}_m$. Let $\varepsilon_n = \frac{a+m}{-k_n}$. Clearly $\varepsilon_n \to 0$, and choosing $\delta_n = \alpha_n$ we have that $\mathbb{P}(A) \leq \delta_n$ implies that $\alpha_n \delta_{k_n} + (1-\alpha_n)\delta_a \leq_{st} \mathbb{P}(A)\delta_{k_n} + (1-\mathbb{P}(A))\delta_a$, so

$$u(k_n \mathbb{1}_A + a\mathbb{1}_{A^c}) \geq 0,$$

that gives for all $\mathbb{Q}$

$$k_n \mathbb{Q}(A) + a\mathbb{Q}(A^c) + c(\mathbb{Q}) \geq 0,$$

which implies

$$\mathbb{Q}(A) \leq \frac{a + c(\mathbb{Q})}{-k_n} \leq \frac{a+m}{-k_n} = \varepsilon_n,$$

which gives the thesis. □

Definition 2.1 (Condition C)**.** Let $k < 0$ and $a > 0$. We say that Condition C holds for (k, a) if there exists $\alpha \in (0,1)$ such that $u(\alpha\delta_k + (1-\alpha)\delta_a) \geq 0$.

Proposition 2.4. *Let $u \colon L^\infty \to \mathbb{R}$ be a concave law determined monetary utility. Then there are the following alternatives:*

a) $u(\xi) = \operatorname{ess\,inf}(\xi)$*, in which case Condition C does not hold for any* (k, a)*,*
b) $u(\xi)$ *has the WC property, in which case Condition C holds for every* (k, a)*,*
c) $u(\xi) \neq \operatorname{ess\,inf}(\xi)$ *and does not have the WC property, in which case Condition C holds only for some* (k, a)*.*

Proof. a) If $u(\xi) = \operatorname{ess\,inf}(\xi)$, then clearly Condition C never holds. The reverse implication is a straightforward consequence of Lemma 2.2.
b) If $u(\xi)$ has the WC property, then the function

$$\lambda \mapsto u\left(\lambda\delta_x + (1-\lambda)\delta_y\right)$$

is continuous also for $\lambda \to 0^+$, as a consequence of the Lebesgue property. Indeed, for any sequence $\lambda_n \to 0^+$, let $\xi_n \sim \lambda_n\delta_x + (1-\lambda_n)\delta_y$, $\xi = y$ $\mathbb{P}$-a.s. and $\xi_n \xrightarrow{\mathbb{P}} \xi$, with $\sup_n \|\xi_n\|_\infty \leq |y|$, so from (2.4) we get

$$u(\lambda_n\delta_x + (1-\lambda_n)\delta_y) \to u(\delta_y).$$

Since $u(k) = k < 0$ and $u(a) = a > 0$, there exists $\alpha \in (0, 1)$ such that $u(\alpha_{\delta_k} + (1-\alpha)\delta_a) = 0$; hence Condition C holds for every (k, a). The reverse implication is a consequence of Lemma 2.3.
c) follows immediately from a) and b). □

Remark. The comparison with condition (3.1) in [41] is of great interest. In our notation, Weber's requirement is that there exists $a_0 > 0$ such that for each $k < 0$, Condition C holds for (k, a_0). For concave law determined monetary utilities, Weber's condition (3.1) is satisfied if and only if $u(\xi)$ has the WC property. The if part follows from Proposition 2.4 item b), while the only if part follows from Lemma 2.3.

In Proposition 2.1 we showed that any concave law determined monetary utility is mixture continuous on dyadic variables for $\lambda \in (0, 1]$. From the trichotomy of Proposition 2.4 it follows that the additional continuity for $\lambda \to 0^+$ is equivalent to the WC property and to the ψ-weak continuity, for some gauge function ψ.

Proposition 2.5. *Let* $u\colon L^\infty \to \mathbb{R}$ *be a concave law determined monetary utility. The following are equivalent:*

a) u *has the WC property.*
b) u *is* ψ*-weakly continuous for some gauge function* ψ*.*
c) For each $x, y \in \mathbb{R}$ *with* $x < y$*, the function* $\lambda \mapsto u(\lambda\delta_x + (1-\lambda)\delta_y)$ *is continuous for* $\lambda \to 0^+$*.*

Proof. First we show that a) $\Rightarrow$ b). Let us consider the case in which u has the dual representation

$$u(\xi) = \inf\left\{\mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q}) \mid \mathbb{Q} \in \mathbf{P}\right\}.$$

From the WC property the sets $\mathcal{S}_k := \{\mathbb{Q} \in \mathbf{P} \mid c(\mathbb{Q}) \leq k\}$ are compact in the $\sigma(L^1, L^\infty)$ topology, for each $k \geq 0$. Let

$$\mathcal{S}_0 := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} \frac{1}{1 + c(\mathbb{Q})} \mid \mathbb{Q} \in \mathbf{P} \right\}.$$

