Weighted Norm Inequalities and Closedness of a Space of Stochastic Integrals

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Summary : Let X be an $\mathbb{R}^d$-valued special semimartingale on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ with canonical decomposition $X = X_0 + M + A$. Denote by $G_T(\Theta)$ the space of all random variables $(\theta \cdot X)_T$, where θ is a predictable X-integrable process such that the stochastic integral $\theta \cdot X$ is in the space $\mathcal{S}^2$ of semimartingales. We investigate under which conditions on the semimartingale X the space $G_T(\Theta)$ is closed in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$, a question which arises naturally in the applications to financial mathematics. Our main results give necessary and/or sufficient conditions for the closedness of $G_T(\Theta)$ in $\mathcal{L}^2(P)$. Most of these conditions deal with BMO-martingales and reverse Hölder inequalities which are equivalent to weighted norm inequalities. By means of these last inequalities, we also extend previous results on the Föllmer-Schweizer decomposition.

Key words : Semimartingales, Stochastic Integrals, Reverse Hölder Inequalities, BMO Space, Weighted Norm Inequalities, Föllmer-Schweizer Decomposition.

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0. Financial introduction.

Despite its rather mathematical title, this paper is concerned with questions which arise from a number of optimization problems in financial applications. It seems therefore appropriate to start with a motivating section to explain the background and the financial interpretation of the results. We emphasize that this section will not contain precise definitions and theorems; the mathematical introduction in the next section will contain more technical details.
Our starting point is a d-dimensional stochastic process $X = (X_t)_{0 \le t \le T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and adapted to a filtration $I\!\!F = (\mathcal{F}_t)_{0 \le t \le T}$ with a fixed time horizon $T \in (0, \infty]$. The process X describes the discounted price evolution of d risky assets in a financial market containing also some riskless asset with discounted price $Y \equiv 1$. Thus, $\mathcal{F}_t$ is the information available at time t and X^i_t is the relative price of asset i at time t, expressed in units of some fixed numeraire. Adaptedness of X simply means that X^i_t is observable at time t. One of the central problems in financial mathematics in such a framework is the pricing and hedging of contingent claims by means of dynamic trading strategies based on X. The prime example of a contingent claim is of course a European call option on some asset i with expiration date T and strike price K, say. The net payoff to its owner at T is obviously the random amount

$$H(\omega) = \max\left(X^i_T(\omega) - K, 0\right) = \left(X^i_T(\omega) - K\right)^+ .$$

More generally, a contingent claim will here simply be an $\mathcal{F}_T$-measurable random variable H describing the net payoff at T of the financial instrument we want to consider. This means that our claims are "European" in the sense that the date of the payoff is fixed, but the amount to be paid out is allowed to depend on the whole history of X up to time T (or even more, if $I\!\!F$ contains additional information). The problems of pricing and hedging H can then be formulated as follows: What price should the seller S of H charge the buyer B at time 0? And having sold H, how can the seller S insure himself against the upcoming random loss at time T?
A natural way to approach these questions is to consider dynamic portfolio strategies of the form $(\theta, \eta) = (\theta_t, \eta_t)_{0 \le t \le T}$, where θ is a d-dimensional predictable process and η is adapted. In such a strategy, θ^i_t describes the number of units of asset i held at time t, and η_t is the amount invested in the riskless asset at time t. Predictability of θ is then a mathematical formulation of the informational constraint that θ is not allowed to anticipate the movement of X. At any time t, the value of the portfolio (θ_t, η_t) is given by

$$V_t = \theta'_t X_t + \eta_t$$

and the cumulative gains from trade up to time t are

$$G_t(\theta) = \int_0^t \theta_s \, dX_s =: (\theta \cdot X)_t .$$

To have this expression well-defined, we assume that X is a semimartingale, and $G(\theta)$ is then the stochastic integral of θ with respect to X. The cumulative costs up to time t incurred by using (θ, η) are given by

$$C_t = V_t - \int_0^t \theta_s \, dX_s = V_t - G_t(\theta).$$

A strategy is called self-financing if its cumulative cost process C is constant in time, and this is equivalent to saying that its value process V is given by

$$V_t = c + \int_0^t \theta_s \, dX_s = c + G_t(\theta), \tag{0.1}$$

where $c = V_0 = C_0$ denotes the initial cost to start the strategy. After time 0, such a strategy is self-supporting: any fluctuations in X can be neutralized by rebalancing θ and η in such a way that no further gains or losses are incurred. Observe that a self-financing strategy is completely determined by c and θ since the self-financing constraint determines V, hence also η.

Now fix a contingent claim H and suppose that there exists a self-financing strategy (c, θ) whose terminal value V_T equals H with probability one. If our market model does not allow arbitrage opportunities, it is immediately clear that the price of H must be given by c, and that θ furnishes a hedging strategy against H. This was the basic insight leading to the celebrated Black-Scholes formula for option pricing; see Black/Scholes (1973) and Merton (1973) who solved this problem for the case where $H = (X_T - K)^+$ is a European call option and X is a one-dimensional geometric Brownian motion. The mathematical structure of the problem and its connections to martingale theory were subsequently worked out and clarified by J. M. Harrison and D. M. Kreps; a detailed account can be found in Harrison/Pliska (1981). Following their terminology, a contingent claim H is called attainable if there exists a self-financing trading strategy whose terminal value equals H with probability one. By (0.1), this means that H can be written as

$$H = H_0 + \int_0^T \xi_s^H \, dX_s \qquad P\text{-a.s.}, \tag{0.2}$$

i.e., as the sum of a constant H_0 and a stochastic integral with respect to X. We speak of a complete market if every contingent claim is attainable. (Recall that we do not give here precise definitions; for a clean mathematical formulation, one has to be rather careful about the integrability conditions imposed on H and ξ^H.)

The importance of the concept of a complete market stems from the fact that it allows the pricing and hedging of contingent claims to be done in a preference-independent fashion. However, completeness is a rather delicate property which typically gets lost if one considers even minor modifications of a basic complete model. For instance,

geometric Brownian motion (the classical Black-Scholes model) becomes incomplete if the volatility is influenced by a second stochastic factor or if one adds a jump component to the model. If one insists on a preference-free approach under incompleteness, one can study the range of possible prices which are consistent with absence of arbitrage in a market containing X, Y and H as traded instruments; see for instance El Karoui/Quenez (1995). An alternative is to introduce subjective criteria according to which strategies are chosen and option prices are computed, and we shall briefly explain two such criteria in the sequel.
For a non-attainable contingent claim, it is by definition impossible to find a strategy with final value $V_T = H$ which is at the same time self-financing. A first possible approach is to insist on the terminal condition $V_T = H$; since η is allowed to be adapted, this condition can always be satisfied by choice of η_T. But since such strategies will not be self-financing, a "good" strategy should now have a "small" cost process C. To measure the riskiness of a strategy, the use of a quadratic criterion was first proposed by Föllmer/Sondermann (1986) for the case where X is a martingale and subsequently extended to the general case in Schweizer (1991). Under certain technical assumptions, such a *locally risk-minimizing* strategy can be characterized by two properties: its cost process C should be a martingale (so that the strategy is no longer self-financing, but still remains mean-self-financing), and this martingale should be orthogonal to the martingale part M of the price process X. Translating this description into conditions on the contingent claim H shows that there exists a locally risk-minimizing strategy for H if and only if H admits a decomposition of the form

$$H = H_0 + \int_0^T \xi_s^H \, dX_s + L_T^H \qquad P\text{-a.s.,} \tag{0.3}$$

where L^H is a martingale orthogonal to M; see Föllmer/Schweizer (1991). The decomposition (0.3) has been called the *Föllmer-Schweizer decomposition* of H; it can be viewed as a generalization to the semimartingale case of the classical Galtchouk-Kunita-Watanabe decomposition from martingale theory. Its financial importance lies in the fact that it directly provides the locally risk-minimizing strategy for H: the risky component θ is given by the integrand ξ^H, and η is determined by the requirement that the cost process C should coincide with $H_0 + L^H$. Note also that the special case (0.2) of an attainable claim simply corresponds to the absence of the orthogonal term L_T^H. In particular cases, one can give more explicit constructions for the decomposition (0.3). In the case of finite discrete time, ξ^H and L^H can be computed recursively backward in time; see Schweizer (1995). If X is continuous, the Föllmer-Schweizer decomposition under P can be obtained as the Galtchouk-Kunita-Watanabe decomposition, computed under the so-called *minimal martingale measure* $\widehat{P}$; see for instance Föllmer/Schweizer (1991).
One drawback of the preceding method is the fact that one has to work with strategies which are not self-financing. To avoid intermediate costs or an unplanned income, a second approach is therefore to insist on the self-financing constraint (0.1). The possible

final outcomes of such strategies are of the form $c+G_T(\theta)$ for some initial capital $c \in I\!R$ and some strategy component θ in the set Θ, say, of all integrands allowed in (0.1). By definition, a non-attainable claim H is not of this form, and so it seems natural to look for a best approximation of H by the terminal value $c+G_T(\theta)$ of some pair (c,θ). The use of a quadratic criterion to measure the quality of this approximation has been proposed by Bouleau/Lamberton (1989) if X is both a martingale and a function of a Markov process, and by Duffie/Richardson (1991) and Schweizer (1994), among others, in more general cases. To find such a *mean-variance optimal* strategy, one therefore has to project H in $\mathcal{L}^2(P)$ on the space $I\!R + G_T(\Theta)$ of attainable claims. In particular, this raises the question whether the space $G_T(\Theta)$ of stochastic integrals is closed in $\mathcal{L}^2(P)$, and this is the main problem studied in this paper.

Before we turn to a more detailed mathematical introduction, let us very briefly describe the main results of the paper. We provide necessary and sufficient conditions for the closedness of $G_T(\Theta)$ in $\mathcal{L}^2(P)$, thus characterizing the existence of mean-variance optimal hedging strategies for arbitrary contingent claims H. Moreover, we also provide new results on the existence and continuity of the Föllmer-Schweizer decomposition, thus ensuring the existence of locally risk-minimizing hedging strategies.

1. Mathematical introduction.

While the previous section is aimed at the finance-oriented part of our readers, this section will discuss in more detail the mathematical aspects of the paper. In particular, we shall here be more careful about definitions and terminology. But in order not to overload this introductory part with too many formal definitions, we still refer to the subsequent sections for unexplained notations.

Consider an $I\!R^d$-valued semimartingale $X=(X_t)_{0\le t\le T}$ defined on a filtered probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},P\big)$ with a fixed time horizon $T\in(0,\infty]$. If X is in $\mathcal{S}^2_{loc}$, then X is special and admits a canonical decomposition

$$X = X_0 + M + A\ .$$

In the present paper, we shall develop an $\mathcal{L}^2$-theory, and so we introduce the space Θ of all predictable X-integrable processes θ such that the stochastic integral

$$G(\theta) := \int \theta dX =: \theta \cdot X$$

is in the space $\mathcal{S}^2$ of semimartingales. As explained in the previous section, a random variable of the form $H = c + G_T(\theta)$ with $c \in I\!R$ and $\theta\in\Theta$ can be interpreted as the final value of a self-financing trading strategy θ which starts with initial capital c, and so the question arises which random variables H are attainable, i.e., can be represented in the above form.

In the typical case of an incomplete financial market, the space of attainable random variables is a proper subspace of $\mathcal{L}^2(\Omega,\mathcal{F}_T,P)$. The problem of determining whether the space

$$G_T(\Theta) := \{(\theta \cdot X)_T \mid \theta \in \Theta\}$$

is *closed* in $\mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ is the central topic of this paper. Note that if $G_T(\Theta)$ or (equivalently) the space $\mathrm{span}(G_T(\Theta), \mathbf{1})$ spanned by $G_T(\Theta)$ and the constant functions is closed in $\mathcal{L}^2(P)$, we may form the orthogonal projection from $\mathcal{L}^2(P)$ onto $\mathrm{span}(G_T(\Theta), \mathbf{1})$ and thus decompose a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ as $H = H^1 + H^2$, where H^1 is attainable while H^2 is orthogonal to $G_T(\Theta)$ and $\mathbf{1}$. As explained in the financial introduction, this provides a mean-variance optimal hedging strategy for H. But quite apart from the motivation for the present study arising from these applications in financial mathematics, one can also consider the problem of characterising the closedness of $G_T(\Theta)$ from a purely mathematical point of view.
In the case where X is a (local) martingale, this question has been studied some time ago. In fact, the right notion of stochastic integration is designed in such a way that the stochastic integral of a local martingale is an isometry between Hilbert spaces, and so the closedness of $G_T(\Theta)$ holds true almost by definition; see Kunita/Watanabe (1967). Actually, there is even a stronger result since Yor (1978) has proved that if Y^n and Y are uniformly integrable martingales such that $(Y^n_\infty)_{n \in I\!N}$ converges weakly to Y_∞ in $\mathcal{L}^1$, and if $Y^n = \phi^n \cdot X$ for all n, then there is a predictable process ϕ such that $Y = \phi \cdot X$. It is a natural question, which might or should have been asked 15 or 20 years ago, to which extent such results for local martingales generalize to semimartingales.
When X is only a semimartingale, further assumptions must be added to study this problem. A usual hypothesis in financial mathematics is a 'no arbitrage' condition, which roughly states that one cannot obtain a positive gain for free. An important consequence is that the finite variation part A of X is absolutely continuous with respect to the variance process $\langle M \rangle$ of the martingale part M; see Ansel/Stricker (1992). According to Delbaen/Schachermayer (1996a), such an absence of arbitrage implies that there is a predictable process λ such that

$$dA_t = d\langle M \rangle_t \, \lambda_t \qquad P\text{-a.s. for all } t \in [0, T],$$

and so we shall assume that λ exists. Moreover, we shall also assume the existence of the so-called mean-variance tradeoff process of X which is defined by

$$K := \int \lambda' d\langle M \rangle \, \lambda,$$

where $'$ denotes transposition. In a discrete-time framework, Schweizer (1995) has proved that $G_T(\Theta)$ is closed if K is uniformly bounded. The same result has been established in continuous time by Monat/Stricker (1994, 1995).
Uniform boundedness of K is equivalent to requiring that the martingale $\lambda \cdot M$ is in H^∞. This is sufficient for the closedness of $G_T(\Theta)$, but quite far from being necessary; see Monat/Stricker (1995) for a counterexample. It turns out that the closedness of $G_T(\Theta)$ is rather related to the question of whether $\lambda \cdot M$ is in BMO and the (intimately related) question of whether the exponential martingale $\mathcal{E}(-\lambda \cdot M)$ or $\mathcal{E}(-\lambda \cdot M + N)$,

for a suitable martingale N strongly orthogonal to M, satisfies the reverse Hölder condition $R_2(P)$. In the case where X is not necessarily continuous, additional care has to be taken to find the right notion for BMO, and it turns out that bmo_2 is the right choice.

The main results of this paper are summarized in the subsequent three theorems.

Theorem A. Let X be an $\mathbb{R}^d$-valued semimartingale such that there is an equivalent local martingale measure Q with $\frac{dQ}{dP} \in \mathcal{L}^2(P)$. Then the following two assertions are equivalent :
i) The process $\lambda \cdot M$ is a martingale in bmo_2.
ii) Condition $D_2(P)$ holds true, i.e., there is a constant $C > 0$ such that for all $\theta \in L^2(M)$

$$\|\theta\|_{L^2(A)} \leq C \, \|\theta\|_{L^2(M)} .$$

If, in addition, X is continuous, then i) and ii) are also equivalent to
iii) $G_T(\Theta)$ is complete with respect to the norm $\|\theta \cdot X\|_{\mathcal{R}^2(P)} \geq \|\theta \cdot X\|_{\mathcal{L}^2(P)}$.

Theorem B. Let X be an $\mathbb{R}^d$-valued *continuous* semimartingale such that there is an equivalent local martingale measure Q with $\frac{dQ}{dP} \in \mathcal{L}^2(P)$. The following assertions are equivalent :
i) $G_T(\Theta)$ is closed in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$.
ii) There is an equivalent local martingale measure Q that satisfies the reverse Hölder inequality $R_2(P)$.
iii) The "variance-optimal" local martingale measure Q^{opt} is equivalent to P and satisfies $R_2(P)$.

Theorem C. Let X be an $\mathbb{R}^d$-valued *continuous* semimartingale such that there is an equivalent local martingale measure Q with $\frac{dQ}{dP} \in \mathcal{L}^2(P)$. The following assertions are equivalent :
i) $G_T(\Theta)$ is closed in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ and there is a Föllmer-Schweizer decomposition for X, i.e., the projection π onto $\text{span}(G_T(\Theta), \mathbf{1})$ with $\text{Ker}(\pi) = M^\perp$ is well-defined and continuous on $\mathcal{L}^2(\Omega, \mathcal{F}, P)$.
ii) The "minimal" martingale measure Q^{min} defined by

$$\frac{dQ^{min}}{dP} = \mathcal{E}(-\lambda \cdot M)_T$$

is well-defined, equivalent to P and satisfies $R_2(P)$.

Let us comment on these three theorems. If we restrict our attention to the case of continuous processes X, they are arranged in ascending order of restrictiveness, i.e., the (equivalent) conditions of theorem C (resp. theorem B) imply the (equivalent) conditions of theorem B (resp. theorem A). The central result is theorem B which – under

the stated hypothesis – gives a necessary and sufficient condition for the closedness of $G_T(\Theta)$. The proofs of these assertions as well as several ramifications and complements will be scattered out through the paper, where we also establish some of the results in greater generality. We also give several examples (some of them rather complicated) to show the limitations of the above theorems.
Note that the difference between the situations described by theorems B and C, respectively, pertains to the difference between the "variance-optimal" and the "minimal" martingale measure. This is another illustration of the phenomenon already encountered in Delbaen/Schachermayer (1996b) and (1995d) that the "variance-optimal measure" which is of the form $\frac{dQ^{opt}}{dP} = \mathcal{E}(-\lambda \cdot M + N)_T$ for a suitably chosen martingale N strongly orthogonal to M in general has better properties than the "minimal" martingale measure which is simply given by $\frac{dQ^{min}}{dP} = \mathcal{E}(-\lambda \cdot M)_T$.

