DIFFERENTIABILITY PROPERTIES OF UTILITY FUNCTIONS

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Abstract. We investigate differentiability properties of monetary utility functions. At the same time we give a counter-example – important in finance – to automatic continuity for concave functions.

Notation and Preliminaries

We use standard notation. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes an atomless probability space. In practice this is not a restriction since the property of being atomless is equivalent to the fact that on $(\Omega, \mathcal{F}, \mathbb{P})$, we can define a random variable that has a continuous distribution function. We use the usual – unacceptable – convention to identify a random variable with its class (defined by a.s. equality). By L^p we denote the standard spaces, i.e. for $0 < p < \infty$, $X \in L^p$ if and only if $\mathbb{E}[|X|^p] < \infty$. L^0 stands for the space of all random variables and L^∞ is the space of bounded random variables. The topological dual of L^∞ is denoted by **ba**, the space of bounded finitely additive measures μ defined on $\mathcal{F}$ with the property that $\mathbb{P}[A] = 0$ implies $\mu(A) = 0$. The subset of normalised non-negative finitely additive measures – the so called finitely additive probability measures – is denoted by $\mathcal{P}^{\mathbf{ba}}$. The subset of countably additive elements of $\mathcal{P}^{\mathbf{ba}}$ (a subset of L^1) is denoted by $\mathcal{P}$.

Definition. *A function $u\colon L^\infty \to \mathbb{R}$ is called a monetary utility function if*

1. *$\xi \in L^\infty$ and $\xi \geq 0$ implies $u(\xi) \geq 0$,*
2. *u is concave,*
3. *for $a \in \mathbb{R}$ and $\xi \in L^\infty$ we have $u(\xi + a) = u(\xi) + a$.*

If u moreover satisfies $u(\lambda\xi) = \lambda u(\xi)$ for $\lambda \geq 0$ (positive homogeniety), we say that u is coherent.

Remark. It is not difficult to prove that monetary concave utility functions satisfy a monotonicity property: $\xi \leq \eta$ implies $u(\xi) \leq u(\eta)$. Consequently we get that $|u(\xi) - u(\eta)| \leq \|\xi - \eta\|_\infty$ and $|u(\xi)| \leq \|\xi\|_\infty$. With a concave monetary utility function u we associate the convex set $\mathcal{A} = \{\xi \mid u(\xi) \geq 0\}$. This set is necessarily closed for the norm topology of L^∞.

Following Föllmer-Schied, [FS], we can characterise a concave monetary utility function by its Fenchel-Legendre transform. This transform, denoted by c, is defined on **ba** but outside $\mathcal{P}^{\mathbf{ba}}$ it takes the value $+\infty$. The function $c : \mathcal{P}^{\mathbf{ba}} \to \mathbb{R}_+ \cup \{+\infty\}$

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is defined as $c(\mu) = \sup\{\mu(-\xi) \mid \xi \in \mathcal{A}\} = \sup\{u(\xi) - \mu(\xi) \mid \xi \in L^\infty\}$. The function c satisfies

(1) c is convex
(2) it is lower semi-continuous for the weak* topology on $\mathcal{P}^{\mathbf{ba}}$
(3) $\inf_\mu c(\mu) = \min_\mu c(\mu) = 0$.

In case u is coherent we have that c is the indicator of a weak*−closed convex set $\mathcal{S}^{\mathbf{ba}} \subset \mathcal{P}^{\mathbf{ba}}$, this means $c(\mu) = 0$ for $\mu \in \mathcal{S}^{\mathbf{ba}}$ and is $+\infty$ elsewhere. In most applications the utility function will satisfy the following continuity property:

Definition. *The concave monetary utility function u satisfies the Fatou property if for uniformly bounded sequences ξ_n of L^∞ the convergence in probability of ξ_n to ξ implies $u(\xi) \geq \limsup u(\xi_n)$.*

Remark. Using the Krein-Smulian theorem one can see that the Fatou property is equivalent to $\mathcal{A}$ being weak*-closed. In this case we get that (see [R] for details on convex analysis):