$\mathcal{S}_0$ is relatively sequentially compact in the $\sigma(L^1, L^\infty)$ topology, since for any sequence $\mathbb{Q}_n \in \mathcal{S}_0$ there are two alternatives: either $\mathbb{Q}_n$ definitely belongs to some $\mathcal{S}_k$, or for some subsequence $c(\mathbb{Q}_{n_j}) \to +\infty$. In both cases, the sequence $\mathbb{Q}_n$ has a $\sigma(L^1, L^\infty)$ convergent subsequence. Denoting with $\mathcal{S}$ the solid closed convex hull of $\mathcal{S}_0$, it follows that $\mathcal{S}$ is compact in the $\sigma(L^1, L^\infty)$ topology and hence uniformly integrable. Let

$$E_{\mathcal{S}} := \{f \in L^1 \mid \exists\, \varepsilon > 0, \text{ s.t. } \varepsilon f \in \mathcal{S}\}.$$

$E_{\mathcal{S}}$ is a Banach space with the Luxemburg norm

$$\|f\|_{\mathcal{S}} := \inf \left\{ \lambda \mid \frac{|f|}{\lambda} \in \mathcal{S} \right\}.$$

Since $\mathcal{S}$ is uniformly integrable, from the de la Vallée-Poussin's criterion there exists a Young function Φ with $\frac{\Phi(x)}{x} \to +\infty$ as $x \to +\infty$ such that

$$\sup_{f \in \mathcal{S}} \mathbb{E}\left[\Phi\left(|f|\right)\right] < +\infty.$$

Hence

$$E_{\mathcal{S}} \subseteq L^\Phi \subset L^1,$$

where we can assume $\Phi \in \Delta_2$, so $L^\Phi = H^\Phi$. Passing to the duals we get

$$L^\infty \subset L^\Psi \subseteq (E_{\mathcal{S}})^*,$$

where Ψ is the convex conjugate of Φ. The dual norm on $(E_{\mathcal{S}})^*$ is given by

$$\|\xi\|_u = \sup_{\mathbb{Q} \in \mathbf{P}} \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}P} \frac{1}{1 + c(\mathbb{Q})} |\xi| \right],$$

and since $\|\xi\|_u \leq 1 \iff u(-|\xi|) \geq -1$, it follows that u is finite on $(E_{\mathcal{S}})^*$ and hence also on the Orlicz heart $H^\Psi \subseteq L^\Psi$. If Ψ satisfies the Δ_2 condition, then from the results of [32] it follows that u is ψ-weakly continuous for $\psi(x) := \Psi(|x|)$, and hence we immediately have b). If instead Ψ does not satisfy the Δ_2 condition, then we consider the gauge function

$$\psi(x) = \sum_{k=1}^{+\infty} \lambda_k \Psi(k|x|), \text{ where } \lambda_k = \frac{1}{2^k \Psi(k^2)}.$$

Let $\xi_n \xrightarrow{\psi} \xi$. From Skorohod's representation, it is possible to assume that

$$\xi_n \to \xi \text{ a.s. and } \mathbb{E}[\psi(|\xi_n|)] \to \mathbb{E}[\psi(|\xi|)]. \tag{2.6}$$

By the continuity of ψ, it follows that $\psi(|\xi_n|) \to \psi(|\xi|)$ a.s., since $\psi(|\xi_n|) \geq 0$ a.s. and $\mathbb{E}[\psi(|\xi_n|)] \to \mathbb{E}[\psi(|\xi|)]$, from Scheffé's lemma it follows that we have $\psi(|\xi_n|) \overset{L^1}{\to} \psi(|\xi|)$, so in particular the family $\psi(|\xi_n|)$ is uniformly integrable. Hence for all $\varepsilon > 0$ there exists $C > 0$ such that for all n

$$\varepsilon \geq \mathbb{E}[\psi(|\xi_n|)\mathbb{1}_{\{\psi(|\xi_n|)\geq C\}}],$$

and since $\psi(x) \geq \lambda_k \Psi(k|x|)$ we have that

$$\varepsilon \geq \lambda_k \mathbb{E}[\Psi(k|\xi_n|)\mathbb{1}_{\{\psi(|\xi_n|)\geq C\}}] \geq \lambda_k \mathbb{E}[\Psi(k|\xi_n|)\mathbb{1}_{\{\Psi(k|\xi_n|)\geq C/\lambda_k\}}],$$

which yields the uniform integrability of the family $\Psi(k|\xi_n|)$. Further the family $\Psi(k|\xi_n - \xi|)$ is also uniformly integrable since $\Psi(|x-y|) \leq \Psi(2|x|) + \Psi(2|y|)$. Hence, under (2.6), it holds that

$$\mathbb{E}[\Psi(k|\xi_n - \xi|)] \to 0, \text{ for each } k > 0,$$

which in turn implies $\|\xi_n - \xi\|_\Psi \to 0$ (see for example Proposition 2.1.10 in [20]). The thesis follows then from the Young inequality since

$$|u(\xi_n) - u(\xi)| \leq \sup_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}[|\xi_n - \xi|] \leq 2 \sup_{\mathbb{Q}\in\mathcal{S}} \left\| \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right\|_\Phi \cdot \|\xi_n - \xi\|_\Psi \to 0.$$