This paper is organized as follows. In section 2, we describe the model and prove the results on the $R_2(P)$ property. This section is written in a very general way and the theorems are stated in terms of spaces that are stable for stopping. Our results generalise known results on the reverse Hölder inequality. Section 3 deals with BMO and/or bmo_2 martingales as well as the connection with the inequality $D_2(P)$. In section 4, we investigate under which conditions the space $G_T(\Theta)$ is closed, and in section 5, we explicitly describe the closure of $G_T(\Theta)$ in some cases. Finally, section 6 extends the definition of the Föllmer-Schweizer decomposition under the assumptions of section 4, and this provides another way of proving the closedness of $G_T(\Theta)$.

Some results of this paper form the subject of a note which has been published in the *Comptes Rendus à l'Académie des Sciences*; see DMSSS (1994).

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2. Preliminaries.

Let us now develop our model. We use the same notations as Schweizer (1994). We recall them here. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T \in (0, +\infty]$ a fixed horizon. We suppose that we have a filtration $(\mathcal{F}_t)_{0 \le t \le T}$ on $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions, that is $(\mathcal{F}_t)_{0 \le t \le T}$ is right-continuous and complete, and we assume moreover that $\mathcal{F} = \mathcal{F}_T$. Let $X = (X_t)_{0 \le t \le T}$ be an $\mathbb{R}^d$-valued semimartingale in $\mathcal{S}^2_{loc}$. This means that if

$$X = X_0 + M + A$$

is the canonical decomposition of X, then $M \in \mathcal{M}^2_{0,loc}$ and the variation $|A^i|$ of the predictable finite variation process of X^i is locally square-integrable for each $i = 1, ..., d$. For all unexplained notations, we refer to Jacod (1979) or Protter (1990).

We recall a definition introduced in Schweizer (1994).

Definition 2.1. X satisfies the *structure condition* (SC) if there exists a predictable $\mathbb{R}^d$-valued process $\lambda = (\lambda_t)_{0 \le t \le T}$ such that

$$dA_t = d\langle M\rangle_t \lambda_t \quad P\text{-a.s. for all } t \in [0,T], \tag{2.1}$$

and

$$K_t := \int_0^t \lambda'_s d\langle M\rangle_s \lambda_s < +\infty \quad P\text{-a.s. for all } t \in [0,T], \tag{2.2}$$

where $'$ denotes the transposition.

We then choose an RCLL version of K and we call it the *mean-variance tradeoff* (MVT) *process* of X.

As easily seen, adding to λ a process that takes values in the orthogonal complement of the infinitesimal range of $d\langle M\rangle$ gives the same result. Hence the process λ is only determined modulo the equivalence class of predictable processes taking almost surely values in the orthogonal complement of the infinitesimal range of $d\langle M\rangle$. The existence of λ as well as the almost sure finiteness of K_T is related to arbitrage properties as shown by Delbaen/Schachermayer (1996a). In the case where X is continuous, it is a necessary condition for the existence of an equivalent local martingale measure. Also in the case where X is continuous, the finiteness of K_T is independent of the choice of probability measure, as shown in Delbaen/Shirakawa (1996) or Choulli/Stricker (1996).

Remark 2.2. For the interpretation of the process K, we refer to Schweizer (1994, 1995).

Definition 2.3. A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0 \le t \le T}$ belongs to $L^2(M)$ if

$$E\left(\int_0^T \theta'_t d\langle M\rangle_t \theta_t\right) < +\infty$$

We define on the space $L^2(M)$ the norm $\| \,.\, \|_{L^2(M)}$ by

$$\|\theta\|^2_{L^2(M)} := \|(\theta \cdot M)_T\|^2_{\mathcal{L}^2(P)} = E\left(\int_0^T \theta'_t d\langle M\rangle_t \theta_t\right).$$

A predictable $\mathbb{R}^d$-valued process $\theta = (\theta_t)_{0 \le t \le T}$ belongs to $L^2(A)$ if the process $\left(\int_0^t |\theta'_s dA_s|\right)_{0 \le t \le T}$ is square-integrable.

We define on the space $L^2(A)$ the norm $\| \, . \, \|_{L^2(A)}$ by

$$\|\theta\|_{L^2(A)} := \left\| \int_0^T |\theta_s' dA_s| \right\|_{\mathcal{L}^2(P)}.$$

Finally, Θ is the space defined by $\Theta := L^2(M) \cap L^2(A)$; $\theta \in \Theta$ is called a $\mathcal{L}^2$-strategy.

If the structure condition holds, then clearly

$$\|\theta\|^2_{L^2(A)} = E\left[\left(\int_0^T |\theta_s' d\langle M\rangle_s \lambda_s|\right)^2\right].$$

Strictly speaking the Banach space $L^2(M)$ is the space of equivalence classes of predictable processes θ with finite $L^2(M)$-norm modulo the subspace of predictable processes θ for which the process $\theta \cdot M$ vanishes almost surely. But we use the usual identification of processes with the associated equivalence class if no confusion can arise. A similar remark applies to $L^2(A)$ and Θ.

Remark 2.4. If θ is X-integrable, we can define the stochastic integral process

$$G_t(\theta) := (\theta \cdot X)_t$$

for all $t \in [0, T]$. Then $G(\theta)$ is a semimartingale in $\mathcal{S}^2$ if and only if $\theta \in \Theta$ and in this case the canonical decomposition is given by $G(\theta) := \theta \cdot M + \theta \cdot A$.

The spaces $G_T(\Theta)$ and $G(\Theta)$ are defined by

$$G_T(\Theta) := \{(\theta \cdot X)_T \mid \theta \in \Theta\} \text{ and } G(\Theta) := \{G(\theta) \mid \theta \in \Theta\}.$$

Note that $G_T(\Theta)$ is a space of variables in $\mathcal{L}^2(P)$ and that $G(\Theta)$ is a space of processes.

We next provide several definitions and inequalities which will be useful in the sequel.

The following concept has been extensively studied in Delbaen/Schachermayer (1994).

Definition 2.5. We say that X *admits an equivalent local martingale measure* if there exists a probability Q equivalent to P such that X is a local martingale under Q.

For the next four definitions we refer to Dellacherie/Meyer (1980).

Definition 2.6. The space $\mathcal{R}^2(P)$ is the space of all RCLL adapted processes H such that

$$\|H\|_{\mathcal{R}^2(P)} := \left\| \sup_{0 \le t \le T} |H_t| \right\|_{\mathcal{L}^2(P)} =: \|H_T^*\|_{\mathcal{L}^2(P)}$$

is finite.

Definition 2.7. We say that M *has the predictable representation property under* P, denoted by $PRP(P)$, if each martingale N relative to $(\mathcal{F}_t)_{0\le t\le T}$ and P can be written

$$N = N_0 + \theta \cdot M$$

where N_0 is $\mathcal{F}_0$-measurable and θ is M-integrable.

Definition 2.8. Let $Y = (Y_t)_{0\le t\le T}$ be a uniformly integrable martingale. Then Y *belongs to* BMO if there is a constant $C > 0$ such that

$$E[|Y_T - Y_{S-}|^2 \mid \mathcal{F}_S] \le C \quad P-\text{a.s.}$$

for every stopping time S.

Definition 2.9. Let $Y = (Y_t)_{0\le t\le T}$ be a locally square-integrable, local martingale. Then Y *belongs to* bmo_2 if there is a constant $C > 0$ such that

$$E[\langle Y\rangle_T - \langle Y\rangle_S \mid \mathcal{F}_S] \le C \quad P-\text{a.s.}$$

for every stopping time S.

We now introduce a new concept which is related to the concepts presented below in Definitions 2.11 and 2.12.

Definition 2.10. We say that X *satisfies the inequality* $D_2(P)$ if there is a constant $C > 0$ such that

$$\|\theta\|_{L^2(A)} \le C\,\|\theta\|_{L^2(M)} \qquad , \qquad \forall\theta \in \Theta.$$

By a truncation argument, the inequality $D_2(P)$ extends immediately from $\theta \in \Theta$ to all $\theta \in L^2(M)$.

The problem whether or not the space $G_T(\Theta)$ is closed is intimately related to properties of BMO-martingales and their exponentials. A good reference for this question is Doléans-Dade/Meyer (1979). For continuous martingales the reader can consult Kazamaki (1994).

Definition 2.11. If L is a uniformly integrable martingale such that $L_0 = 1$ and $L_T > 0 \quad P$-a.s, then we say that L *satisfies the reverse Hölder inequality under* P, denoted by $R_p(P)$, where $1 < p \le +\infty$, if and only if there is a constant C such that for every t, we have

$$E\left[\left(\frac{L_T}{L_t}\right)^p \mid \mathcal{F}_t\right] \le C.$$

For $p = +\infty$, we require that $\dfrac{L_T}{L_t}$ is bounded by C (see definition 3.1. of Kazamaki (1994)).
We remark that if L satisfies $R_p(P)$, $1 < p < \infty$, then for the same constant C as in the definition, we have for every stopping time S that

$$L_S^p \leq E[L_T^p \mid \mathcal{F}_S] \leq C L_S^p.$$

In particular the martingale L is bounded in $\mathcal{L}^p(P)$. We remark that a martingale which satisfies the inequality $R_\infty(P)$ is necessarily bounded but there are martingales which satisfy the inequality $R_\infty(P)$ such that $\inf L_t$ is not necessarily bounded from below by a constant $\delta > 0$. A condition dual to $R_p(P)$ is the inequality $A_q(P)$ (see definition 2.2. of Kazamaki (1994)).

Definition 2.12. If L is a uniformly integrable martingale such that $L_0 = 1$ and $L_T > 0 \quad P$-a.s, we say that L *satisfies the Muckenhoupt inequality* denoted by $A_q(P)$ for some $1 \leq q < +\infty$, if and only if there is a constant C such that for every t

$$E\left[\left(\frac{L_t}{L_T}\right)^{\frac{1}{q-1}} \mid \mathcal{F}_t\right] \leq C.$$

If $q = 1$, we require that $\dfrac{L_t}{L_T}$ is bounded by C.

Again, we remark that with the same constant C, the inequality holds for arbitrary stopping times S.

Definition 2.13. Let Z be a positive process. Z *satisfies condition (J)* if there exists a constant $C > 0$ such that

$$\frac{1}{C} Z_- \leq Z \leq C Z_-.$$

In the (French) paper Doléans-Dade/Meyer (1979), this condition is called condition (S) since it involves the jumps ("sauts") of Z. To avoid confusion with the structure condition (SC) in Definition 2.1, we have relabelled it here as (J).
Let us now recall some definitions and notations related to changes of law. If Y is a semimartingale, $Y_0 = 0$, then its stochastic exponential, denoted by $\mathcal{E}(Y)$, is the semimartingale

$$\mathcal{E}(Y)_t := \exp\left(Y_t - \frac{1}{2}\langle Y^c\rangle_t\right) \prod_{0<s\leq t} (1 + \Delta Y_s) e^{-\Delta Y_s}.$$

If Z is a semimartingale such that $\inf_{0\leq s\leq T} Z_s > 0$ (for instance if Z is a strictly positive local martingale), then its stochastic logarithm, denoted by $\mathcal{L}(Z)$, is the semimartingale

$$\mathcal{L}(Z) := \frac{1}{Z_-} \cdot Z.$$

Now let Q be an equivalent probability measure and define

$$Z_t := E_P\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right] \text{ and } \widehat{Z}_t = E_Q\left[\frac{dP}{dQ} \mid \mathcal{F}_t\right] = \frac{1}{Z_t}.$$

From Bayes' rule

$$E_Q[f \mid \mathcal{F}_t]Z_t = E_P[fZ_T \mid \mathcal{F}_t]$$

it easily follows that Z satisfies $R_p(P)$ if and only if $\widehat{Z}$ satisfies $A_q(Q)$ where of course $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p \leq +\infty$.

The following theorem relates BMO and $R_p(P)$ (see Doléans-Dade/Meyer (1979), proposition 5 and 6).

Theorem 2.14. The following assertions are equivalent for a strictly positive martingale Z, $Z_0 = 1$:
(1) $\mathcal{L}(Z)$ is in $BMO(P)$ and there exists a constant $h > 0$ such that $1 + \Delta\mathcal{L}(Z) \geq h$.
(2) $\mathcal{L}(\widehat{Z})$ is in $BMO(Q)$ and there exists a constant $h > 0$ such that $1 + \Delta\mathcal{L}(\widehat{Z}) \geq h$.
(3) Z satisfies condition (J) and $R_p(P)$ for some $p > 1$.
(4) $\widehat{Z}$ satisfies condition (J) and $A_q(Q)$ for some $q < +\infty$.
In addition, (3) is satisfied for $1 < p < \infty$ iff (4) is satisfied for $q = \frac{p}{p-1}$.

The next theorem states that the set of exponents p such that Z satisfies $R_p(P)$ is necessarily open. Of course, a similar argument holds for $A_q(P)$ (see Doléans-Dade/Meyer (1979) proposition 4).

Theorem 2.15. Assume Z is a strictly positive martingale with $Z_0 = 1$. If Z satisfies condition (J) and $R_p(P)$ $(p > 1)$, then there is $p' > p$ such that Z satisfies $R_{p'}(P)$.

A basic property, that we will need later on, is that if Z satisfies $R_p(P)$ then the conditional expectation with respect to Q is a continuous operator on $L^q(P)$. More precisely, we have (see Doléans-Dade/Meyer (1979) proposition 2 and the corollary on page 318 combined with proposition 4) the subsequent result :

Theorem 2.16. Assume Z is a strictly positive martingale with $Z_0 = 1$. For $1 < p < +\infty$, assertions (1) and (2) below are equivalent
(1) Z satisfies $R_p(P)$.
(2) There is a constant C such that for each Q-martingale N, and for $q = \dfrac{p}{p-1}$ and $\lambda > 0$

$$\lambda^q P[N_T^* > \lambda] \leq CE_P[|N_T|^q].$$

Moreover under the additional assumption that Z satisfies condition (J) the weak inequality (2) implies the following strong inequality

(3) There is a constant K such that for each Q-martingale N, and for $q = \frac{p}{p-1}$

$$E_P[(N_T^*)^q] \leq K E_P[|N_T|^q].$$

Below we will give a generalization of this theorem. As we deal in this paper with the case $p = 2$ only, we do not focus our attention to possible extensions of this generalization to the case $p \neq 2$, $p > 1$.

The symbol $\mathcal{V}$ denotes a vector space of bounded continuous adapted processes. If $Y \in \mathcal{V}$, we suppose that $Y_0 = 0$. We require $\mathcal{V}$ to be stable for stopping, i.e. if S is a stopping time and if Y is in $\mathcal{V}$, then $Y^S \in \mathcal{V}$. For each stopping time S, we denote by $\mathcal{V}_S$ the vector space $\{Y_S \mid Y \in \mathcal{V}\}$. The space ${}_S\mathcal{V}$ is the space $\{Y_T - Y_S \mid Y \in \mathcal{V}\}$. We remark that this notation is consistent with the notation for stopping and starting a process. We remark that $\mathcal{V}$ denotes a vector space of adapted processes while $\mathcal{V}_S$ and ${}_S\mathcal{V}$ denote spaces of ($\mathcal{F}_S$ -resp. $\mathcal{F}_T$- measurable) random variables. Since $\mathcal{V}$ is stable for stopping, we have for every stopping time S and every set $A \in \mathcal{F}_S$ that $\mathbf{1}_A \ {}_S\mathcal{V} \subset \ {}_S\mathcal{V} \subset \mathcal{V}_T$. Clearly $\mathcal{V}_0 = \{0\}$. The set $I\!M(\mathcal{V})$ denotes the set of all probability measures Q that are absolutely continuous with respect to P and for which the elements $Y \in \mathcal{V}$ become Q-martingales. The symbol $I\!M^e(\mathcal{V})$ is reserved for the elements of $I\!M(\mathcal{V})$ that are equivalent to P.
We shall simply write $I\!M^e$ and $I\!M$ instead of $I\!M^e(\mathcal{V})$ and $I\!M(\mathcal{V})$ if there is no danger of confusion.
It is easily seen that if Q is absolutely continuous with respect to P and if L denotes the càdlàg martingale

$$L_t = E_P\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right],$$

then $Q \in I\!M(\mathcal{V})$ if and only if for every $Y \in \mathcal{V}$, the process YL is a martingale or, which is the same because $\mathcal{V}$ is stable for stopping, $E[L_T Y_T] = 0$. More generally, we define $I\!M^s$ as the affine space of measures μ absolutely continuous with respect to P such that $\mu(\Omega) = 1$ and

$$E_P\left[Y_T \frac{d\mu}{dP}\right] = 0$$

for all $Y \in \mathcal{V}$. If we denote by L the càdlàg martingale

$$L_t = E_P\left[\frac{d\mu}{dP} \mid \mathcal{F}_t\right],$$

then this is equivalent to the property that $E[L_T] = 1$ and LY is a martingale for each $Y \in \mathcal{V}$. Without further notice, we will identify an absolutely continuous measure μ with its Radon-Nikodym derivative $\frac{d\mu}{dP}$. In this setting, $I\!M$ and $I\!M^s$ are closed sets of $\mathcal{L}^1(P)$ and if $I\!M^e$ is non empty, then it is $\mathcal{L}^1(P)$-dense in $I\!M$.