To prove that b) $\Rightarrow$ c), let u be $\tilde{\Psi}$-weak continuous and let $x, y \in \mathbb{R}$ with $x < y$. Then

$$\lambda\delta_x + (1-\lambda)\delta_y \overset{\tilde{\Psi}}{\to} \delta_y \text{ for } \lambda \to 0^+,$$

so from $\tilde{\Psi}$-weak continuity it follows that

$$u(\lambda\delta_x + (1-\lambda)\delta_y) \to u(\delta_y).$$

To prove that c) $\Rightarrow$ a), we assume by contradiction that u does not have the WC property. From Proposition 2.4 we know that the Condition C fails for some $(\bar{k}, \bar{a})$, with $\bar{k} < 0$ and $\bar{a} > 0$. Since from Proposition 2.1 the mapping $\lambda \mapsto u(\lambda\delta_{\bar{k}} + (1-\lambda)\delta_{\bar{a}})$ is continuous for $\lambda \in (0,1]$, it must hold that

$$\lim_{\lambda\to 0^+} u(\lambda\delta_{\bar{k}} + (1-\lambda)\delta_{\bar{a}})) < 0 < \bar{a},$$

giving a contradiction with c). □

We notice that as a consequence of the Fatou property, it always holds that $F_n \in \mathcal{N}$, $supp(F_n) \subseteq K$ for some compact K and $F_n \to F$ weakly implies that $F \in \mathcal{N}$. From the preceding theorem it follows that if u has the WC property, then the acceptance set $\mathcal{N}$ is ψ-weakly closed for some gauge function ψ.

3 Monetary concave utility functions with CxLS

From now on we assume that the monetary concave law determined utility $u\colon L^\infty \to \mathbb{R}$ has the property of convex level sets at the level of distributions (CxLS), that is

$$u(F) = u(G) = \gamma \Rightarrow u(\lambda F + (1-\lambda)G) = \gamma, \text{ for each } \lambda \in (0,1).$$

We recall that $\mathcal{N} = \{\text{Law}(\xi) \mid u(\xi) \geq 0\}$ is the acceptance set of u at the level of distributions. We have the following:

Lemma 3.1. *Let u be a monetary concave law determined utility function with CxLS. Then:*

a) $\mathcal{N}$ and $\mathcal{N}^c$ are convex with respect to mixtures.
b) Let $u(\xi) \neq \operatorname{ess\,inf}(\xi)$. Then there exists a $k_0 < 0$ such that Condition C holds for (k_0, a), for each $a > 0$.

Proof. a) Let $F, G \in \mathcal{N}$ and let ξ, η such that $\text{Law}(\xi) = F$ and $\text{Law}(\eta) = G$. Take $A \in \mathcal{F}$ with $\mathbb{P}[A] = \alpha$ and assume the variables ξ, η and the set A to be independent. To construct such ξ, η and A is always possible since on an atomless probability space we have an iid sequence of uniform random variables (see [18]). Without loss of generality, assume that $u(\xi) = u(\eta) + \beta$, with $\beta \geq 0$, and let $\xi' := \xi - \beta$. From translation invariance $u(\xi') = u(\xi) - \beta = u(\eta)$. Let $F' = \text{Law}(\xi')$; then $\xi \mathbb{1}_A + \eta \mathbb{1}_{A^c}$ has law $\alpha F + (1-\alpha)G$ and $\xi' \mathbb{1}_A + \eta \mathbb{1}_{A^c}$ has law $\alpha F' + (1-\alpha)G$. Since $\xi \mathbb{1}_A + \eta \mathbb{1}_{A^c} = \xi' \mathbb{1}_A + \eta \mathbb{1}_{A_c} + \beta \mathbb{1}_A \geq \xi' \mathbb{1}_A + \eta \mathbb{1}_{A_c}$, from monotonicity and CxLS, $u(\xi \mathbb{1}_A + \eta \mathbb{1} A^c) \geq u(\xi' \mathbb{1}_A + \eta \mathbb{1}_{A^c}) = u(\eta) \geq 0$, that gives the convexity of $\mathcal{N}$ with respect to mixtures. A similar argument applies to $\mathcal{N}^c$.
b) From Proposition 2.4, it follows that there exists $k_0 < 0$, $a_0 > 0$ and $\alpha_0 \in (0,1)$ such that $u(\alpha_0 \delta_{k_0} + (1-\alpha_0)\delta_{a_0}) \geq 0$. We have to prove that for each $a > 0$, there exists a suitable $\alpha \in (0,1)$ such that $u(\alpha \delta_{k_0} + (1-\alpha)\delta_a) \geq 0$. If $a \geq a_0$, then by monotonicity $\alpha = \alpha_0$ satisfies the thesis. Let then $0 < a < a_0$. Note first that since $u(\alpha_0 \delta_{k_0} + (1-\alpha_0)\delta_{a_0}) \geq 0$ and $u(0) = 0$, from CxLS and a) it follows that for each $\lambda \in (0,1)$