An important role will be played by the element of $I\!M^s \cap \mathcal{L}^2$ that has minimal $\mathcal{L}^2(P)$-norm, which we call the *variance optimal measure* and which we denote by Q^{opt}.
This measure was previously studied by Schweizer (1995) as well as by Delbaen/ Schachermayer (1996b). It is shown there that $I\!M^s \cap \mathcal{L}^2(P)$ is non empty if and only if the constant function 1 is not in the $\mathcal{L}^2$-closure of $\mathcal{V}_T$. If we adopt the convention that a bar denotes the closure in $\mathcal{L}^2(P)$, then $I\!M^s \cap \mathcal{L}^2(P)$ is non empty if and only if $1 \notin \overline{\mathcal{V}}_T$. In this case, there is an element μ in $I\!M^s \cap \mathcal{L}^2(P)$ with minimal norm and it is given by

$$\frac{d\mu}{dP} = \frac{1-f}{1-E[f]},$$

where f is the orthogonal projection of 1 onto the closed subspace $\overline{\mathcal{V}}_T$ of $\mathcal{L}^2(P)$.
The $\mathcal{L}^2$-norm of $\dfrac{d\mu}{dP}$ is given by

$$\left\| \frac{d\mu}{dP} \right\|_{\mathcal{L}^2(P)} = \frac{1}{\operatorname{dist}(1, \mathcal{V}_T)} = \frac{1}{(1-E[f])^{1/2}} = \frac{1}{\sin\varphi},$$

where φ is the positive angle between 1 and $\overline{\mathcal{V}}_T$. Exactly as in theorem 3.1 of Delbaen/Schachermayer (1995b), one shows that due to the continuity of elements in $\mathcal{V}$, the measure μ is necessarily nonnegative, i.e. $\mu \in I\!M \cap \mathcal{L}^2(P)$.

Lemma 2.17. If the variance optimal measure $Q^{opt} \in I\!M^e(\mathcal{V})$ exists and the càdlàg martingale L defined as

$$L_t = E\left[\frac{dQ^{opt}}{dP} \mid \mathcal{F}_t\right]$$

satisfies $R_2(P)$, then L satisfies condition (J).

Proof. Since L satisfies $R_2(P)$, L is a square integrable martingale. Hence we can define for each $f_T \in \overline{\mathcal{V}}_T$ the Q^{opt}-martingale

$$f_t := E_{Q^{opt}}\left[f_T \mid \mathcal{F}_t\right]$$

Moreover if (f_T^n) is a sequence in $\mathcal{V}_T$ converging to f_T with respect to the $\mathcal{L}^2(P)$-norm, then the sequence (f_t^n) converges uniformly in t with respect to the norm of $\mathcal{L}^1(Q^{opt})$ and hence in probability to (f_t). As each (f_t^n) is a continuous martingale, the Q^{opt}-martingale (f_t) is continuous whenever $f_T \in \overline{\mathcal{V}}_T$. In particular if f_T is the orthogonal projection of 1 onto $\overline{\mathcal{V}}_T$, then (f_t) is a continuous Q^{opt}-martingale. Since

$$L_T = \frac{dQ^{opt}}{dP} = \frac{1-f}{1-E[f]}$$

the Q^{opt}-martingale $\widetilde{Z}_t = E_{Q^{opt}}\left[L_T \mid \mathcal{F}_t\right]$ is continuous too. By Bayes'rule

$$\widetilde{Z}_t = \frac{E_P\left[\widetilde{Z}_T^2 \mid \mathcal{F}_t\right]}{L_t} = \frac{E_P\left[L_T^2 \mid \mathcal{F}_t\right]}{L_t}$$

Suppose now that L satisfies $R_2(P)$, then

$$1 \leq \frac{E_P\left[L_T^2 \mid \mathcal{F}_t\right]}{L_t^2} \leq C$$

and hence

$$L_t \leq \widetilde{Z}_t \leq C L_t.$$

Since $\widetilde{Z}_t$ is continuous, it follows that L satisfies condition (J).

In Delbaen/Schachermayer (1995b), it is shown that if $I\!M^e \cap \mathcal{L}^2(P) \neq \emptyset$, then $Q^{opt} = \mu \in I\!M^e$. The theorem below investigates the inequality $R_2(P)$ for μ and part of its proof uses the same method as theirs. For simplicity of notation, we assume that $\mathcal{F}_0$ is trivial.

Theorem 2.18. If $\mathcal{V}$ is a space of bounded continuous adapted processes such that for each $Y \in \mathcal{V}$ we have $Y_0 = 0$, if $\mathcal{V}$ is stable for stopping (as described above), if $\mathcal{F}_0$ is trivial, then are equivalent

(1) The variance optimal measure $Q^{opt} \in I\!M^e(\mathcal{V})$ exists and the càdlàg martingale L defined as

$$L_t = E\left[\frac{dQ^{opt}}{dP} \mid \mathcal{F}_t\right]$$

satisfies $R_2(P)$.

(2) There is $Q \in I\!M^e(\mathcal{V}) \cap \mathcal{L}^2(P)$ such that the càdlàg martingale Z defined as

$$Z_t = E\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right]$$

satisfies the inequality $R_2(P)$.

(3) There is a constant C such that for every $Y \in \mathcal{V}$

$$\|Y_T^*\|_{\mathcal{L}^2(P)} \leq C \|Y_T\|_{\mathcal{L}^2(P)}.$$

(4) There is a constant C such that for every $Y \in \mathcal{V}$ and every $\lambda \geq 0$

$$\lambda P[Y_T^* > \lambda]^{1/2} \leq C \|Y_T\|_{\mathcal{L}^2(P)}.$$

(5) There is a constant $C > 0$ such that for every stopping time S, every $A \in \mathcal{F}_S$ and every $U_T \in {}_S\mathcal{V}$

$$\|\mathbf{1}_A - U_T\|_{\mathcal{L}^2(P)} \geq C P[A]^{1/2}.$$

In addition, if one of the above equivalent conditions is fulfilled, then Q^{opt} satisfies $R_p(P)$ for some $p > 2$.

Remarks 2.19. i) In condition (5), we can of course restrict the inequality to elements U_T in ${}_{S_A}\mathcal{V}$ i.e. elements constructed with the stopping time $S_A = S$ on A and $S_A = T$

on A^c. These elements can be written as $\mathbf{1}_A(Y_T - Y_S)$ where $Y \in \mathcal{V}$. We remark that condition (5) expresses that there is a lower bound $\varphi_0 = \arcsin C$ such that for each $A \in \mathcal{F}_S$, the angle between $\mathbf{1}_A$ and the space ${}_S\mathcal{V}$ is bounded below by φ_0.
ii) If in theorem 2.18 we take for Q an equivalent probability measure that defines a density process that satisfies $R_2(P)$ but that not necessarily satisfies condition (J), if for $\mathcal{V}$ we take the space of all continuous bounded martingales for Q, then (3) of theorem 2.18 extends, at least for continuous martingales, proposition 2.16. The trick is that the density process of the variance minimal measure for $\mathcal{V}$ satisfies $R_2(P)$ and condition (J)!

Proof of theorem 2.18. It is clear that (1) implies (2). By theorem 2.15 and lemma 2.17 , (1) implies (3) and (2) implies (4), the constant C being valid for every Q-uniformly integrable martingale. The strong inequality in (3) certainly implies the weak inequality in (4). We now prove the equivalence of (4) and (5), after which we show that (5), together with (4), implies (1).
(4) $\Longrightarrow$ (5)
This is done by using a reflection argument. Fix a stopping time S, $A \in \mathcal{F}_S$ and a process U of the form $U = X - X^{S_A} = \mathbf{1}_A(X - X^S)$ where $X \in \mathcal{V}$. Define $\nu := \inf\{t \mid U_t > \frac{1}{2}\} \wedge T$ and let

$$Y_t = \begin{cases} U_t & \text{for} \quad t \le \nu \\ 2U_\nu - U_t & \text{for} \quad t > \nu, \end{cases}$$

i.e. Y is U reflected at time ν. Then $Y \in \mathcal{V}$ and

$$|Y_T| = |U_T|\mathbf{1}_{\{\nu = T\}} + |1 - U_T|\mathbf{1}_{\{\nu < T\}} \le |1 - U_T|$$

since $U_T \le \frac{1}{2}$ on $\{\nu = T\}$. On A^c, we have $U = 0$, hence $\nu = T$ and $Y_T = 0$; thus we obtain $|Y_T| \le |\mathbf{1}_A - U_T|$, and the weak inequality in (4) implies

$$\begin{aligned} \|\mathbf{1}_A - U_T\|_{\mathcal{L}^2(P)} &\ge \|Y_T\|_{\mathcal{L}^2(P)} \\ &\ge C^{-1}\frac{1}{2}P\left[Y_T^* \ge \frac{1}{2}\right]^{1/2} \\ &\ge \frac{C^{-1}}{2}P[\nu < T]^{1/2} \\ &= \frac{C^{-1}}{2}P\left[U_T^* > \frac{1}{2}\right]^{1/2}. \end{aligned}$$

On the other hand,

$$\|U_T - \mathbf{1}_A\|_{\mathcal{L}^2(P)} \ge \frac{1}{2}P[A \cap \{U_T^* \le 1/2\}]^{1/2}$$

and hence

$$\|U_T - \mathbf{1}_A\|_{\mathcal{L}^2(P)} \ge \delta P[A]^{1/2} \text{ where } \delta = \frac{1}{\sqrt{2}}\min\left(\frac{C^{-1}}{2}, \frac{1}{2}\right).$$

(5) $\Longrightarrow$ (4)
For fixed $Y \in \mathcal{V}$ and $\lambda > 0$, let us define $S = \inf\{t \mid |Y_t| > \lambda\}$. The element $U_T = -\text{sign}(Y_S)(Y_T - Y_S)$ is clearly in ${}_S\mathcal{V}$ and hence for $A = \{S < T\} = \{Y_T^* > \lambda\}$ we have

$$\left\| \mathbf{1}_A - \frac{U_T}{\lambda} \right\|_{\mathcal{L}^2(P)} \geq C P[A]^{1/2}$$

or, what is the same

$$C\lambda P[Y_T^* > \lambda]^{1/2} \leq \|\lambda \mathbf{1}_A - U_T\|_{\mathcal{L}^2(P)}.$$

But $\lambda \mathbf{1}_A - U_T = \lambda \mathbf{1}_A + \text{sign}(Y_S)(Y_T - Y_S) = Y_T \mathbf{1}_A \text{sign}(Y_S)$ and hence

$$C\lambda P[Y_T^* > \lambda]^{1/2} \leq \|\lambda \mathbf{1}_A - U_T\|_{\mathcal{L}^2(P)} \leq \|Y_T\|_{\mathcal{L}^2(P)}.$$

(5) $\Longrightarrow$ (1)
This is the most technical part. The proof mimics the proof of theorem 1.3 in Delbaen/Schachermayer (1996b). Since we do not assume a priori that there is an element $Q \in I\!\!M^e \cap \mathcal{L}^2(P)$, there are some extra technical difficulties. We start with two lemmas. The first should be folklore (see lemma 3.4 in Delbaen/Schachermayer (1995b)). The second exploits that the angle between $\mathbf{1}_A$ and ${}_S\mathcal{V}$ is bounded from below.

Lemma 2.20. If $U = (U_t)_{0 \leq t \leq T}$ is a non-negative square integrable martingale, if $U_0 > 0$, if the stopping time $\tau = \inf\{t \mid U_t = 0\}$ is predictable and announced by a sequence of stopping times $(\tau_n)_{n \geq 1}$, then

$$E\left[\frac{U_\tau^2}{U_{\tau_n}^2} \mid \mathcal{F}_{\tau_n}\right] \to +\infty$$

on the $\mathcal{F}_{\tau-}$-measurable set $\{U_\tau = 0\}$.

Lemma 2.21. If condition (5) holds with a constant C , then for each stopping time S there is an element $g \in \mathcal{L}^2_+(P)$ such that $E[g \mid \mathcal{F}_S] = 1$, $E[g^2 \mid \mathcal{F}_S] \leq C^{-2}$ and $E[gU] = 0$ for each $U \in {}_S\mathcal{V}$.

Proof of lemma 2.2. We proceed exactly as in theorem 3.1 in Delbaen/Schachermayer (1995b). Let f be the projection of 1 onto the space $\overline{{}_S\mathcal{V}}$. For each $A \in \mathcal{F}_S$, the spaces $\mathbf{1}_A \,.\, \overline{{}_S\mathcal{V}}$ and $\mathbf{1}_{A^c} \,.\, \overline{{}_S\mathcal{V}}$ form an orthogonal decomposition of $\overline{{}_S\mathcal{V}}$ and hence $f\mathbf{1}_A$ is the orthogonal projection of $\mathbf{1}_A$ onto $\overline{{}_{S_A}\mathcal{V}} = \mathbf{1}_A \,.\, \overline{{}_S\mathcal{V}}$. This shows that $E[f^2\mathbf{1}_A] = E[f\mathbf{1}_A f\mathbf{1}_A] = E[f\mathbf{1}_A\mathbf{1}_A] = E[f\mathbf{1}_A]$. The inequality in condition (5) shows that $\|\mathbf{1}_A - f\mathbf{1}_A\|^2_{\mathcal{L}^2(P)} \geq C^2 P[A]$ and hence $E[\mathbf{1}_A - f\mathbf{1}_A] = E[\mathbf{1}_A(1-f)^2] \geq C^2 P[A]$ for all $A \in \mathcal{F}_S$, i.e. $1 - E[f \mid \mathcal{F}_S] \geq C^2$.
We now define

$$g = \frac{1-f}{1 - E[f \mid \mathcal{F}_S]}.$$

The computation above shows that $E[f^2 \mid \mathcal{F}_S] = E[f \mid \mathcal{F}_S]$ and hence

$$\|g\|^2_{\mathcal{L}^2(P)} = E\left[\frac{1}{1 - E[f \mid \mathcal{F}_S]}\right] \leq C^{-2}.$$

Now, for each $A \in \mathcal{F}_S$ and each $U \in {}_S\mathcal{V}$, we have $\mathbf{1}_A U \in {}_S\mathcal{V}$ and hence

$$E[\mathbf{1}_A(1-f)U] = 0.$$

An easy approximation argument on the bounded function

$$\frac{1}{1 - E[f \mid \mathcal{F}_S]}$$

then shows that $E[gU] = 0$ for all $U \in {}_S\mathcal{V}$.
The positivity of g is shown exactly as in theorem 3.1 of Delbaen/Schachermayer (1995b).
This completes the proof of lemma 2.21.

Proof of th. 2.18 continued : Let us come back to the end of the proof of theorem 2.18. If we denote by f the orthogonal projection of 1 onto the space $\overline{\mathcal{V}}_T$, then as seen above, the optimal measure Q^{opt} is nonnegative and is given by

$$\frac{dQ^{opt}}{dP} = \frac{1-f}{1-E[f]}.$$

The next step is to construct a continuous process that resembles the process $\widetilde{Z}$ as in Delbaen/Schachermayer (1995b). There is a sequence of elements Y^n in $\mathcal{V}$ such that $\left\|Y^n_T - Y^{n+1}_T\right\|_{\mathcal{L}^2(P)} \leq 3^{-n}$ and such that $Y^n_T \longrightarrow f$ in $\mathcal{L}^2(P)$. From the weak inequality, we deduce that

$$\sum_{n\geq 1} P\left[\sup_{0\leq t\leq T} |Y^n_t - Y^{n+1}_t| > 2^{-n}\right] < +\infty$$

and hence the sequence Y^n_t converges uniformly in t a.s. to a continuous process that we denote by f_t. Clearly $f_T = f$. Define

$$\widetilde{Z}_t = \frac{1-f_t}{1-E[f_t]}.$$

If we denote by L the density process

$$L_t = E_P\left[\frac{dQ^{opt}}{dP} \mid \mathcal{F}_t\right] = E_P[\widetilde{Z}_T \mid \mathcal{F}_t]$$

then for each element Y in $\mathcal{V}$, we have that $L_t Y_t = E_P[L_T Y_T \mid \mathcal{F}_t]$. Since L_T and L_t are in $\mathcal{L}^2(P)$, it follows that also $L_t\widetilde{Z}_t = E_P[L_T\widetilde{Z}_T \mid \mathcal{F}_t] = E_P[L_T^2 \mid \mathcal{F}_t]$. If τ denotes the stopping time $\tau = \inf\{t \mid L_t\widetilde{Z}_t = 0\}$, then we have

$$0 = \int_{\tau<T} L_T^2 dP$$

and hence $L_T = 0$ on $\{\tau < T\}$. This implies that $L_\tau = 0$ on $\{\tau < T\}$. From the continuity of $\widetilde{Z}$, it follows that necessarily $\widetilde{Z}_\tau \geq 0$. Suppose now that

$$A = \{\widetilde{Z}_\tau > 0\} \cap \{\tau < T\}$$

has strictly positive measure. Because $L_T = \widetilde{Z}_T = 0$ on $\{\tau < T\}$ we have that $f_T = 1$ on A. Hence the function $(1 - f_\tau)\mathbf{1}_A \in \overline{{}_{\tau_A}V}$. Let g be the positive element constructed in lemma 2.21. for the stopping time τ_A. Since $E[g\mathbf{1}_A(1-f_\tau)] = 0$ and since $(1-f_\tau) > 0$ on A, we have that $E[g\mathbf{1}_A] = 0$, a contradiction to $E[g \mid \mathcal{F}_{\tau_A}] = 1$. It follows that also $\widetilde{Z}_\tau = 0$ and hence $\inf\{t \mid L_t = 0\} = \inf\{t \mid \widetilde{Z}_t = 0\} = \tau$. We now proceed exactly as in the proof of theorem 1.3 of Delbaen/Schachermayer (1996b). The stopping time τ is predictable and announced by a sequence $(\tau_n)_{n\geq 1}$. If

$$E\left[\left(\frac{L_T}{L_{\tau_n}}\right)^2 \mid \mathcal{F}_{\tau_n}\right]$$

would be greater than C^{-2}, then we use the element g constructed for the stopping time τ_n and whose existence is given by lemma 2.21. The element $L_{\tau_n} g$ would give an element in $I\!M^s$ with smaller $\mathcal{L}^2(P)$-norm. This reasoning shows that $L_T > 0$ according to lemma 2.20, and that for every stopping time S, we have

$$E\left[\left(\frac{L_T}{L_S}\right)^2 \mid \mathcal{F}_S\right] \leq C^{-2}.$$

This completes the proof of theorem 2.18.

The existence of an element in $I\!M^e \cap \mathcal{L}^2(P)$ is taken care of by the following theorem (see Stricker (1990)).

Theorem 2.22. If $\mathcal{V}$ is a space of bounded continuous adapted processes, if $\mathcal{V}$ is stable for stopping (as described above), then $I\!M^e \cap \mathcal{L}^2(P)$ is non-empty if and only if

$$\overline{\mathcal{V}_T} \cap \mathcal{L}^2_+(P) = \{0\}.$$

One can improve slightly the above theorem as follows (see Yan (1980)). This result is formulated in the same language as (5) of theorem 2.18.

Theorem 2.23. If $\mathcal{V}$ is a space of bounded continuous adapted processes, if $\mathcal{V}$ is stable for stopping (as described above), then $I\!M^e \cap \mathcal{L}^2(P)$ is non-empty if and only if for every $A \in \mathcal{F}_T$, we have $\mathbf{1}_A \notin \overline{\mathcal{V}_T}$.

Proof. Suppose that there is $f \in \overline{\mathcal{V}_T} \cap \mathcal{L}^2_+(P)$, $P[f > 0] > 0$. For each such element, let us denote by A_f the set $A_f = \{f > 0\}$. If $(f_n)_{n\geq 1}$ is a sequence of such elements then $f = \sum 2^{-n} \|f_n\|^{-1}_{\mathcal{L}^2(P)} f_n \in \overline{\mathcal{V}_T}$ and $A_f = \cup_{n\geq 1} A_{f_n}$. Hence there is a maximal set of this form. Call it A_f where f is the associated function. Take a sequence η_n strictly decreasing to 0 such that $P[f > \eta_n] > 0$. For each n, take ε_n so that $\varepsilon_n \leq \frac{1}{2}\eta_n^2$ and choose $Y_T^n \in \mathcal{V}_T$ so that $\|Y_T^n - f\|_{\mathcal{L}^2(P)} < \varepsilon_n^2$. It then follows that $P[|f - Y_T^n| > \varepsilon_n] \leq \varepsilon_n^2$. Hence $Y_T^n > f - \varepsilon_n > \eta_n - \varepsilon_n$ on a set of measure at least $P[f > \eta_n] - \varepsilon_n^2$. The element $(\eta_n - \varepsilon_n)^{-1} Y_T^n = g_n$ is still in $\mathcal{V}_T$ and satisfies $\{g_n > 1\}$ on the set $\{f > \eta_n\} \backslash \{|Y_T^n - f| > \varepsilon_n\}$ which has measure greater than $P[f > \eta_n] - \varepsilon_n^2$. We stop the process g_n when it hits the level 1, i.e. $V^n = (Y^n)^\tau$ where

$$\tau = \inf\{t \mid g_n \geq 1\}$$

Clearly
(i) $V_T^n = 1$ on $\{f > \eta_n\} \backslash \{|Y_T^n - f| > \varepsilon_n\}$;
(ii) Since A_f is maximal, $(V_T^*)^+ \leq \mathbf{1}_{A_f}$;
(iii) $(V_T^n)^- \leq (\eta_n - \varepsilon_n)^{-1} (Y_T^n)^-$.
If n tends to $+\infty$, (i) and (ii) show that $(V_T^n)^+ \longrightarrow \mathbf{1}_{A_f}$ whereas (iii) shows that $\|(V_T^n)^-\|_{\mathcal{L}^2(P)} \leq (\eta_n - \varepsilon_n)^{-1} \|(Y_T^n)^-\|_{\mathcal{L}^2(P)} \leq (\eta_n - \varepsilon_n)^{-1} \varepsilon_n^2$ which tends to 0. This shows that $\mathbf{1}_{A_f} \in \overline{\mathcal{V}_T}$. This completes the proof of theorem 2.21.

3. The inequality $\mathbf{D_2(P)}$ and its relation to BMO.

Throughout this section, we do not assume that X is continuous.

The inequality $D_2(P)$ is an assumption which arises naturally when one studies the closedness of $G_T(\Theta)$. Indeed, to prove that the limit of a sequence $(G_T(\theta^n))_{n\geq 0}$ which converges in $\mathcal{L}^2(P)$ belongs to $G_T(\Theta)$, we would like to show that the sequence $(\theta^n)_{n\geq 0}$ converges to some θ in $L^2(M)$ and $L^2(A)$. Now, convergence in $L^2(M)$ is rather easy to study since a sequence $(\theta^n)_{n\geq 0}$ converges in $L^2(M)$ if and only if $((\theta^n \cdot M)_T)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{L}^2(P)$. Convergence in $L^2(A)$ is more difficult to prove. So an idea to solve this problem is to find an assumption under which convergence in $L^2(M)$ will imply convergence in $L^2(A)$, that is $L^2(M) \subseteq L^2(A)$ or, equivalently, $\Theta = L^2(M)$. We first show that the inequality $D_2(P)$ is a sufficient condition for the structure condition (SC); see Definition 2.1.

Lemma 3.1. If the inequality $D_2(P)$ holds, then λ exists and K is square-integrable.

Proof. The inequality $D_2(P)$ implies that if $\theta \cdot M = 0$, then $\theta \cdot A = 0$ so by the multidimensional Radon-Nikodym theorem (see Delbaen/Schachermayer (1996a)), there exists a predictable $\mathbb{R}^d$-valued process λ such that $dA = d\langle M\rangle \lambda$. For each n, let $\theta^n = \lambda \mathbf{1}_{\{\|\lambda\| \leq n\} \cap [\![0, \tau_n [\![}$ where τ_n is the predictable stopping time

$$\tau_n := \inf\left\{t \mid \int_0^t d|\langle M\rangle|_s \geq n\right\}.$$

Clearly $\theta^n dA = \lambda' d\langle M\rangle \lambda \mathbf{1}_{\{\|\lambda\|\le n\}\cap [\![0,\tau_n[\![}$ and $D_2(P)$ implies that for all n

$$\begin{aligned} E\left[\left(\int_{\{\|\lambda\|\le n\}\cap[\![0,\tau_n[\![}\lambda' d\langle M\rangle \lambda\right)^2\right] &\le C^2 E\left[\int_{\{\|\lambda\|\le n\}\cap[\![0,\tau_n[\![}\lambda' d\langle M\rangle \lambda\right] \\ &\le C^2 E\left[\left(\int_{\{\|\lambda\|\le n\}\cap[\![0,\tau_n[\![}\lambda' d\langle M\rangle \lambda\right)^2\right]^{1/2}. \end{aligned}$$

Since both quantities are finite, we find

$$E\left[\left(\int_{\{\|\lambda\|\le n\}\cap[\![0,\tau_n[\![}\lambda' d\langle M\rangle \lambda\right)^2\right]^{1/2} \le C^2.$$

When n tends to $+\infty$, we obtain that K_T is square-integrable. This completes the proof of lemma 3.1.

The next lemma gives an equivalent reformulation of $D_2(P)$.

Lemma 3.2. The inequality $D_2(P)$ holds if and only if $L^2(M) \subseteq L^2(A)$, i.e. if and only if $\Theta = L^2(M)$.

Proof. Since $\Theta = L^2(M)$ is equivalent to saying that $L^2(M) \subseteq L^2(A)$, the "only if" part is obvious. Conversely, suppose that $L^2(M) \subseteq L^2(A)$. By means of the multidimensional Radon-Nikodym theorem (see Delbaen/Schachermayer (1996a))) it is easy to see that A is absolutely continuous with respect to $\langle M\rangle$. So we conclude that the graph of the identity mapping from $L^2(M)$ into $L^2(A)$ is closed in $L^2(M)\times L^2(A)$. Hence the identity is continuous, and this proves the "if" part.

The existence of λ and the square-integrability of K are necessary conditions for $D_2(P)$, but far from being sufficient. The necessary and sufficient condition for $D_2(P)$ given by the next theorem is substantially stronger.

Theorem 3.3. The inequality $D_2(P)$ holds if and only if λ exists and $\lambda\cdot M$ is in bmo_2.

To prove theorem 3.3, we need an auxiliary result. Recall that h^1_0 denotes the space of all locally square-integrable local martingales Y null at 0 such that $\langle Y\rangle_T^{1/2}$ is integrable.

Lemma 3.4. If $Z \in \mathcal{M}^2$ and $R \in \mathcal{M}^2_0$, then $\int Z_- dR$ is in h^1_0 and

$$\left\|\int Z_- dR\right\|_{h^1} \le 2\|Z_T\|_{\mathcal{L}^2}\|R\|_{\mathcal{M}^2}.$$

In particular, choosing $R := \displaystyle\int \theta dM$ with $\theta \in L^2(M)$ gives

$$\left\|\int Z_-\theta dM\right\|_{h^1} \le 2\|Z_T\|_{\mathcal{L}^2}\|\theta\|_{L^2(M)}.$$

Proof. Since

$$\left\langle \int Z_- dR \right\rangle_T = \int_0^T Z_{u-}^2 d\left\langle R\right\rangle_u \leq \left(\sup_{0\leq u\leq T} |Z_u|\right)^2 \left\langle R\right\rangle_T,$$

we get

$$\left\| \int Z_- dR \right\|_{h^1} = \left\| \left\langle \int Z_- dR \right\rangle_T^{\frac{1}{2}} \right\|_{\mathcal{L}^1} \leq 2\|Z_T\|_{\mathcal{L}^2}\|R\|_{\mathcal{M}^2}$$

by the Cauchy-Schwarz and Doob inequalities.

Proof of theorem 3.3. 1) Suppose first that $\lambda \cdot M$ is in bmo_2. Take any bounded positive random variable Y and denote by Z an RCLL version of the martingale $Z_t = E[Y \mid \mathcal{F}_t]$. Fix $\theta \in L^2(M)$ and set $\zeta := Z_-\theta$ so that $\int \zeta dM$ is in h_0^1 by lemma 3.4. By Fefferman's inequality and the end of lemma 3.4, we then obtain

$$\begin{aligned} E\left[Y\int_0^T |\theta' d\left\langle M\right\rangle \lambda|_u\right] &= E\left[\int_0^T Z_{u-}|d\theta'\left\langle M\right\rangle \lambda|_u\right] \\ &= E\left[\int_0^T d\left|\left\langle (Z_-\theta)\cdot M, \lambda\cdot M\right\rangle\right|_u\right] \\ &\leq \sqrt{2}\left\|(Z_-\theta)\cdot M\right\|_{h_0^1}\left\|\lambda\cdot M\right\|_{bmo_2} \\ &\leq \sqrt{8}\left\|Y\right\|_{\mathcal{L}^2}\|\lambda\cdot M\|_{bmo_2}\|\theta\|_{L^2(M)}. \end{aligned}$$

Since Y was arbitrary, we conclude that

$$\|\theta\|_{L^2(A)} = \left\| \int_0^T |\theta' d\left\langle M\right\rangle \lambda|_u \right\|_{\mathcal{L}^2} \leq \sqrt{8}\|\lambda\cdot M\|_{bmo_2}\|\theta\|_{L^2(M)}$$

and this proves the "if" part.
2) Now suppose that the inequality $D_2(P)$ holds. Then, in view of lemma 3.2, $L^2(M) = \Theta$. Moreover, $K_T = \left\langle \lambda \cdot M\right\rangle_T$ is in $\mathcal{L}^1$ by lemma 3.1. Fix $t \in [0,T]$ and a bounded $\mathcal{F}_t$-measurable random variable V and define $\psi := \lambda V \mathbf{1}_{]\!]t,T]\!]}$ so that $\psi \in \Theta$, since $K_T \in \mathcal{L}^1$. If Y is any bounded random variable, then Y can be written as

$$Y = E[Y \mid \mathcal{F}_0] + (\xi \cdot M)_T + L_T$$

by the Galtchouk-Kunita-Watanabe projection theorem, where ξ is in $L^2(M)$ and $L \in \mathcal{M}_0^2$ is strongly orthogonal to $\theta \cdot M$ for every $\theta \in L^2(M)$. By the definition of λ and ψ, this implies

$$\begin{aligned} \left|E\left[Y(\psi\cdot M)_T\right]\right| &= \left|E\left[V\int_t^T \xi_u' d\left\langle M\right\rangle_u \lambda_u\right]\right| \\ &\leq \|V\|_{\mathcal{L}^2}\left\|\xi\mathbf{1}_{]\!]t,T]\!]}\right\|_{L^2(A)} \\ &\leq \|V\|_{\mathcal{L}^2}C\|\xi\|_{L^2(M)} \\ &\leq \|V\|_{\mathcal{L}^2}C\|Y\|_{\mathcal{L}^2}, \end{aligned}$$

where the second inequality follows from $D_2(P)$. Since Y was arbitrary, we deduce that

$$C^2\|V\|^2_{\mathcal{L}^2} \geq \|(\psi \cdot M)_T\|^2_{\mathcal{L}^2} = E\left[\int_t^T V^2\lambda'_u d\left\langle M\right\rangle_u \lambda_u\right] = E\left[V^2 E\left[K_T - K_t \mid \mathcal{F}_t\right]\right].$$

Since V was arbitrary chosen in $\mathcal{L}^2(\mathcal{F}_t, P)$, we conclude that

$$E\left[K_T - K_t \mid \mathcal{F}_t\right] \leq C^2 \quad P-\text{a.s.},$$

and so $\lambda \cdot M$ is in bmo_2. This completes the proof of theorem 3.3.

We now turn to the second part of this section where we return to our question of closedness of $G_T(\Theta)$ in $\mathcal{L}^2(P)$. Given $\theta \in \Theta$, there are two ways to look at the stochastic integral $\theta \cdot X$: either we consider the entire process $G(\theta) = (\theta \cdot X)_{0\leq t\leq T}$ or we only look at the final result, i.e. the random variable $G_T(\theta) = (\theta \cdot X)_T$.
If we adopt the first point of view, we consider two other norms on Θ : for $\theta \in \Theta$, we define

$$|\!|\!|\theta|\!|\!| = \|\theta\|_{L^2(M)} + \|\theta\|_{L^2(A)}$$

and as in definition 2.6 above

$$\|\theta\|_{G(\Theta)} = \|\theta \cdot X\|_{\mathcal{R}^2(P)}.$$

Both concepts define norms on the vector space Θ with the property that these norms equal 0 for $\theta \in \Theta$ if and only if the process $(\theta \cdot X)_{0\leq t\leq T}$ vanishes almost surely.
On the other hand, we consider on the vector space $G_T(\Theta)$ the norm $\| \,.\, \|_{\mathcal{L}^2(P)}$.

Consider the diagram

$$(\Theta, |\!|\!| \,.\, |\!|\!|) \xrightarrow{i} (\Theta, \| \,.\, \|_{G(\Theta)}) \xrightarrow{j} (G_T(\Theta), \| \,.\, \|_{\mathcal{L}^2(P)})$$

where i denotes the identical map and j the canonical map which associates to $\theta \in \Theta$ the random variable $G_T(\theta)$.
The continuity of i follows from Doob's inequality and the continuity of j is obvious. Also note that the definition of Θ was designed in such a way that Θ is complete with respect to $|\!|\!| \,.\, |\!|\!|$, i.e., $(\Theta, |\!|\!| \,.\, |\!|\!|)$ is a Banach space. As the maps i and j are surjective, we deduce from the open mapping theorem that the problem whether Θ is complete with respect to $\| \,.\, \|_{G(\Theta)}$ and whether $G_T(\Theta)$ is complete with respect to $\| \,.\, \|_{\mathcal{L}^2(P)}$ is therefore equivalent to the question whether i, resp. $j \circ i$, are open maps.
To take full advantage of this information, we want to know whether j is one-to-one, i.e. whether, for $\theta \in \Theta$, $G_T(\theta) = 0$ implies that the entire process $G(\theta)$ vanishes almost surely. Fortunately, this is the case under a very mild condition.

Lemma 3.5. Assume that X is a (not necessarily continuous) semimartingale in $\mathcal{S}^2_{loc}$ which is a local martingale under some equivalent measure Q with square-integrable density $\frac{dQ}{dP}$. Then the map j is one-to-one.

Proof. Let us take $\theta \in \Theta$ such that $G_T(\theta) = 0$. If Z is defined by

$$Z_t := E\left[\frac{dQ}{dP} \mid \mathcal{F}_t\right] \quad 0 \le t \le T$$

then Z is a strictly positive square-integrable P-martingale and $G(\theta)Z$ is a P-local martingale. Moreover, $G(\theta)^*$ as well as Z^* are in $\mathcal{L}^2(P)$, by Doob's inequality. Hence the maximal function $(G(\theta)Z)^*$ is P-integrable so that $G(\theta)Z$ is a $H^1(P)$-martingale. By hypothesis, $G_T(\theta) = 0$ so that the P-martingale $G(\theta)Z$ vanishes identically. As Z is strictly positive almost surely, we conclude that the process $G(\theta)$ also vanishes almost surely. This completes the proof of lemma 3.5.