$$u(\lambda \alpha_0 \delta_{k_0} + (1-\lambda)\delta_0 + \lambda(1-\alpha_0)\delta_{a_0}) \geq 0.$$

By choosing

$$\lambda = \frac{a}{(1-\alpha_0)a_0 + \alpha_0 a},$$

it follows that

$$\lambda \alpha_0 \delta_{k_0} + (1-\lambda)\delta_0 + \lambda(1-\alpha_0)\delta_{a_0} \leq_{cv} \lambda \alpha_0 \delta_{k_0} + (1-\lambda\alpha_0)\delta_a,$$

which implies that $u(\lambda \alpha_0 \delta_{k_0} + (1-\lambda\alpha_0)\delta_a) \geq 0$. □

Remark. Without the hypothesis of CxLS item b) is false, as can be seen by considering $u(\xi) = \frac{3}{4} \operatorname{ess\,inf}(\xi) + \frac{1}{4}\mathbb{E}[\xi]$ and $k_0 = -a$.

Lemma 3.1 shows that when $u(\xi) \neq \operatorname{ess\,inf}(\xi)$, the quantity

$$K := \inf \{k < 0 \mid \forall a > 0, \exists \alpha \in (0,1) \text{ with } u(\alpha\delta_k + (1-\alpha)\delta_a) \geq 0\} \quad (3.1)$$

is well defined, and Proposition 2.4 shows that $K = -\infty$ if and only if u has the WC property. Moreover, Condition C holds on the set $(K, 0) \times (0, +\infty)$ or on the set $[K, 0) \times (0, +\infty)$.

Lemma 3.2. *Let K be as in (3.1), and let $k \in (K, 0)$ and $a > 0$. Then*

$$C(k,a) := \{\alpha \mid u(\alpha\delta_k + (1-\alpha)\delta_a) \geq 0\}$$

is a closed interval. Moreover, letting

$$\alpha(k,a) := \max C(k,a), \quad (3.2)$$

it holds that $\alpha(k,a)$ is non decreasing with respect to k and a, and

$$u(\alpha(k,a)\delta_k + (1-\alpha(k,a))\delta_a) = 0.$$

Proof. The first part of the thesis and the last equality follow from Proposition 2.1. By the assumption $k \in (K, 0)$ it follows that $0 < \alpha(k,a) < 1$. From the monotonicity of u we have

$$k \leq k',\ a \leq a' \Rightarrow C(k,a) \subseteq C(k',a'),$$

which yields the monotonicity of $\alpha(k,a)$. □

We now parallel the construction of [41], including also the case $K > -\infty$. We begin by defining $\varphi\colon (K, +\infty) \to \mathbb{R}$ as in [41]. We set $\varphi(0) = 0$. For $k \in (K, 0)$, we define $\varphi(k)$ implicitly by means of

$$\varphi(k)\alpha(k,1) + (1 - \alpha(k,1)) = 0,$$

hence

$$\varphi(k) = -\frac{1-\alpha(k,1)}{\alpha(k,1)} = 1 - \frac{1}{\alpha(k,1)} < 0, \quad (3.3)$$

which is non decreasing in k, by Lemma 3.2. For $a > 0$, we fix a reference point $k_0 \in (K, 0)$ and define $\varphi(a)$ implicitly by means of

$$\varphi(k_0)\alpha(k_0,a) + \varphi(a)(1 - \alpha(k_0,a)) = 0,$$

hence

$$\varphi(a) = -\frac{\varphi(k_0)\alpha(k_0,a)}{1-\alpha(k_0,a)} = \varphi(k_0)\left[1 + \frac{1}{\alpha(k_0,a) - 1}\right] > 0, \quad (3.4)$$

which is also non decreasing in a, since $\varphi(k_0) < 0$. It can be easily checked that $\varphi(1) = 1$, independently on the choice of the reference point k_0. Thus the affine functional $L_\varphi\colon \mathcal{M}_{1,c} \to \mathbb{R}$ given by

$$L_\varphi(\mu) = \int \varphi \,\mathrm{d}\mu$$

with φ defined in (3.3) and (3.4) vanishes whenever we have that either $\mu = \alpha(k_0, a)\delta_{k_0} + (1 - \alpha(k_0, a))\delta_a$ or $\mu = \alpha(k, 1)\delta_k + (1 - \alpha(k, 1))\delta_1$ holds. In the following lemma we prove that L_φ vanishes also on all dyadic variables of the more general form $\mu = \alpha(k, a)\delta_k + (1 - \alpha(k, a))\delta_a$, with $\alpha(k, a)$ defined in (3.2).