Proposition 3.6. Assume that X is a (not necessarily continuous) semimartingale in $\mathcal{S}^2_{loc}$.
(i) The normed space $(\Theta, \|\ .\ \|_{G(\Theta)})$ is complete if and only if the map i is an isomorphism, i.e. if and only if there is a constant $C > 0$ such that

$$\forall \theta \in \Theta, \quad \|\!|\theta|\!\| \le C \left\|\theta\right\|_{G(\Theta)} .$$

(ii) Assume in addition that there is an equivalent local martingale measure Q for X with square-integrable density. Then the normed space $(G_T(\Theta), \|\ .\ \|_{\mathcal{L}^2(P)})$ is complete, that is, $G_T(\Theta)$ is closed in $\mathcal{L}^2(P)$, if and only if the map $j \circ i$ is an isomorphism, i.e., if and only if there is a constant $C > 0$ such that

$$\forall \theta \in \Theta, \quad \|\!|\theta|\!\| \le C \left\|G_T(\theta)\right\|_{\mathcal{L}^2(P)} .$$

Proof. Immediate from lemma 3.5 and Banach's isomorphism theorem.

Now the question arises whether the property described in part (ii) of proposition 3.6 is related to the inequality $D_2(P)$ studied in the first part of this section. To answer this question, it is important to distinguish the continuous case from the general case. In the former, we get an interesting connection between the closedness of $G_T(\Theta)$ in $\mathcal{L}^2(P)$ and the inequality $D_2(P)$ (see theorem 3.7 below). In the general case, however, there is no hope for a positive result as shown by example 3.9 below.

Theorem 3.7. Suppose that X is a semimartingale in $\mathcal{S}^2_{loc}$ such that A, the predictable part of X, is continuous. If $j \circ i\ :\ \Theta \longrightarrow G_T(\Theta)$ is one-to-one and if $G_T(\Theta)$ is closed in $\mathcal{L}^2(P)$ then the inequality $D_2(P)$ is satisfied.
In particular, $D_2(P)$ holds true if $G_T(\Theta)$ is closed, A is continuous and there is an equivalent local martingale measure with square-integrable density.

For the proof we need the following easy result.

Lemma 3.8. Suppose that A is continuous. Let $\theta \in \Theta$ and $\eta > 0$. Then there exists a predictable process ε with values in $\{-1, +1\}$ such that

$$\forall t \in [0, T], \quad \left| \int_0^t \varepsilon_s \theta'_s dA_s \right| \leq \eta.$$

Proof of lemma 3.8. We can assume that $\theta \cdot A$ is increasing. If it is not the case, we multiply $\theta' dA$ by its sign. Then, we define a sequence $(T_n)_{n \geq 0}$ of stopping times by setting

$$T_0 = 0 \quad \text{and} \quad T_{n+1} = \inf \left\{ t \geq T_n \mid \int_0^T \mathbf{1}_{]\!]T_n, t]\!]}(s) \theta'_s dA_s \geq \eta \right\}.$$

Since A is a finite variation process, the sequence $(T_n)_{n \geq 0}$ is finite. Finally, we set $\varepsilon = 1$ on $[\![T_{2n}, T_{2n+1}[\![$ and $\varepsilon = -1$ elsewhere. This completes the proof of lemma 3.8.

Proof of theorem 3.7. Now let $\theta \in \Theta$ and take ε as in lemma 3.8. ¿From Doob's inequality

$$\|G_T(\varepsilon\theta)\|_{\mathcal{L}^2} \leq \|\theta\|_{L^2(M)} + \eta.$$

Therefore, from proposition 3.6, we deduce

$$\begin{aligned} \|\theta\|_{L^2(M)} + \|\theta\|_{L^2(A)} &= \|\varepsilon\theta\|_{L^2(M)} + \|\varepsilon\theta\|_{L^2(A)} \\ &\leq C \|G_T(\varepsilon\theta)\|_{\mathcal{L}^2} \\ &\leq C(\|\theta\|_{L^2(M)} + \eta). \end{aligned}$$

When η tends to 0, we obtain the inequality $D_2(P)$.

Let us comment on the hypothesis that A is continuous. Of course, this is satisfied if X is continuous. But if X has only jumps at totally inaccessible stopping times, we still can see that A remains continuous. On the other hand when X jumps also at predictable stopping times the assumption that A is continuous is not satisfactory. Indeed suppose that X jumps at a predictable time τ and suppose that A is continuous. Since τ is predictable , this implies $E[\Delta X_\tau \mid \mathcal{F}_{\tau-}] = 0$. But an economic interpretation of A is related to the so-called "price of risk" process. Assuming that A is continuous at τ would then be interpreted as "the risk at time τ is not rewarded". In economic term such an assumption would mean that the risk at time τ can be "diversified", a concept used in many texts but without a precise definition.

We now pass to the general case : the subsequent example shows that for processes with jumps, theorem 3.7 does not hold true anymore.

Example 3.9. There is a bounded stochastic process $X = (X_0, X_1, X_2)$ admitting a bounded equivalent martingale measure such that
(i) the inequality $D_2(P)$ fails ;
(ii) $G_2(\Theta)$ is closed in $\mathcal{L}^2(P)$.

First consider the following building block for the construction of the example. Let $0 < \varepsilon \leq 1$ and define the stochastic process $Y^\varepsilon = (Y_0^\varepsilon, Y_1^\varepsilon)$ by $Y_0^\varepsilon \equiv 0$ and

$$Y_1^\varepsilon = \begin{cases} -1 & \text{with probability } \frac{\varepsilon}{2+\varepsilon} \\ 1+\varepsilon & \text{with probability } \frac{2}{2+\varepsilon} \end{cases}$$

so that $E[Y_1^\varepsilon] = 1$.
If $(\mathcal{F}_0, \mathcal{F}_1)$ denotes the filtration generated by Y^ε, then the predictable part of Y^ε is given by $A_0^\varepsilon = 0$, $A_1^\varepsilon = E[Y_1^\varepsilon] = 1$, and the martingale part by $M_0^\varepsilon = 0$ and

$$M_1^\varepsilon = \begin{cases} -2 & \text{with probability } \frac{\varepsilon}{2+\varepsilon} \\ \varepsilon & \text{with probability } \frac{2}{2+\varepsilon} \end{cases}$$

An elementary calculation gives

$$\|A_1\|_{\mathcal{L}^2(P)} = 1, \;\; \|M_1\|_{\mathcal{L}^2(P)} = \sqrt{2\varepsilon}, \;\; \|Y_1\|_{\mathcal{L}^2(P)} = \sqrt{1+2\varepsilon}.$$

As $\varepsilon > 0$ tends to 0, the ratio $\|A_1\|_{\mathcal{L}^2(P)} \;/\; \|M_1\|_{\mathcal{L}^2(P)}$ tends to infinity while the ratio $(\|A_1\|_{\mathcal{L}^2(P)} + \|M_1\|_{\mathcal{L}^2(P)}) \;/\; \|Y_1\|_{\mathcal{L}^2(P)}$ tends to one and therefore remains bounded.
How is this related to the inequality $D_2(P)$ and the closedness of $G_T(\Theta)$ in $\mathcal{L}^2(P)$? Of course, both properties are satisfied for Y^ε as the space Θ is simply one-dimensional (the only stochastic integrals of Y^ε are the scalar multiples of Y^ε). But the constant C in the definition of $D_2(P)$ deteriorates as ε tends to 0, as for each $\theta \in \Theta$, $\theta \neq 0$,

$$\frac{\|\theta\|_{L^2(A)}}{\|\theta\|_{L^2(M)}} = \frac{\|A_1^\varepsilon\|_{\mathcal{L}^2(P)}}{\|M_1^\varepsilon\|_{\mathcal{L}^2(P)}} = (2\varepsilon)^{-1/2}.$$

On the other hand, the constant in proposition 3.6 (ii) above does not deteriorate as ε tends to 0, as

$$\frac{|\!|\!|\theta|\!|\!|}{\|G_1(\theta)\|_{\mathcal{L}^2(P)}} = \frac{\|A_1^\varepsilon\|_{\mathcal{L}^2(P)} + \|M_1^\varepsilon\|_{\mathcal{L}^2(P)}}{\|Y_1^\varepsilon\|_{\mathcal{L}^2(P)}} = \frac{1+(2\varepsilon)^{1/2}}{(1+2\varepsilon)^{1/2}} \xrightarrow[\varepsilon \to 0]{} 1.$$

Finally, to transform this quantitative phenomenon into a qualitative one, it suffices to glue a sequence of the above building blocks together. This is most easily done in the following way : let $X_0 = X_1 = 0$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (to maintain our usual setting) and let $\mathcal{F}_1$ be generated by a partition $(B_n)_{n\geq 1}$ of Ω such that $P[B_n] > 0$, for each n. Fix a sequence $\varepsilon_n > 0$ tending to 0 and define

$$X_2 = \begin{cases} -1 & \text{on a subset of } B_n \text{ of probability } \frac{\varepsilon_n}{2+\varepsilon_n} P[B_n] \\ 1+\varepsilon_n & \text{on a subset of } B_n \text{ of probability } \frac{2}{2+\varepsilon_n} P[B_n] \end{cases}$$

It is straightforward to check that X satisfies the required properties.

We now construct a series of three counter-examples which are arranged in ascending order of complexity.

The first example is similar to example 7.5.3 of Durrett (1984); we also refer to a more sophisticated example in Kazamaki (1994, example 3.4).
The third example uses an idea from Schachermayer (1993) and Delbaen/Schachermayer (1995d). We shall try to harmonize the present notation with that of Delbaen/ Schachermayer (1996d).
For a continuous semimartingale X with canonical decomposition

$$X = X_0 + M + A = X_0 + M + \langle M \rangle \cdot \lambda$$

we shall call the local martingale $L = \mathcal{E}(-\lambda \cdot M)$ the *density process associated to* X. In order not to obscure the subsequent calculations with irrelevant constants we adopt the following notation : we write $a_n \approx b_n$ if there is a constant $0 < c < \infty$ such that $a_n = cb_n$, for all $n \in I\!N$.

Example 3.10. For $1 < p_0 < +\infty$, we construct a continuous real semimartingale $X = (X_t)_{t \in [0,\infty]}$ with canonical decomposition $X = M + A = M + \langle M \rangle \cdot \lambda$ such that the associated density process $L = \mathcal{E}(-\lambda \cdot M)$ has the following properties :
(i) L satisfies the predictable representation property (PRP).
(ii) For $1 < p < p_0$ the martingale L satisfies $R_p(P)$. In particular L is bounded in $\mathcal{L}^p(P)$ and the martingale $\lambda \cdot M$ is in BMO.
(iii) The martingale L is unbounded in $\mathcal{L}^{p_0}(P)$ as $\|L_\infty\|_{\mathcal{L}^{p_0}(P)} = \infty$. In particular, inequality $R_{p_0}(P)$ is not satisfied for L.

Proof. Let $\widetilde{W}$ denote a one-dimensional standard Brownian motion based on $(\Omega, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t \in I\!R_+}, P)$ and $\widetilde{X}$ the semimartingale

$$\widetilde{X} = \widetilde{W}_t - t.$$

In this case $\lambda \equiv -1$ and the associated density process $\widetilde{L} = \mathcal{E}(\widetilde{W})$ simply equals standard geometric Brownian motion.
The next step will also be used for the examples below : fix two parameters $a > 0$, $a \neq 1$, and $0 < \gamma <$min $(1, a^{-1})$ and define inductively a sequence $(\tau_n)_{n \geq 0}$ of stopping times by letting $\tau_0 = 0$ and

$$\tau_n = \inf \left\{ t > \tau_{n-1} \mid \frac{\widetilde{L}_t}{\widetilde{L}_{\tau_{n-1}}} = a \text{ or } b \right\}$$

where we define $b := \dfrac{1 - a\gamma}{1 - \gamma}$. Note that $0 < b < +\infty$ and $b \neq 1$. The martingale property implies that

$$1 = E[\widetilde{L}_{\tau_1}] = aP[\widetilde{L}_{\tau_1} = a] + bP[\widetilde{L}_{\tau_1} = b].$$

The real number b was chosen such that we obtain
(3.1) $$P[\widetilde{L}_{\tau_1} = a] = \gamma \text{ and } P[\widetilde{L}_{\tau_1} = b] = 1 - \gamma.$$

Define the random number $N = N(\omega)$ as

$$N = \inf \left\{ n \mid \frac{\widetilde{L}_{\tau_n}}{\widetilde{L}_{\tau_{n-1}}} = b \right\}$$

and let τ denote the stopping time $\tau = \tau_N$. We now stop the processes $\tilde{X}$ and $\tilde{L}$ at time τ and indicate this by dropping the tildes, i.e., $L = \widetilde{L}^{\tau}$, $X = \widetilde{X}^{\tau}$, and we denote by $\mathcal{F}$ and $(\mathcal{F}_t)_{t\in[0,\infty]}$ the σ-algebra and the (saturated and right-continuous) filtration generated by X (or equivalently by L).
By iterating the argument in (3.1) above one easily obtains that, for $n \geq 1$,

(3.2) $$P(\tau = \tau_n) = (1-\gamma)\gamma^{n-1} \approx \gamma^n$$

and

(3.3) $$L_\infty L_\tau = ba^{n-1} \approx a^n \quad \text{on } \{\tau = \tau_n\}.$$

Finally note that there are constants $c > 0$ and $C > 0$, depending only on a and γ, such that, for every $n \in \mathbb{N}$ and random times S, T taking values in the stochastic interval $[\![\tau_{n-1}, \tau_n]\!]$ we have that

(3.4) $$c \leq \frac{L_S}{L_T} \leq C \quad P\text{-a.s.}$$

Now we fix the parameters a and γ by letting $a > 1$, e.g. $a = 2$, and $0 < \gamma < a^{-1}$ such that $a^{p_0}\gamma = 1$, which is obviously possible as $p_0 > 1$. Let us check that L meets our requirements :
(i) is rather obvious,
(iii) : it suffices to simply calculate the $\mathcal{L}^{p_0}(P)$-norm of $L_\infty = L_\tau$

$$\|L_\infty\|^{p_0}_{\mathcal{L}^{p_0}(P)} = E\left[\left(\sum_{n=1}^{\infty} L_\tau \mathbf{1}_{\{\tau=\tau_n\}}\right)^{p_0}\right] \approx \sum_{n=1}^{\infty} (a^n)^{p_0}\gamma^n = +\infty.$$

(ii) : as regards $R_p(P)$ for $1 < p < p_0$ first note that the same computation as above reveals that

$$\|L_\infty\|^{p}_{\mathcal{L}^{p}(P)} \approx \sum_{n=1}^{\infty} (a_n)^p\gamma^n < \infty.$$

Next note that our construction is homogeneous with respect to the multiplicative structure of $\mathbb{R}_+$ in the following sense : if $A \in \mathcal{F}_{\tau_n}$ is a set of positive measure contained in $\{\tau > \tau_n\}$ and if P_A denotes the renormalized restriction of P to A, then the process

$$\left(\frac{L_{t+\tau_n}}{L_{\tau_n}}\right)_{t\geq 0} = \left(\frac{L_{t+\tau_n}}{a^n}\right)_{t\geq 0}$$

under P_A is identical in law to the original process $(L_t)_{t\geq 0}$ under P. In particular, for every $n \geq 1$,

(3.5) $$\frac{E[L^p_\infty \mid \mathcal{F}_{\tau_n}]}{L^p_{\tau_n}} = E[L^p_\infty]\mathbf{1}_{\{\tau>\tau_n\}},$$

which shows inequality $R_p(P)$ to hold true for all stopping times S of the form $S = \tau_n$.

To verify $R_p(P)$ for an arbitrary stopping time S, it is easy to see that we may assume that there is $n \geq 1$ such that S takes its values (except for infinity) in $]\!]\tau_{n-1}, \tau_n]\!]$. Indeed, the sets $\{S \in]\!]\tau_{n-1}, \tau_n]\!]\}$ are in $\mathcal{F}_S$.
So assume that $[\![S]\!] \subseteq]\!]\tau_{n-1}, \tau_n]\!] \cup [\![\infty]\!]$ and use (3.4) and (3.5) above to estimate

$$\begin{aligned}
\left\| E\left[\frac{L_\infty^p}{L_S^p} \mid \mathcal{F}_S\right]\right\|_\infty &\leq c^{-1}\left\| E\left[\frac{L_\infty^p}{L_{\tau_n}^p} \mid \mathcal{F}_S\right]\right\|_\infty \\
&= c^{-1}\left\| E\left[E\left[\frac{L_\infty^p}{L_{\tau_n}^p} \mid \mathcal{F}_{\tau_n}\right] \mid \mathcal{F}_S\right]\right\|_\infty \\
&\leq c^{-1}\left\| E\left[\frac{L_\infty^p}{L_{\tau_n}^p} \mid \mathcal{F}_{\tau_n}\right]\right\|_\infty \\
&\leq c^{-1} E[L_\infty^p].
\end{aligned}$$

This shows that L satisfies $R_p(P)$, thus finishing the proof of the assertions for example 3.10.

The next step is to construct an example with similar features as the first one, but such that the $\mathcal{L}^{p_0}(P)$-norm of L is finite and only the inequality $R_{p_0}(P)$ fails for L.

Example 3.11. For $1 < p_0 < \infty$ we construct a continuous real semimartingale $X = (X_t)_{t\in[0,\infty]}$ with canonical decomposition $X = M + A = M + \langle M\rangle \cdot \lambda$ such that the associated density process $L = \mathcal{E}(-\lambda \cdot M)$ has the following properties :
(i) L satisfies the predictable representation property (PRP).
(ii) For $1 < p < p_0$ the martingale L satisfies $R_p(P)$. In particular L is bounded in $\mathcal{L}^p(P)$ and $\lambda \cdot M$ is in BMO.
(iii) The martingale L is bounded in $\mathcal{L}^{p_0}(P)$, but L does not satisfy $R_{p_0}(P)$.

Proof. If $\widetilde{W}$ again denotes a standard Brownian motion, define now

$$\widetilde{X}_t = \begin{cases} \widetilde{W}_t & \text{for} \quad t \in [0,1] \\ \widetilde{W}_t - (t-1) & \text{for} \quad t \in [1,\infty[. \end{cases}$$

Choose a partition $(A_k)_{k\geq 1}$ of Ω into sets of $\mathcal{F}_1$ satisfying $P(A_k) = 2^{-k}$.
Note that the density process $\widetilde{L}$ associated to $\widetilde{X}$ now equals

$$\widetilde{L}_t = \begin{cases} 1 & \text{for} \quad t \in [0,1] \\ \mathcal{E}(\widetilde{W}_t - \widetilde{W}_1) & \text{for} \quad t \in [1,\infty[. \end{cases}$$

Define the stopping times τ_n and the random number N for the process $\widetilde{L}$ exactly as above ; only for the definition of τ we apply a small modification. Define τ to equal $\tau_{N\wedge k}$ on each A_k.
With this modification done define again X and L by stopping $\widetilde{X}$ and $\widetilde{L}$ at time τ and consider these processes with respect to the filtrations they generate.

The verification of the associated properties of this example now is a straightforward modification of the above arguments and left to the reader.

The next example, which again is a variation of the same theme, is more tricky. This time, it is crucial to drop the property that M (or equivalently L) satisfies the predictable representation property. In this case the density process $L = \mathcal{E}(-\lambda \cdot M)$ associated to $X = M + \langle M \rangle \cdot \lambda$ is not the only candidate for (the density process of) an equivalent martingale measure for the semimartingale X ; if Z is any positive local martingale, $Z_0 = 1$, strongly orthogonal to L such that ZL, the pointwise product process, is not only a local martingale but a true uniformly integrable martingale, then $Z_\infty L_\infty$ is the density of a measure Q under which X is a local martingale (see Ansel/Stricker (1992)). It was shown in Schachermayer (1993) and Delbaen/Schachermayer (1996d) that, for a properly chosen Z, the process ZL may have better properties than the process L. This also turns out to be the case in the present context in a rather striking way.

Example 3.12. For $1 < p_0 < \infty$ we construct a continuous real semimartingale $X = (X_t)_{t \in [0,\infty]}$ with canonical decomposition $X = M + A = M + \langle M \rangle \cdot \lambda$ and a continuous real uniformly bounded martingale Z, strongly orthogonal to M, such that, for $L = \mathcal{E}(-\lambda \cdot M)$ denoting the density process associated to X, the following properties are satisfied :
(i) The process ZL is a martingale satisfying the predictable representation property (PRP), while this property fails for the martingales M, L and Z.
(ii) For $1 < p < p_0$ the martingale L satisfies $R_p(P)$. In particular L is bounded in $\mathcal{L}^p(P)$ and $\lambda \cdot M$ is in BMO.
(iii) The martingale L is unbounded in $\mathcal{L}^{p_0}(P)$ as $\|L_\infty\|_{\mathcal{L}^{p_0}(P)} = \infty$. In particular, inequality $R_{p_0}(P)$ fails for L. (iv) There are constants $0 < c < C < \infty$ such that $c \leq ZL \leq C$; whence the product martingale ZL satisfies $R_\infty(P)$.

Proof. Choose $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \in \mathbb{R}_+}, P)$ such that there are two independent standard Brownian motions W' and W'' defined on this stochastic base. Let $L' = \mathcal{E}(W')$ and $L'' = \mathcal{E}(W'')$.
Fix the parameters $a' > 1$, $0 < \gamma' < (a')^{-1}$, $a'' = (a')^{-1}$ and $0 < \gamma'' < 1$. We choose these parameters such that we have $(a')^{p_0} \gamma' \gamma'' = 1$, which obviously is possible as $p_0 > 1$.
Now define stopping times $(\tau'_n)_{n \geq 0}$ and $(\tau''_n)_{n \geq 0}$ by letting $\tau'_0 = \tau''_0 = 0$ and

$$\tau'_{n+1} = \inf \left\{ t > \tau'_n \mid \frac{L'_t}{L'_{\tau'_n}} = a' \text{ or } b' \right\}$$

and

$$\tau''_{n+1} = \inf \left\{ t > \tau''_n \mid \frac{L''_t}{L''_{\tau''_n}} = a'' \text{ or } b'' \right\},$$

where $b' = \dfrac{1-a'\gamma'}{1-\gamma'}$ and $b'' = \dfrac{1-a''\gamma''}{1-\gamma''}$.
The idea of the example is to patch the processes L' and L'' together by intertwining the stochastic intervals $]\!]\tau'_{n-1}, \tau'_n]\!]$ and $]\!]\tau''_{n-1}, \tau''_n]\!]$. Define inductively the random times $(\tau_n)_{n\geq 0}$ and $(\sigma_n)_{n\geq 0}$, which are stopping times for the filtration $(\mathcal{G}_t)_{t\in \mathbb{R}_+}$, by letting $\sigma_0 = \tau_0 = 0$ and, for, $n \geq 1$,

$$\tau_n = \sigma_{n-1} + \tau'_n \quad \text{and} \quad \sigma_n = \tau_n + \tau''_n.$$

Note that $0 = \tau_0 = \sigma_0 < \tau_1 < \sigma_1 < \tau_2 < \ldots$. Next define the processes $\widetilde{X}$, $\widetilde{L}$ and $\widetilde{Z}$ by specifying their values on the stochastic intervals $]\!]\sigma_{n-1}, \tau_n]\!]$ and $]\!]\tau_n, \sigma_n]\!]$ inductively for $n = 1, 2, \ldots$:
If $t = t(\omega) \geq 0$ is such that $\sigma_{n-1} + t \leq \tau_n$ let

$$\begin{aligned}
\widetilde{X}_{\sigma_{n-1}+t} - \widetilde{X}_{\sigma_{n-1}} &= (W'_{\tau'_{n-1}+t} - W'_{\tau'_{n-1}}) - t, \\
\widetilde{L}_{\sigma_{n-1}+t} - \widetilde{L}_{\sigma_{n-1}} &= L'_{\tau'_{n-1}+t} - L'_{\tau'_{n-1}}, \\
\widetilde{Z}_{\sigma_{n-1}+t} - \widetilde{Z}_{\sigma_{n-1}} &= 0.
\end{aligned}$$

If $t = t(\omega) \geq 0$ is such that $\tau_n + t \leq \sigma_n$ let

$$\begin{aligned}
\widetilde{X}_{\tau_n+t} - \widetilde{X}_{\tau_n} &= 0, \\
\widetilde{L}_{\tau_n+t} - \widetilde{L}_{\tau_n} &= 0, \\
\widetilde{Z}_{\tau_n+t} - \widetilde{Z}_{\tau_n} &= L''_{\tau_{n-1}+t} - L''_{\tau_{n-1}}.
\end{aligned}$$

Loosely speaking, the processes $\widetilde{X}$ and $\widetilde{L}$ are constant on the intervals of the form $[\![\tau_n, \sigma_n]\!]$ and move only on the intervals of the form $[\![\sigma_{n-1}, \tau_n]\!]$, where they behave like $W'_t - t$ and L'_t resp. on the corresponding intervals $[\![\tau'_{n-1}, \tau'_n]\!]$. Similarly, $\widetilde{Z}$ is constant on the intervals of the form $[\![\sigma_{n-1}, \tau_n]\!]$ and moves on the intervals of the form $[\![\tau_n, \sigma_n]\!]$ as L'' does on $[\![\tau''_{n-1}, \tau''_n]\!]$. Define the random numbers $N(\omega)$ and $M(\omega)$ as

$$N = \inf\left\{ n \mid \frac{L'_{\tau'_n}}{L'_{\tau'_{n-1}}} = b' \right\} = \inf\left\{ n \mid \frac{\widetilde{L}_{\tau_n}}{\widetilde{L}_{\tau_{n-1}}} = b' \right\}$$

and

$$M = \inf\left\{ n \mid \frac{L''_{\tau''_n}}{L''_{\tau''_{n-1}}} = b'' \right\} = \inf\left\{ n \mid \frac{\widetilde{Z}_{\sigma_n}}{\widetilde{Z}_{\sigma_{n-1}}} = b'' \right\}$$

and define $\tau = \tau_N$, $\sigma = \sigma_M$; finally, stop the processes $\widetilde{X}$, $\widetilde{L}$ and $\widetilde{Z}$ at time $\sigma \wedge \tau$ and indicate this by dropping the tildes, i.e. $X = \widetilde{X}^{\sigma\wedge\tau}$, $L = \widetilde{L}^{\sigma\wedge\tau}$, $Z = \widetilde{Z}^{\sigma\wedge\tau}$. Define $\mathcal{F}$ and $(\mathcal{F}_t)_{t\in[0,\infty]}$ to be the σ-algebra and the (right-continuous, saturated) filtration generated by L and Z. Note that neither L nor Z alone generate $\mathcal{F}$ and $(\mathcal{F}_t)_{t\in[0,\infty]}$ while the product ZL does generate them.

It is rather obvious that L and Z are martingales with respect to the filtration $(\mathcal{F}_t)_{t\in[0,\infty]}$ and that L is the density process associated to X. Assertion (i) follows from the remark in the preceding paragraph.
Similarly as in the previous examples note that there are constants $c < C < \infty$ depending only on the parameters a', a'', γ' and γ'' such that, for each $n \geq 1$ and random times S, T taking their values in $[\![\sigma_{n-1}, \sigma_n]\!]$ we have

$$c \leq \frac{L_S}{L_T} \leq C, \quad c \leq \frac{Z_S}{Z_T} \leq C, \quad c \leq \frac{(ZL)_S}{(ZL)_T} \leq C.$$

Making the crucial observation that because of $a'a'' = 1$ we have that $(ZL)_{\sigma_n} = 1$ on $\{\sigma_n < \tau \wedge \sigma\}$ we conclude that, for arbitrary stopping times S, T we have

$$c \leq \frac{(ZL)_S}{(ZL)_T} \leq C,$$

which readily proves (iv).
To prove (iii) note that

$$P[\tau \wedge \sigma = \tau_n] \approx P[\tau \wedge \sigma = \sigma_n] \approx (\gamma'\gamma'')^n$$

and that the values of L_∞ on $\{\tau \wedge \sigma = \tau_n\}$ as well as on $\{\tau \wedge \sigma = \sigma_n\}$ are -up to constant factors- equal to $(a')^n$. Hence we may calculate

$$\|L_\infty\|^{p_0}_{\mathcal{L}^{p_0}(P)} = \left\| \sum_{n=1}^{\infty} L_{\tau\wedge\sigma}(\mathbf{1}_{\{\tau\wedge\sigma=\tau_n\}} + \mathbf{1}_{\{\tau\wedge\sigma=\sigma_n\}}) \right\|^{p_0}_{\mathcal{L}^{p_0}(P)} \approx \sum_{n=1}^{\infty} (a')^{np_0}(\gamma'\gamma'')^n = +\infty,$$

which shows (iii). The analogous calculation for $1 < p < p_0$ reveals that

$$\|L_\infty\|_{\mathcal{L}^p(P)} < \infty$$

and similar arguments as the ones used for the first example show that L in fact satisfies $R_p(P)$, thus showing (ii).
This finishes the construction of example 3.12.

We have seen that for the closedness of $G_T(\Theta)$ in $\mathcal{L}^2(P)$, the inequality $D_2(P)$ is in general neither necessary nor sufficient. If we study the closedness of $G(\Theta)$ in $\mathcal{R}^2(P)$, we have a necessary and sufficient condition when A is continuous.

Theorem 3.13. Let X be an $\mathbb{R}^d$-valued semimartingale such that there is an equivalent local martingale measure Q with $\frac{dQ}{dP} \in \mathcal{L}^2(P)$ and such that the predictable part A of X is continuous. Then the space $G(\Theta)$ is closed in $\mathcal{R}^2(P)$ if and only if the inequality $D_2(P)$ holds.

We need an auxiliary result to prove theorem 3.13. The following lemma is a slight variant of Proposition 2 of Yor (1985), adapted for our present purposes. The main difference is that we do not assume that the local martingale M is continuous. Recall that the canonical decomposition of X is $X = M + \langle M \rangle \cdot \lambda$.

Lemma 3.14. Suppose that $N := \lambda \cdot M$ is in bmo_2. If A is continuous, then there is a constant C such that

$$E\left[\langle \theta \cdot X + Z \rangle_T\right] \leq C \left\| G(\theta) + Z \right\|^2_{\mathcal{R}^2(P)}$$

for all $\theta \in \Theta$ and $Z \in \mathcal{M}_0^2$ strongly orthogonal to M.

Proof. Define the processes $\widetilde{L} := \theta \cdot M + Z$ and

$$L := \widetilde{L} + \langle \widetilde{L}, N \rangle = \widetilde{L} + \theta \cdot A = \theta \cdot X + Z = G(\theta) + Z.$$

By Itô's formula,

$$L_t^2 = 2 \int_0^t L_{s-}\, dL_s + [L]_t$$

and therefore

$$\begin{aligned} E\left[\langle L \rangle_T\right] &= E[[L]_T] \\ &\leq 2 \left(E\left[(L_T^*)^2 \right] + E\left[\sup_{t \in [0,T]} \left| \int_0^t L_{s-}\, d\widetilde{L}_s \right| \right] + E\left[\left(\int L_- \left| d\langle \widetilde{L}, N \rangle \right| \right)_T \right] \right). \end{aligned}$$

Since A is continuous, we have $[\widetilde{L}] = [L]$ and so the Burkholder-Davis-Gundy inequality yields

$$E\left[\sup_{t \in [0,T]} \left| \int_0^t L_{s-}\, d\widetilde{L}_s \right| \right] \leq CE\left[\left(\int_0^T L_{s-}^2\, d[\widetilde{L}]_s \right)^{\frac{1}{2}} \right] \leq CE\left[L_T^* [L]_T^{\frac{1}{2}} \right].$$

Moreover, L is in $\mathcal{S}^2$ and $\widetilde{L}$ is in $\mathcal{M}_0^2$, and so $\int L_-\, d\widetilde{L}$ is in h_0^1 by the same argument as in lemma 3.4. Hence Fefferman's inequality implies

$$\begin{aligned} E\left[\left| \left\langle \int L_-\, d\widetilde{L}, N \right\rangle \right|_T \right] &\leq \sqrt{2} \|N\|_{bmo_2} \left\| \int L_-\, d\widetilde{L} \right\|_{h^1} \\ &= CE\left[\left(\int_0^T L_{s-}^2\, d\langle \widetilde{L} \rangle_s \right)^{\frac{1}{2}} \right] \\ &\leq CE\left[L_T^* \langle L \rangle_T^{\frac{1}{2}} \right], \end{aligned}$$

since $\langle \widetilde{L} \rangle = \langle L \rangle$ by the continuity of A. Putting these estimates together, we obtain

$$\begin{aligned} E\left[\langle L \rangle_T + [L]_T\right] &\leq C \left(E\left[(L_T^*)^2 \right] + E\left[L_T^* \left([L]_T^{\frac{1}{2}} + \langle L \rangle_T^{\frac{1}{2}} \right) \right] \right) \\ &\leq C \left(E\left[(L_T^*)^2 \right] + \left(E\left[(L_T^*)^2 \right] E[[L]_T + \langle L \rangle_T] \right)^{\frac{1}{2}} \right) \end{aligned}$$

and therefore, from classical results on 2^{nd} degree inequalities

$$E\left[\langle L\rangle_T\right] = E[[L]_T] \leq CE\left[(L_T^*)^2\right].$$

This completes the proof of lemma 3.14.

Proof of theorem 3.13. "if" part. Suppose that $D_2(P)$ is satisfied. Let $(G(\theta^n))_{n\geq 0}$ a sequence of $G(\Theta)$ which converges in $\mathcal{R}^2(P)$. Then it is a Cauchy sequence on the space $\mathcal{R}^2(P)$, so that

$$\|G(\theta^n) - G(\theta^m)\|_{\mathcal{R}^2(P)} < \varepsilon,$$

provided that m and n are large enough.
Since $D_2(P)$ is satisfied, it follows from lemma 3.1 that we can define the process N by $N_t := (\lambda \cdot M)_t$ and for each θ in Θ, we have

$$\langle \theta \cdot X, N\rangle_t = (\theta \cdot A)_t.$$

Hence, by lemma 3.14, we deduce that

$$E\left[\langle (\theta^n - \theta^m)\cdot X\rangle_T\right] \leq \varepsilon,$$

for m and n large enough. Since

$$E\left[\langle (\theta^n - \theta^m)\cdot X\rangle_T\right] = \|\theta^n - \theta^m\|_{L^2(M)},$$

the sequence $(\theta^n)_{n\geq 0}$ is a Cauchy sequence in $(L^2(M), \|\,.\,\|_{L^2(M)})$, so that it converges in $L^2(M)$ to a process θ. Thanks to $D_2(P)$, the convergence of $(\theta^n)_{n\geq 0}$ to θ in $L^2(M)$ implies the same convergence in $L^2(A)$. Finally,

$$\begin{aligned}
\|G(\theta^n) - G(\theta)\|_{\mathcal{R}^2(P)} &= \left\| \sup_{t\in[0,T]} |((\theta^n - \theta)\cdot X)_t| \right\|_{\mathcal{L}^2(P)} \\
&\leq \left\| \sup_{t\in[0,T]} |((\theta^n - \theta)\cdot M)_t| \right\|_{\mathcal{L}^2(P)} + \left\| \sup_{t\in[0,T]} |((\theta^n - \theta)\cdot A)_t| \right\|_{\mathcal{L}^2(P)} \\
&\leq 2\,\|\theta^n - \theta\|_{L^2(M)} + \|\theta^n - \theta\|_{L^2(A)}
\end{aligned}$$

from Doob's inequality. Therefore, the sequence $(G(\theta^n))_{n\geq 0}$ converges to $G(\theta)$ in $\mathcal{R}^2(P)$, which completes the proof of the "if" part.