Lemma 3.3. *Let K be as in* (3.1)*, $\alpha(k, a)$ as in* (3.2) *and φ as in* (3.3) *and* (3.4)*. Let $k \in (K, 0)$ and $a > 0$. Then*

$$\alpha(k, a)\varphi(k) + (1 - \alpha(k, a))\varphi(a) = 0.$$

Proof. If $k = k_0$ or if $a = 1$ the thesis is immediate from (3.3) and (3.4), so we assume that $k \neq k_0$ and $a \neq 1$. Let

$$\Delta = \left\{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+ \mid \sum_{i=1}^{3} \lambda_i \leq 1 \right\},$$

and let $\Phi\colon \Delta \to \mathcal{M}_{1,c}(\mathbb{R})$ be defined by

$$\Phi(\lambda_1, \lambda_2, \lambda_3) = (1 - \sum_{i=1}^{3} \lambda_i)\delta_{k_0} + \lambda_1\delta_1 + \lambda_2\delta_a + \lambda_3\delta_k.$$

Φ is an affine bijective mapping of Δ onto the finite dimensional face of $\mathcal{M}_{1,c}$ given by the measures with support in k_0, k, 1, a. Let

$$\mathcal{D} := \{(\lambda_1, \lambda_2, \lambda_3) \in \Delta \mid \Phi(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{N}\}$$
$$\mathcal{C} := \{(\lambda_1, \lambda_2, \lambda_3) \in \Delta \mid \Phi(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{N}^c\}.$$

From Lemma 3.1 it follows that $\mathcal{D}$ and $\mathcal{C}$ are convex, $\mathcal{D}$ is closed and $\mathcal{C}$ is relatively open in Δ. We consider the following points in Δ:

$$x^1 := (1 - \alpha(k_0, 1), 0, 0)$$
$$x^2 := (0, 1 - \alpha(k_0, a), 0)$$
$$x^3 := (1 - \alpha(k, 1), 0, \alpha(k, 1))$$
$$x^4 := (0, 1 - \alpha(k, a), \alpha(k, a)).$$

By (3.2), it follows that $x^i \in \mathcal{D}$ and each x^i is in the relative closure of $\mathcal{C}$ with respect to Δ. Our aim is to show that x^4 is an affine combination of x^1, x^2, x^3. Since by definition $\int \varphi \, \mathrm{d}\Phi(x^i) = 0$ for $i = 1, \ldots, 3$, this would imply that $\int \varphi \, \mathrm{d}\Phi(x^4) = 0$, which is the thesis. In order to show that x^4 is an affine combination of x^1, x^2, x^3, we apply the separation theorem to the convex and disjoint sets $\mathcal{D}$ and $\mathcal{C}$. There exists a nontrivial linear $f : \mathbb{R}^3 \to \mathbb{R}$ and $s \in \mathbb{R}$ such that $f(x) \leq s$ on $\mathcal{C}$ and $f(x) \geq s$ on $\mathcal{D}$. Since $(0, 0, 0) \in \mathcal{C}$ and $f(0, 0, 0) = 0$, it follows that $s \geq 0$. If $s = 0$, we would get $f(x^i) = 0$ for $i = 1, \ldots, 3$ since x^i is in the relative closure of C, which in turn would imply that $f = 0$, which is a contradiction; hence $s > 0$. Thus the points x^1, x^2, x^3, x^4 lie on the nontrivial hyperplane $f(x) = s$; from the linear independence of x^1, x^2, x^3 it follows that x^4 is an affine combination of x^1, x^2, x^3. □

Corollary 3.4. *Let K be as in* (3.1) *and φ as in* (3.3) *and* (3.4)*. Let $k, k_0 \in (K, 0)$ and $a > 0$. If ξ is supported by $k, k_0, 1, a$, then $u(\xi) < 0$ if and only if $\mathbb{E}[\varphi(\xi)] < 0$.*

Lemma 3.5. *Let K be as in* (3.1) *and φ as in* (3.3) *and* (3.4)*. Let ξ be supported by finitely many points, all greater than K. Then $u(\xi) < 0$ if and only if $E[\varphi(\xi)] < 0$.*

Proof. Let $k_0, k_1, \dots, k_N < 0 \le a_1, \dots, a_M$ be distinct points. Similar to the proof of Lemma 3.3, we define

$$\begin{aligned}
&\Delta = \{(\lambda_0, \dots, \lambda_N, \gamma_1, \dots, \gamma_M) \mid \sum \lambda_i + \sum \gamma_j = 1, \lambda_i \ge 0, \gamma_j \ge 0\},\\
&\Phi(\underline{\lambda}, \underline{\gamma}) = \sum_{i=0}^{N} \lambda_i \delta_{k_i} + \sum_{j=1}^{M} \gamma_j \delta_{a_j}, \mathcal{D} = \{(\underline{\lambda}, \underline{\gamma}) \in \Delta \mid \Phi(\underline{\lambda}, \underline{\gamma}) \in \mathcal{N}\},\\
&\mathcal{C} = \{(\underline{\lambda}, \underline{\gamma}) \in \Delta \mid \Phi(\underline{\lambda}, \underline{\gamma}) \in \mathcal{N}^c\}.
\end{aligned}$$

As in the proof of Lemma 3.3, $\mathcal{D}$ and $\mathcal{C}$ are convex, $\mathcal{D}$ is closed and $\mathcal{C}$ is relatively open in Δ. By the separation theorem, reasoning as in the proof of Lemma 3.3, there is an affine functional $g \colon \Delta \to \mathbb{R}$ such that $g(x) < 0$ for $x \in \mathcal{C}$ and $g(x) \ge 0$ for $x \in \mathcal{D}$. Let $G := \{x \mid g(x) = 0\}$. Then $G \cap \Delta$ is compact and convex and its extremal points lie on the edges of Δ. Let us denote with $\overline{k}_i$ and $\overline{a}_i$ the corners of Δ corresponding respectively to δ_{k_i} and δ_{a_i}. There are four type of edges: $[\overline{k}_i, \overline{k}_{i'}]$, $[\overline{a}_j, \overline{a}_{j'}]$, $[\overline{k}_i, 0]$, $[\overline{k}_i, \overline{a}_j]$, with $a_j > 0$. Analysing the different possibilities, it follows that the extremal points of $G \cap \Delta$ corresponds to δ_0 or to dyadic variables of the form $\alpha(k, a)\delta_k + (1 - \alpha)(k, a)\delta_a$, with $\alpha(k, a)$ given by (3.2). It follows that $\int \varphi \,\mathrm{d}\Phi(x) = 0$ for all extremal points of $G \cap \Delta$, and hence $\int \varphi \,\mathrm{d}\Phi(x) = 0$ for each $x \in G \cap \Delta$. Let $F = \{x \in \mathbb{R}^{N+M+1} \mid \int \varphi \,\mathrm{d}\Phi(x) = 0, \sum_{i=0}^{N+M} x_i = 1\}$. F and G are affine spaces of dimension $N + M - 1$ and $G \subset F$, so that $G = F$. Hence $x \in C \iff g(x) < 0 \iff \int \varphi \,\mathrm{d}\Phi(x) < 0$, which gives the thesis.

□

Lemma 3.6. *The function $\varphi \colon (K, +\infty) \to \mathbb{R}$ is concave on $(K, 0)$ and on $[0, +\infty)$ and right continuous.*

Proof. Take $k_1, k_2 < 0$ and let $\alpha_1 := \alpha(k_1, 1)$ and $\alpha_2 := \alpha(k_2, 1)$, so that

$$u(\alpha_1 \delta_{k_1} + (1 - \alpha_1)\delta_1) = u(\alpha_2 \delta_{k_2} + (1 - \alpha_2)\delta_1) = 0.$$

From CxLS, for each $\gamma \in (0, 1)$, it holds that

$$u(\gamma(\alpha_1 \delta_{k_1} + (1 - \alpha_1)\delta_1) + (1 - \gamma)(\alpha_2 \delta_{k_2} + (1 - \alpha_2)\delta_1)) = 0,$$

or equivalently

$$u\left(\gamma \alpha_1 \delta_{k_1} + (1 - \gamma)\alpha_2 \delta_{k_2} + \left[\gamma(1 - \alpha_1) + (1 - \gamma)(1 - \alpha_2)\right] \delta_1\right) = 0.$$