"only if" part. Let us now suppose that $G(\Theta)$ is closed in $\mathcal{R}^2(P)$. Consider the mapping

$$\begin{aligned}
k\ :\ (\Theta, \|\,.\,\|_{L^2(M)} + \|\,.\,\|_{L^2(A)}) &\longrightarrow (G(\Theta), \|\,.\,\|_{\mathcal{R}^2(P)}) \\
\theta &\longmapsto G(\theta) = \theta\cdot X.
\end{aligned}$$

Then k is one-to-one and continuous by Doob's inequality. Due to the closedness of $G(\Theta)$ in $\mathcal{R}^2(P)$, the inverse mapping is also continuous, so that the norms $\| \,.\, \|_{L^2(M)} + \| \,.\, \|_{L^2(A)}$ and $\| \,.\, \|_{\mathcal{R}^2(P)}$ are equivalent : there are $C_1 > 0$ and $C_2 > 0$ such that

$$\forall \theta \in \Theta, \;\; C_1(\|\theta\|_{L^2(M)} + \|\theta\|_{L^2(A)}) \leq \|G(\theta)\|_{\mathcal{R}^2(P)} \leq C_2(\|\theta\|_{L^2(M)} + \|\theta\|_{L^2(A)}).$$

Let $\theta \in \Theta$ and $\eta > 0$, and choose a process ε as in lemma 3.8. Then Doob's inequality yields

$$\begin{aligned} \|G(\varepsilon\theta)\|_{\mathcal{R}^2(P)} &= \left\| \sup_{t\in[0,T]} |((\varepsilon\theta)\cdot X)_t| \right\|_{\mathcal{L}^2(P)} \\ &\leq \left\| \sup_{t\in[0,T]} |((\varepsilon\theta)\cdot M)_t| + \sup_{t\in[0,T]} |((\varepsilon\theta)\cdot A)_t| \right\|_{\mathcal{L}^2(P)} \\ &\leq 2\,\|\varepsilon\theta\|_{L^2(M)} + \|\varepsilon\theta\|_{L^2(A)} \\ &\leq 2\,\|\theta\|_{L^2(M)} + \eta. \end{aligned}$$

Hence

$$\begin{aligned} \|\theta\|_{L^2(M)} + \|\theta\|_{L^2(A)} &= \|\varepsilon\theta\|_{L^2(M)} + \|\varepsilon\theta\|_{L^2(A)} \\ &\leq \frac{1}{C_1}\,\|G(\varepsilon\theta)\|_{\mathcal{R}^2(P)} \\ &\leq \frac{1}{C_1}\Big(2\,\|\theta\|_{L^2(M)} + \eta\Big) \end{aligned}$$

When η tends to 0, we obtain the inequality $D_2(P)$, and this completes the proof of the "only if" part.

4. Necessary and sufficient conditions for the closedness of $G_T(\Theta)$.

In this section we will suppose that X is a continuous semimartingale for which an equivalent local martingale measure with square integrable density exists. The symbol $\mathcal{V}$ stands for the space of stochastic integrals $\theta\cdot X$ such that θ is a simple integrand and $\theta\cdot X$ remains bounded. As shown in section 3, a necessary condition for the closedness of $G_T(\Theta)$ is that the mapping $j\circ i \,:\, \Theta \longrightarrow G_T(\Theta)$ is one-to-one and that $D_2(P)$ holds. The following theorem solves the problem of the closedness of $G_T(\Theta)$ for continuous semimartingales completely.

Theorem 4.1. Let X denote a continuous semimartingale, then are equivalent :
(1) There is an equivalent local martingale measure with square integrable density and $G_T(\Theta)$ is closed in $\mathcal{L}^2(P)$.
(2) There is a square integrable local martingale measure Q that satisfies the inequality $R_2(P)$.

(3) The variance optimal measure Q^{opt} is in $I\!M^e \cap L^2(P)$ and satisfies $R_2(P)$.
(4) $\exists C$ such that for all $Y \in \mathcal{V}$ we have $\|Y_T^*\|_{\mathcal{L}^2(P)} \leq C\,\|Y_T\|_{\mathcal{L}^2(P)}$.
(4') $\exists C$ such that for all $\theta \in \Theta$ we have

$$\|(\theta \cdot X)_T^*\|_{\mathcal{L}^2(P)} = \|\theta\|_{G(\Theta)} \leq C\,\|(\theta \cdot X)_T\|_{\mathcal{L}^2(P)}\,.$$

(5) $\exists C$ such that for all $Y \in \mathcal{V}$ and all $\lambda \geq 0$ we have $\lambda P[Y_T^* > \lambda]^{1/2} \leq C\,\|Y_T\|_{\mathcal{L}^2(P)}$.
(5') $\exists C$ such that for all $\theta \in \Theta$ and all $\lambda \geq 0$ we have

$$\lambda P[(\theta \cdot X)_T^* > \lambda]^{1/2} \leq C\,\|(\theta \cdot X)_T\|_{\mathcal{L}^2(P)}\,.$$

(6) $\exists C > 0$ such that for every stopping time S and every $A \in \mathcal{F}_S$ we have

$$\|\mathbf{1}_A - U\|_{\mathcal{L}^2(P)} \geq CP[A]^{1/2} \quad \text{for every } U \in {}_S\mathcal{V}.$$

(6') $\exists C > 0$ such that for every stopping time S, every $A \in \mathcal{F}_S$ and every $\theta \in \Theta$ with $\theta = \theta\mathbf{1}_{]\!]S,T]\!]}$ we have $\|\mathbf{1}_A - (\theta \cdot X)_T\|_{\mathcal{L}^2(P)} \geq CP[A]^{1/2}$.

Proof. The theorem is almost a reformulation of the results of section 2. A local martingale measure for X is the same as a martingale measure for $\mathcal{V}$. Since the appropriate spaces of simple stochastic integrals are dense in the spaces of stochastic integrals, we simply deduce from theorem 2.18 that the properties (2), (3), (4), (4'),(5), (5'), (6), (6') are all equivalent. Let us now show that (1) implies all the other properties. If there is an equivalent martingale measure with square-integrable density, then proposition 3.6 applies and the $\mathcal{R}^2(P)$-norm and the $\mathcal{L}^2(P)$-norm are equivalent (both to the $L^2(M)$-norm in fact). As a result one obtains (4') and hence all the other equivalent conditions. Conversely if (2) up to (6') hold, we have to deduce that the space $G_T(\Theta)$ is closed. By assumption there is a local martingale measure with square-integrable density that satisfies the inequality $R_2(P)$. So let Q be this martingale measure and put $E\left[\dfrac{dQ}{dP} \mid \mathcal{F}_t\right] = L_t$. Then L_t is necessarily of the form $L = \mathcal{E}(-\lambda \cdot M + U)$ where U is a local martingale strongly orthogonal to M, i.e. $\langle M, U\rangle = 0$ (see for instance Ansel/Stricker (1992)). The lemma below shows that $-\lambda \cdot M + U$ is in bmo_2. Since M and U are strongly orthogonal, we have $\langle -\lambda \cdot M + U\rangle = \langle \lambda \cdot M\rangle + \langle U\rangle$ and hence the local martingale $-\lambda \cdot M$ is also in bmo_2, which by the way is the same as BMO since M is continuous. Therefore X satisfies $D_2(P)$ and the norm on Θ is equivalent to the $L^2(M)$-norm. From lemma 3.14 we deduce that the $L^2(M)$-norm on Θ is dominated by the $\mathcal{R}^2(P)$-norm on $G(\Theta)$. This norm is by hypothesis equivalent to the $\mathcal{L}^2(P)$-norm on $G_T(\Theta)$. We finally find that the norm on Θ is equivalent to the $\mathcal{L}^2(P)$-norm on $G_T(\Theta)$ and hence by proposition 3.6, the space $G_T(\Theta)$ is closed.
This completes the proof of theorem 4.1 (modulo the subsequent lemma).

Lemma 4.2. If L is a uniformly integrable martingale with $L_T > 0$ and $L_0 = 1$ that satisfies the inequality $R_2(P)$, then necessarily L is of the form $\mathcal{E}(N)$ where N is in bmo_2.

Proof. The process L remains strictly positive and hence the process $\left(\frac{1}{L_{u-}}\right)_{0\leq t\leq T}$ is locally bounded. The square-integrability of the process L implies that the local martingale N defined by $dN_u = \frac{1}{L_{u-}}dL_u$ is locally square-integrable so that it makes sense to talk about $\langle N\rangle$. The process L is therefore of the form $L = \mathcal{E}(N)$ with N locally square-integrable.
For $s \geq 0$ fixed we define the sequence of stopping times $(T_n)_{n\geq 0}$ by

$$T_0 = s, \quad T_n = \inf\left\{t > T_{n-1} \mid \frac{L_t}{L_{T_{n-1}}} \leq \frac{1}{2}\right\} \wedge T.$$

Let C be the $R_2(P)$ constant of L, i.e. for all t we have

$$E\left[\left(\frac{L_T}{L_t}\right)^2 \mid \mathcal{F}_t\right] \leq C^2.$$

We first show that there is $\gamma < 1$, only depending on C, such that for all n,

$$P[T_n < \infty \mid \mathcal{F}_{T_{n-1}}] \leq \gamma.$$

This follows easily from the fact that on $\{T_{n-1} < T\}$

$$1 = E\left[\frac{L_{T_n}}{L_{T_{n-1}}} \mid \mathcal{F}_{T_{n-1}}\right] = E\left[\frac{L_{T_n}}{L_{T_{n-1}}}\mathbf{1}_{\{T_n<T\}} \mid \mathcal{F}_{T_{n-1}}\right] + E\left[\frac{L_{T_n}}{L_{T_{n-1}}}\mathbf{1}_{\{T_n=T\}} \mid \mathcal{F}_{T_{n-1}}\right]$$

The first term is smaller then $\frac{1}{2}P[T_n < T \mid \mathcal{F}_{T_{n-1}}]$ whereas the second can be estimated from above using the Cauchy-Schwarz inequality. We obtain

$$1 \leq \frac{1}{2}P[T_n < T \mid \mathcal{F}_{T_{n-1}}] + C\left(1 - P[T_n < T \mid \mathcal{F}_{T_{n-1}}]\right)^{1/2}$$

This implies the existence of $\gamma < 1$ such that $P[T_n < T \mid \mathcal{F}_{T_{n-1}}] \leq \gamma$ and where γ clearly depends only on C.
For $t \geq T_{n-1}$ set $U_t = \frac{\mathcal{E}(N)_t}{\mathcal{E}(N)_{T_{n-1}}}$ and note that $dU = U_- dN$. Since for $t \leq T_n$, $2U_{t-} \geq 1$ we have

$$\begin{aligned} E\left[\langle N\rangle_{T_n} - \langle N\rangle_{T_{n-1}} \mid \mathcal{F}_{T_{n-1}}\right] &= E\left[[N]_{T_n} - [N]_{T_{n-1}} \mid \mathcal{F}_{T_{n-1}}\right] \\ &\leq E\left[\int_{T_{n-1}}^{T_n} 4U_{s-}^2 d[N]_s \mid \mathcal{F}_{T_{n-1}}\right] \\ &\leq 4E\left[U_{T_n}^2 \mid \mathcal{F}_{T_n}\right]. \end{aligned}$$

It follows that

$$E\left[\langle N\rangle_{T_n} - \langle N\rangle_{T_{n-1}} \mid \mathcal{F}_{T_{n-1}}\right] \leq 4C^2.$$

Now we finally can estimate $E\left[\langle N\rangle_T - \langle N\rangle_s \mid \mathcal{F}_s\right]$ by the series

$$\begin{aligned}\sum_{k\geq 0} E\left[\langle N\rangle_{T_k} - \langle N\rangle_{T_{k-1}} \mid \mathcal{F}_s\right] &\leq \sum_{k\geq 0} E\left[E\left[\langle N\rangle_{T_k} - \langle N\rangle_{T_{k-1}} \mid \mathcal{F}_{T_{k-1}}\right] \mid \mathcal{F}_s\right] \\ &\leq \sum_{k\geq 0} E\left[E\left[\langle N\rangle_{T_k} - \langle N\rangle_{T_{k-1}} \mid \mathcal{F}_{T_{k-1}}\right] \mathbf{1}_{\{T_{k-1}<T\}} \mid \mathcal{F}_s\right] \\ &\leq 4C^2 \sum_{k\geq 0} E\left[\mathbf{1}_{\{T_{k-1}<T\}} \mid \mathcal{F}_s\right].\end{aligned}$$

Since

$$\begin{aligned} E\left[\mathbf{1}_{\{T_k<T\}} \mid \mathcal{F}_s\right] &= E\left[\mathbf{1}_{\{T_{k-1}<T\}}\, E\left[\mathbf{1}_{\{T_k<T\}} \mid \mathcal{F}_{T_{k-1}}\right] \mid \mathcal{F}_s\right] \\ &\leq E[\mathbf{1}_{\{T_{k-1}<T\}}\gamma \mid \mathcal{F}_s], \end{aligned}$$

we find that $E[\mathbf{1}_{\{T_{k-1}<T\}} \mid \mathcal{F}_s] \leq \gamma^{k-1}$ and hence

$$E\left[\langle N\rangle_T - \langle N\rangle_s \mid \mathcal{F}_s\right] \leq 4C^2 \sum_{k\geq 0} \gamma^k \leq \frac{4C^2}{1-\gamma}.$$

This completes the proof of lemma 4.2.

5. On the closure of $G_T(\Theta)$ in $\mathcal{L}^2(P)$.

Throughout this section, we do not assume that X is continuous.

When X admits an equivalent local martingale measure Q and when M has the predictable representation property under P, we shall determine the closure of $G_T(\Theta)$ in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. If the density of the equivalent martingale measure is square-integrable, the closure of $G_T(\Theta)$ is the space of square-integrable random variables H such that $E_Q[H \mid \mathcal{F}_0] = 0$. On the contrary, when the density of the equivalent local martingale measure is not square-integrable and if we assume moreover that X is continuous, we can prove that the closure of $G_T(\Theta)$ is the whole space $\mathcal{L}^2(\Omega, \mathcal{F}, P)$, under the assumption that $\mathcal{F}_0$ is trivial. These results are related to the results obtained by Delbaen/Schachermayer (1996c). We start with an auxiliary proposition.

Proposition 5.1. Suppose that M satisfies the predictable representation property under P and that there exists an equivalent martingale measure Q for X. Then
(1) For every bounded $\mathcal{F}_T$-measurable random variable U_T, there exists a sequence $(\theta^n)_{n\geq 0} \in \Theta$ such that $\theta^n \cdot X$ is a bounded Q-martingale and

$$\left(E_Q[U_T \mid \mathcal{F}_0] + (\theta^n \cdot X)_T\right)_{n\geq 0}$$

converges to U_T in $\mathcal{L}^2(P)$ and $\mathcal{L}^2(Q)$.
(2) $\overline{\mathcal{L}^2(\Omega, \mathcal{F}_0, P) + G_T(\Theta)} = \mathcal{L}^2(\Omega, \mathcal{F}, P)$.

Proof. (1) Let U_T be a random variable in $\mathcal{L}^\infty(\mathcal{F}_T)$. Since M has the PRP(P), X satisfies the PRP(Q) so that there exists a predictable, X-integrable process θ such that

$$U_T = E_Q[U_T \mid \mathcal{F}_0] + (\theta \cdot X)_T.$$

If $U_t := E_Q[U_T \mid \mathcal{F}_0] + (\theta \cdot X)_t = E_Q[U_T \mid \mathcal{F}_t]$, then U is uniformly bounded and therefore, $\theta \cdot X$ is in $\mathcal{S}^2_{loc}(P)$. So we can define an increasing sequence of stopping times $(T_n)_{n\geq 0}$ which tends to T and such that $\theta^n := \theta \mathbf{1}_{]\!]0,T_n]\!]}$ is in Θ. From the definition of T_n, the sequence $(U^n_T)_{n\geq 0} := (U_{T_n})_{n\geq 0}$ converges to U_T in $\mathcal{L}^2(P)$ and $\mathcal{L}^2(Q)$ because this sequence is bounded.

(2) Let H be a random variable in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ which is orthogonal to $\mathcal{L}^2(\Omega, \mathcal{F}_0, P) + G_T(\Theta)$. If U_T is a bounded random variable, part (1) allows us to build a sequence $(U^n_T)_{n\geq 0}$ which converges to U_T in $\mathcal{L}^2(P)$ and such that $U^n_T = U^n_0 + (\theta^n \cdot X)_T$ with $\theta^n \in \Theta$ and $U^n_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$. So

$$E_P[HU_T] = \lim_{n\to+\infty} E_P[HU^n_T] = \lim_{n\to+\infty} E_P[H(U^n_0 + (\theta^n \cdot X)_T)] = 0.$$

These equalities imply that $H = 0 \quad P-$a.s., that is

$$\overline{\mathcal{L}^2(\Omega, \mathcal{F}_0, P) + G_T(\Theta)} = \mathcal{L}^2(\Omega, \mathcal{F}, P).$$

By means of proposition 5.1, we can easily prove the next result.

Theorem 5.2. If M satisfies the predictable representation property under P and if X admits an equivalent local martingale measure Q with a square-integrable density, then $\overline{G_T(\Theta)} = \{H \in \mathcal{L}^2(\Omega, \mathcal{F}, P) \mid E_Q[H \mid \mathcal{F}_0] = 0\}$.

Proof. Let H be a random variable in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$, such that $E_Q[H \mid \mathcal{F}_0] = 0$. We already know that $\overline{\mathcal{L}^2(\Omega, \mathcal{F}_0, P) + G_T(\Theta)} = \mathcal{L}^2(\Omega, \mathcal{F}, P)$, so

$$H = \lim_{n\to+\infty} \left(H^n_0 + (\theta^n \cdot X)_T\right),$$

where $H^n_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$ and $\theta^n \in \Theta$. Since the density of Q is square-integrable, we can take the conditional expectation with respect to $\mathcal{F}_0$ under Q in the last equality and we obtain

$$\lim_{n\to+\infty} H^n_0 = 0,$$

which implies that H is in $\overline{G_T(\Theta)}$.

In the case where the density of the equivalent local martingale measure is no longer square-integrable , we can also characterize entirely the closure of $G_T(\Theta)$ in $\mathcal{L}^2(\Omega, \mathcal{F}, P)$, under the assumption that $\mathcal{F}_0$ is trivial.

Theorem 5.3. Let X be a càdlàg semimartingale which admits an equivalent local martingale measure Q. Assume that M satisfies the predictable representation property under P and that the density of Q is not square-integrable. Then, if $\mathcal{F}_0$ is trivial, $\overline{G_T(\Theta)} = \mathcal{L}^2(\Omega, \mathcal{F}, P)$.

Proof. Denote by $\mathcal{H}$ the hyperplane in $\mathcal{L}^\infty(P)$

$$\mathcal{H} = \{U \in \mathcal{L}^\infty(\Omega, \mathcal{F}, P) \mid E_Q[U] = 0\}.$$

As the density of Q is not square-integrable we have that $\mathcal{H}$ is dense in $\mathcal{L}^\infty(P)$ with respect to the norm-topology induced by $\| \,.\, \|_{\mathcal{L}^2(P)}$ on $\mathcal{L}^\infty(P)$.
Proposition 5.1 implies that $G_T(\Theta)$ is $\| \,.\, \|_{\mathcal{L}^2(P)}$ -dense in $\mathcal{H}$, we just have seen that $\mathcal{H}$ is $\| \,.\, \|_{\mathcal{L}^2(P)}$-dense in $\mathcal{L}^\infty(P)$ and, of course, $\mathcal{L}^\infty(P)$ is $\| \,.\, \|_{\mathcal{L}^2(P)}$-dense in $\mathcal{L}^2(P)$.
Hence $G_T(\Theta)$ is dense in $\left(\mathcal{L}^2(P), \| \,.\, \|_{\mathcal{L}^2(P)}\right)$.

Remark 5.4. It is easy to construct an example such that theorem 5.3 fails if we drop the assumption that $\mathcal{F}_0$ is trivial.

6. The Föllmer-Schweizer decomposition and property $R_2(P)$ for the minimal martingale measure.

Throughout this section we assume X is a continuous semimartingale with canonical decomposition

$$X = X_0 + M + A.$$

We extend some results of Schweizer (1994) and Monat/Stricker (1995) and prove that X admits a Föllmer-Schweizer decomposition if and only if the minimal martingale measure exists and satisfies $R_2(P)$.

Definitions 6.1. (i) Given a semimartingale X as above, we say that a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ admits a *Föllmer-Schweizer decomposition*, denoted by F-S decomposition in what follows, if it can be written

$$H = H_0 + (\xi \cdot X)_T + L_T \quad P\text{-a.s.} \tag{6.1}$$

where H_0 is an $\mathcal{F}_0$-measurable random variable, $\xi \in \Theta$ and $L = (L_t)_{0 \le t \le T}$ is a martingale in $\mathcal{M}_0^2$, strongly orthogonal to M.
(ii) The semimartingale X admits a *Föllmer-Schweizer decomposition* if there are unique continuous projections π_0, π_1 and $\pi_2 : \mathcal{L}^2(P) \to \mathcal{L}^2(P)$ such that every $H \in \mathcal{L}^2(P)$ admits a Föllmer-Schweizer decomposition

$$H = \pi_0(H) + \pi_1(H) + \pi_2(H) = H_0 + (\theta \cdot X)_T + L_T$$

where $H_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, P)$, $\theta \in \Theta$ and $(L_t)_{0 \le t \le T}$ is a martingale in $\mathcal{M}_0^2$, strongly orthogonal to M.

For the next definition we refer to Föllmer/Schweizer(1991).

Definition 6.2. Suppose X is a continuous semimartingale satisfying the structure condition (SC). If $(\mathcal{E}\left(-\lambda \cdot M\right)_t)_{0\leq t\leq T}$ is a martingale, then the measure Q^{min} with density $\frac{dQ}{dP} := \mathcal{E}\left(-\lambda \cdot M\right)_T$ is called the *minimal martingale measure.*

Theorem 6.3. Suppose X is a continuous semimartingale satisfying the structure condition (SC). Then X admits a Föllmer-Schweizer decomposition if and only if Q^{min} exists and satisfies $R_2(P)$.

Proof. We first prove the "only if" part.
Suppose that X admits a Föllmer-Schweizer decomposition and denote by π_0, π_1, π_2 the corresponding projections in $\mathcal{L}^2(P)$.
Let $(T_n)_{n\geq 0}$ be an increasing sequence of stopping times converging stationarily to T and such that for each $n \geq 0$, K_{T_n} is uniformly bounded. It follows from Schweizer (1994) and Monat/Stricker (1995) that for every $H \in \mathcal{L}^2(\Omega, \mathcal{F}_{T_n}, P)$ there is a Föllmer-Schweizer decomposition $H = H_0 + (\theta \cdot X)_T + L_T$ such that the following formulae are valid :
(6.2) $\quad H_0 = \pi_0(H) = E_{Q^{min}}(H \mid \mathcal{F}_0)$
(6.3) $\quad H_0 + (\theta \cdot X)_t + L_t = E_{Q^{min}}(H \mid \mathcal{F}_t)$ for $t \in [0, T]$
As by assumption, π_0 is continuous on $\mathcal{L}^2(P)$ and coincides with $E_{Q^{min}}(\cdot \mid \mathcal{F}_0)$ on each $\mathcal{L}^2(\Omega, \mathcal{F}_{T_n}, P)$ we obtain that $E_{Q^{min}}(\cdot \mid \mathcal{F}_0)$ is a continuous linear functional on $\mathcal{L}^2(\Omega, \mathcal{F}_{T_n}, P)$, whence $(Z_t^{min})_{0\leq t\leq T} := (\mathcal{E}(-\lambda \cdot M)_t)_{0\leq t\leq T}$ is a bounded martingale in $\mathcal{L}^2(P)$. Therefore the minimal martingale measure exists and formula (6.2) holds for every $H \in \mathcal{L}^2(P)$.
To show the boundedness of the projectors

$$P_t := E_{Q^{min}}(\cdot \mid \mathcal{F}_t)$$

as operators from $\mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ to $\mathcal{L}^2(\Omega, \mathcal{F}_t, P)$, write

$$P_t = P_t \circ \pi_0 + P_t \circ \pi_1 + P_t \circ \pi_2.$$

As regards $P_t \circ \pi_0 = \pi_0$ this operator clearly is uniformly bounded in t. Similarly we have according to the contraction property for P-martingales

$$\forall t \in [0, T] \;\|P_t \circ \pi_2\| \leq \|\pi_2\|$$

where $\|\cdot\|$ denotes the operator norm on $\mathcal{L}^2(\Omega, \mathcal{F}_T, P)$. Finally we claim that there is a constant $C > 0$ such that
(6.4) $\quad \|P_t \circ \pi_1\| \leq C\|\pi_1\|.$
Indeed this follows from the fact that, by the assumption of the continuity of the projection π_1, we have that $\pi_1(\mathcal{L}^2(\Omega, \mathcal{F}_T, P)) = G_T(\Theta)$ is closed in $\mathcal{L}^2(\Omega, \mathcal{F}_T, P)$. Hence

we know from proposition 3.6 that there exists a constant $C > 0$ such that for each $\theta \in \Theta$ we have

$$\|(\theta \cdot X)^*\|_{\mathcal{L}^2(P)} \leq C\|(\theta \cdot X)_T\|_{\mathcal{L}^2(P)}$$

which readily implies (6.4). This shows the uniform boundedness of the family of projections

$$P_t = E_{Q^{min}}(\cdot \mid \mathcal{F}_t).$$

This uniform boundedness is easily seen to be tantamount to condition $R_2(P)$ for the minimal density Z^{min} (see for instance Doléans-Dade/Meyer (1979) page 318).
Finally the boundedness of the operators P_t also shows that (6.3) holds true not only for $H \in \mathcal{L}^2(\Omega, \mathcal{F}_{T_n}, P)$ but for arbitrary $H \in \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$. This completes the proof of the "only if" part.

Now we prove the "if" part.
We suppose that the minimal density satisfies $R_2(P)$. In particular it is a square integrable martingale. To prove that the decomposition is unique, we can and shall assume that $H = 0$. If

$$H_0 + \int_0^T \theta_s dX_s + L_T$$

is a F-S decomposition of H, then $H_0 = 0$ because $H_0 = E_{Q^{min}}[H \mid \mathcal{F}_0]$. So

$$\int_0^T \theta_s dX_s + L_T = 0.$$

From the continuity of X, taking the bracket with L in the previous equality yields $L_T = 0$. Finally, $\theta \cdot X$ is a Q-martingale such that $(\theta \cdot X)_T = 0$, so $\theta \cdot X \equiv 0$. Since X is continuous and $\theta \cdot X$ is a P-semimartingale in $\mathcal{S}^2$, the last equality implies that $\theta = 0$ in $L^2(M)$, which completes the proof of the uniqueness.

Now let us prove that X admits a Föllmer-Schweizer decomposition. Recall that the minimal density satisfies $R_2(P)$ and is continuous, so the stochastic logarithm $\mathcal{L}(Z^{min})$ is in $BMO(P)$ by theorem 2.14, $\Theta = L^2(M)$ and $D_2(P)$ holds. Denote by $\mathcal{M}_0^{\perp}$ the space of martingales $L \in \mathcal{M}_0^2$ strongly orthogonal to M and consider the Banach space $B = \mathcal{L}^2(\Omega, \mathcal{F}_0, P) \times \Theta \times \mathcal{M}_0^{\perp}$ equipped with the norm $\|(H_0, \theta, L)\| := \|H_0\|_{\mathcal{L}^2(P)} + \|\theta\|_{L^2(M)} + \|L_T\|_{\mathcal{L}^2(P)}$. The mapping $\phi : B \to \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ defined by $\phi(H_0, \theta, L) := H_0 + (\theta \cdot X)_T + L_T$ is continuous. The uniqueness of the Föllmer-Schweizer decomposition means that ϕ is one to one. We know that $H_0 = E_{Q^{min}}(H \mid \mathcal{F}_0)$. Hence $\|H_0\|_{\mathcal{L}^2(P)} \leq \|H\|_{\mathcal{L}^2(P)}$ as Q^{min} and P coincide on $\mathcal{F}_0$. According to lemma 3.14 we have $E(\langle \theta \cdot X + L \rangle_T) \leq C\,\|(\theta \cdot X + L)_T^*\|_{\mathcal{L}^2(P)}$. Since Z^{min} satisfies $R_2(P)$ and is continuous, theorem 2.16 tells us that $\|(\theta \cdot X + L)_T^*\|_{\mathcal{L}^2(P)} \leq C\,\|(\theta \cdot X + L)_T\|_{\mathcal{L}^2(P)}$. Hence we obtain

(6.5) $$\|H_0\|_{\mathcal{L}^2(P)} + \|\theta\|_{L^2(M)} + \|L_T^*\|_{\mathcal{L}^2(P)} \leq C\|H_0 + (\theta \cdot X)_T + L_T\|$$

It follows that ϕ^{-1} defined on $\phi(B)$ is continuous and therefore $\phi(B)$ is complete. From Schweizer (1994) and Monat/Stricker (1995) we know that for every $n \geq 0$ we have that $\mathcal{L}^2(\Omega, \mathcal{F}_{T_n}, P) \subset \phi(B)$. Since $\phi(B)$ is complete, we obtain $\phi(B) = \mathcal{L}^2(\Omega, \mathcal{F}_T, P)$ and the proof of the theorem is complete.

We end this section with an example which is somewhat different in spirit than the material presented above. So far we saw results following roughly the pattern : $G_T(\Theta)$ has nice closedness properties iff the semimartingale $X = M + A$ is not too far from being a (local) martingale, i.e. A is somehow small compared to M. But this type of result only holds true if we add an assumption of type : "X admits an equivalent local martingale measure". The next example shows that some hypothesis of the latter type is indeed indispensable. We shall see that if we turn completely around and consider the case where M is small compared to A (which typically excludes the existence of an equivalent local martingale measure for X), then again $G_T(\Theta)$ may be closed.
For example, if $(\mathcal{F}_t)_{0\le t\le\infty}$ is the filtration generated by a standard Brownian motion and we simply let the process X be strictly increasing and deterministic, e.g. $X_t = \arctan(t)$, then $G_\infty(\Theta)$ equals the entire space $\mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$ and therefore is of course closed. This easily follows from the arguments given in the example below, which presents a slightly more complicated situation. Note that in the subsequent example there does not exist an equivalent martingale measure for Y and the structure condition (SC) does not hold true.

Example 6.4. Let $Y_t := W_t + t$, where $(W_t)_{0\le t<\infty}$ is a one-dimensional standard Brownian motion with natural filtration $(\mathcal{F}_t)_{0\le t<\infty}$. Now consider the predictable process ϕ defined by $\phi_t = (1+t^2)^{-1}$ and set $X := \phi \cdot Y$. Then the process X extends to a semimartingale at infinity and its natural filtration is $(\mathcal{F}_t)_{0\le t\le\infty}$ where $\mathcal{F}_\infty$ is the sigma-algebra generated by $\cup_{0\le t<\infty}\mathcal{F}_t$. We claim that $G_\infty(\Theta) = \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$. In particular every random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$ has a F-S decomposition. However this decomposition is not unique and K does not exist.
To prove that $G_\infty(\Theta) = \mathcal{L}^2(\Omega, \mathcal{F}_\infty, P)$, it will suffice to prove that there is a constant $c > 0$ such that for every $n \in \mathbb{N}$ and for every $f \in \mathcal{L}^2(\Omega, \mathcal{F}_n, P)$ there is an integrand $\theta \in \Theta$ such that
(6.6) $\quad (\theta \cdot X)_\infty = f$ and $\|\theta\|_{L^2(M)} + \|\theta\|_{L^2(A)} \le c \|f\|_{\mathcal{L}^2(P)}$.
In order to prove this inequality fix an integer n and let $(n_i)_{i\ge1}$ be a strictly increasing sequence of positive integers such that $\sum_{k=1}^{\infty}(n_1 \dots n_k)^{-1/2} < \infty$. We set $n_0 := n$, $\theta^{(0)} := f\phi^{-1}\mathbf{1}_{]\!]n,n+1]\!]}$, $g_0 := (\theta^{(0)} \cdot M)_\infty$ and for $i \ge 1$ $s_i := 1 + n_1 + \ldots + n_i$, $\theta^{(i)} := -\frac{g_{i-1}}{n_i}\phi^{-1}\mathbf{1}_{]\!]s_i,s_{i+1}]\!]}$, $g_i := (\theta^{(i)} \cdot M)_\infty$, $\theta := \sum_{i=0}^{\infty}\theta^{(i)}$.
Then $\left\|\theta^{(i)}\right\|_{L^2(A)} = \|g_{i-1}\|_{\mathcal{L}^2(P)}$, $\|\theta\|_{L^2(M)} = \|g_i\|_{\mathcal{L}^2(P)} = \frac{\|g_{i-1}\|_{\mathcal{L}^2(P)}}{\sqrt{n_i}}$. Hence

$$\|\theta\|_{L^2(M)} + \|\theta\|_{L^2(A)} \le 2\|f\|_{\mathcal{L}^2(P)} \sum_{k=0}^{\infty}(n_1 \dots n_k)^{-1/2}.$$

Thus inequality (6.6) is proved and the proof of the example is now complete.

7. Conclusion.

This paper gives necessary and sufficient conditions on a discounted asset price X for the subspace of attainable claims to be closed in the space $\mathcal{L}^2(P)$ of square-integrable random variables. This closedness is important for applications in financial mathematics since it allows the construction of mean-variance optimal hedging strategies for arbitrary square-integrable contingent claims. Mathematically, our results involve weighted norm inequalities, and the condition on X (apart from continuity) is that the variance-optimal local martingale measure for X should be equivalent to the original measure and satisfy the reverse Hölder inequality with exponent 2. Our techniques also allow us to extend existing results on the Föllmer-Schweizer decomposition, and this can in turn be used for the construction of locally risk-minimizing hedging strategies.

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