Let

$$\lambda = \frac{\gamma\alpha_1}{\gamma\alpha_1 + (1-\gamma)\alpha_2}$$

and $k = \lambda k_1 + (1-\lambda)k_2$. Then

$$\gamma\alpha_1\delta_{k_1} + (1-\gamma)\alpha_2\delta_{k_2} \leq_{cv} (\gamma\alpha_1 + (1-\gamma)\alpha_2)\delta_k,$$

hence the isotonicity of u with respect to the concave order implies

$$u\left((\gamma\alpha_1 + (1-\gamma)\alpha_2)\delta_k + \left[\gamma(1-\alpha_1) + (1-\gamma)(1-\alpha_2)\right]\delta_1\right) \geq 0,$$

and from Lemma 3.5 we get

$$(\gamma\alpha_1 + (1-\gamma)\alpha_2)\varphi(k) + [\gamma(1-\alpha_1) + (1-\gamma)(1-\alpha_2)] \geq 0,$$

or

$$\begin{aligned}
\varphi(k) &\geq -\frac{(1-\gamma)(1-\alpha_2) + \gamma(1-\alpha_1)}{\gamma\alpha_1 + (1-\gamma)\alpha_2} \\
&= \frac{\gamma\alpha_1}{\gamma\alpha_1 + (1-\gamma)\alpha_2}\left(-\frac{1-\alpha_1}{\alpha_1}\right) + \frac{(1-\gamma)\alpha_2}{\gamma\alpha_1 + (1-\gamma)\alpha_2}\left(-\frac{1-\alpha_2}{\alpha_2}\right) \\
&= \lambda\varphi(k_1) + (1-\lambda)\varphi(k_2).
\end{aligned}$$

A similar argument can be used to establish concavity on $[0, +\infty)$.
Let $a_n \downarrow 0$ and let $\alpha_n := \alpha(k_0, a_n)$, with α given by (3.2). From the monotonicity of α, the sequence α_n is non increasing; we denote with α_0 its limit. Let $B_{\alpha_n} \in \mathcal{F}$ with $\mathbb{P}(B_{\alpha_n}) = \alpha_n$, and let $\xi_n = k_0 \mathbb{1}_{B_{\alpha_n}} + a_n \mathbb{1}_{B^c_{\alpha_n}}$. Then $\xi_n \xrightarrow{\mathbb{P}} \xi$, with $\xi = k_0 \mathbb{1}_{B_{\alpha_0}}$, and from the Fatou property $u(\xi) \geq \limsup u(\xi_n) \geq 0$, since $\xi_n \in \mathcal{A}$. Hence $\mathbb{E}_{\mathbb{P}}[\xi] = k_0 \mathbb{P}[B_{\alpha_0}] \geq u(\xi) \geq 0$, which gives $\alpha_0 = 0$. Since $\varphi(a_n) = \frac{-\alpha_n \varphi(k_0)}{1-\alpha_n}$, it follows that $\varphi(a_n) \to 0$ when $a_n \to 0$. □

From now on we define

$$\varphi(K) = \lim_{x \downarrow K} \varphi(x).$$

Lemma 3.7. *If $\xi \geq K$ a.s., then $u(\xi) < 0$ if and only if $E[\varphi(\xi)] < 0$.*

Proof. Let $\xi_n > K$, ξ_n finitely supported, with $\xi_n \downarrow \xi$ and $\|\xi_n - \xi\|_\infty \to 0$. Then

$$u(\xi) < 0 \iff \exists n \text{ s.t. } u(\xi_n) < 0 \iff \exists n \text{ s.t. } \mathbb{E}[\varphi(\xi_n)] < 0 \iff \mathbb{E}[\varphi(\xi)] < 0,$$

where the second equivalence follows from Lemma 3.5 and the last equivalence from the right continuity of φ in 0. □

Lemma 3.8. *φ is concave on $(K, +\infty)$ and hence continuous on $(K, +\infty)$.*

Proof. The proof proceeds along the lines of Lemma 3.6, using Lemma 3.7 instead of Lemma 3.5 to have the stronger thesis. □

We can finally prove the announced characterisation:

Theorem 3.9. *Let $u\colon L^\infty \to \mathbb{R}$ be a monetary, concave law determined utility with CxLS. Then there exists a concave $\overline{\varphi}\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that $u(\xi) \geq 0$ if and only if $E[\overline{\varphi}(\xi)] \geq 0$.*

Proof. If $u(\xi) = \operatorname{ess\,inf}(\xi)$, then

$$\overline{\varphi}(x) = \begin{cases} -\infty & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

satisfies the thesis. If $u(\xi) \neq \operatorname{ess\,inf}(\xi)$, define

$$\overline{\varphi} = \begin{cases} -\infty & \text{if } x < K \\ \varphi(x) & \text{if } x \geq K \end{cases}$$

with φ and K defined as before. If $\xi \geq K$ a.s., then Lemma 3.7 gives that $u(\xi) \geq 0$ if and only if $E[\varphi(\xi)] \geq 0$, which is the thesis. If $\mathbb{P}(\xi < K) > 0$, then $\mathbb{E}[\overline{\varphi}(\xi)] = -\infty$. In order to show that $u(\xi) < 0$, let $\mathcal{B}$ be the algebra generated by the events $\{\xi < K\}$ and $\{\xi \geq K\}$, and let $\eta = \mathbb{E}[\xi \mathbb{1}_{\xi<K} + \xi \mathbb{1}_{\xi \geq 0} | \mathcal{B}]$. Then

$$u(\eta) \geq u(\xi \mathbb{1}_{\xi<K} + \xi \mathbb{1}_{\xi \geq 0}) \geq u(\xi),$$

and $u(\eta) < 0$ by definition of K, so that $u(\xi) < 0$, which completes the proof. □

Example 3.1 (Essential infimum)**.** Let $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, with

$$\varphi(x) = \begin{cases} -\infty & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases}$$

Then $\mathcal{A} = \{\xi | \xi \geq 0\}$ and $u(\xi) = \operatorname{ess\,inf}(\xi)$.

Example 3.2 (Finite Shortfall)**.** Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ concave and increasing with $\varphi(0) = 0$. Then

$$\mathcal{A} = \{\xi \in L^\infty | \mathbb{E}[\varphi(\xi)] \geq 0\}$$

is the acceptance set of a concave law determined utility u with CxLS. In the particular case $\varphi(x) = x$ we have $u(\xi) = \mathbb{E}[\xi]$. Let us remark that the function $\varphi(x) = x$ for $x \leq 0$ and $\varphi(x) = 0$ for $x \geq 0$ also defines the essential infimum.

Example 3.3 (Truncated shortfall)**.** Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ concave and increasing with $\varphi(0) = 0$ and $K < 0$. Set $\overline{\varphi}\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$

$$\overline{\varphi}(x) := \begin{cases} -\infty & \text{if } x < K \\ \varphi(x) & \text{if } x \geq K. \end{cases}$$

Then $\mathcal{A} = \{\xi | \mathbb{E}[\varphi(\xi) \geq 0], \xi \geq K\}$.

Example 3.4 (Truncated mean)**.** Let

$$\overline{\varphi}(x) := \begin{cases} x & \text{if } x \geq -1 \\ -\infty & \text{if } x < -1. \end{cases}$$

Then

$$\mathcal{A} = \{\xi \in L^\infty \text{ s.t. } \xi \geq -1 \text{ and } \mathbb{E}[\xi] \geq 0\},$$

and

$$u(\xi) = \min(\mathbb{E}[\xi], 1 + \operatorname{ess\,inf}(\xi)).$$

It is easy to see that mixture continuity for $\lambda \to 0^+$ fails, so from Proposition 2.5 the WC property does not hold. Indeed the penalty function c is given by

$$c(\mathbb{Q}) = 1 - \operatorname{ess\,inf} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$$

and does not have $\sigma(L^1, L^\infty)$ compact lower level sets, since $c(\mathbb{Q}) \leq 1$.

Example 3.5. Let $K < 0$ and

$$\varphi(x) = \begin{cases} \log(x - K) - \log(-K) & \text{if } x > K \\ -\infty & \text{if } x \leq K. \end{cases}$$

Example 3.6.

$$\varphi(x) = \begin{cases} \sqrt{x+1} & \text{if } x \geq -1 \\ -\infty & \text{if } x < -1. \end{cases}$$

Note that in the last two examples $\varphi'_+(K) = +\infty$, so they do not belong to the family of "truncated shortfalls" considered in Example 3.3.

A Appendix

Lemma A.1. *Let $\phi_i \colon [0,1] \to \mathbb{R}$ be non decreasing and convex functions with $D_0 < \phi_i < D_1$, and let c_i be bounded from below. Then $\phi(x) := \inf_i \big(\phi_i(x) + c_i\big)$ is Lipschitz on compact subsets of $(0,1)$.*

Proof. Let $0 < \varepsilon < 1$, $\varepsilon \leq x < y \leq 1 - \varepsilon$, and $z = y + \varepsilon$. Then $z \leq 1$ and

$$y = (1-\lambda)z + \lambda x, \text{ with } \lambda = \frac{\varepsilon}{\varepsilon + y - x}.$$

Hence $\phi_i(y) \leq (1-\lambda)\phi_i(z) + \lambda\phi_i(x)$, that gives

$$\phi_i(y) - \phi_i(x) \leq (1-\lambda)[\phi_i(z) - \phi_i(x)] \leq \frac{y-x}{\varepsilon}(D_1 - D_0).$$

It follows that the family $\{\phi_i(x) + c_i\}_i$ is equi-Lipschitz on $[\varepsilon, 1-\varepsilon]$, which implies that also $\phi(x) = \sup_i \big(\phi_i(x) + c_i\big)$ is Lipschitz on $[\varepsilon, 1-\varepsilon]$; letting $\varepsilon \to 0$ gives the thesis. ☐

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