(1) $u(\xi) = \inf\{\mathbb{E}_\mathbb{Q}[\xi] + c(\mathbb{Q}) \mid \mathbb{Q} \in \mathcal{P}\}$,
(2) the set $\{(\mu, t) \mid t \geq c(\mu); \mu \in \mathcal{P}^{\mathbf{ba}}\}, t \in \mathbb{R}$ is the weak* closure (in $\mathbf{ba} \times \mathbb{R}$) of the set $\{(\mathbb{Q}, t) \mid t \geq c(\mathbb{Q}); \mathbb{Q} \in \mathcal{P}, t \in \mathbb{R}\}$,
(3) the previous property can be written as: for each $\mu \in \mathcal{P}^{\mathbf{ba}}$ there is a generalised sequence or net $(\mathbb{Q}_\alpha)_\alpha$ in $\mathcal{P}$ so that $\mathbb{Q}_\alpha$ converges weak* to μ and $c(\mu) = \lim_\alpha c(\mathbb{Q}_\alpha)$. This property is even equivalent to the Fatou property.
(4) If u is coherent and satisfies the Fatou property, then the set $\mathcal{S} = \mathcal{S}^{\mathbf{ba}} \cap L^1$ satisfies $u(\xi) = \inf\{\mathbb{E}_\mathbb{Q}[\xi] \mid \mathbb{Q} \in \mathcal{S}\}$ and $\mathcal{S}$ is weak* dense in $\mathcal{S}^{\mathbf{ba}}$. This property is even equivalent to the Fatou property.

The aim of this paper is to clarify some issues on the differentiability of monetary concave utility functions u. The differentiability of utility functions is related to equilibrium prices and plays a big role in economic theory. The Gateaux differentiability of utility functions was used in the paper of Gerber and Deprez, [DG]. They pointed out – without referring to any topology on the underlying space – that premium calculation principles can be derived from such utility functions and they also gave examples. This is in line with general micro-economic principles. Although Gateaux differentiability can be defined without any reference to a topology, the topological properties of the underlying space cannot be avoided. This will become clear when we give some examples. For the moment let us remark that the points where a continuous concave function defined on a *separable* Banach space, is diferentiable, form a dense G_δ set. This famous theorem (due to Mazur, [Ph]) does not generalise to general spaces. Especially for L^∞ we will give natural examples of coherent utility functions (with the Fatou property) that are nowhere differentiable.

The study of the differentiability of a monetary utility function can be restricted to the point 0. Indeed if g_0 is another point, then the differentiability at g_0 is the same problem as for the point $g_0 - u(g_0)$. This already means that we may suppose that $u(g_0) = 0$. So let us introduce the new monetary utility function defined as $u_0(\xi) = u(\xi + g_0)$, the corresponding penalty function is then given by $c_0(\mu) = c(\mu) + \mu(g_0)$. From convex analysis, see[R] and [Ph], we learn that a concave monetary utility function is Gateaux differentiable at ξ if and only if there is exactly one element $\mu \in \mathcal{P}^{\mathbf{ba}}$ such that $u(\xi) = \mu(\xi) + c(\mu)$. This unique element is then the Gateaux derivative. If u is supposed to be coherent, we must have that

μ is an extreme (even exposed) point in the set $\mathcal{S}^{\mathbf{ba}}$. For $\xi = 0$ this means that there is exactly one element μ such that $c(\mu) = 0$. For coherent utility functions we immediately get that u can only be Gateaux differentiable at 0 if $\mathcal{S}^{\mathbf{ba}}$ is reduced to one point, i.e. u is linear.

Some of the proofs below use a trick, called homogenisation. This allows the concave utility function to be replaced by a coherent one, on the cost of enlarging the space Ω. We will sketch how this works, leaving most of the elementary details to the reader. The trick is probably not new, it is certainly not very deep but it has some "didactical" values.

We replace the set Ω by the set $\Omega_1 = \Omega \cup \{p\}$ where p is an element not in Ω, e.g. $p = \{\Omega\}$. The sigma algebra, $\mathcal{F}_1$, on Ω_1 is generated by the sets $A \in \mathcal{F}$ and the set $\Omega \subset \Omega_1$. The probability $\mathbb{P}$ is replaced by the probability $\mathbb{P}_1$ defined as $\mathbb{P}_1[A_1] = \frac{1}{2}\mathbb{P}[A_1 \cap \Omega] + \frac{1}{2}\mathbf{1}_{A_1}(p)$. Probabilities on Ω_1 are convex combinations of probabilities of Ω and the Dirac measure Δ_p concentrated in the extra point p. The utility function defined on $L^\infty(\Omega_1) = L^\infty(\Omega) \times \mathbb{R}$, is defined via the acceptance cone $\mathcal{A}_1$. The latter is defined as the norm closed cone generated by the elements of the form $(f, 1)$ where $f \in \mathcal{A}$. An element of the cone $\mathcal{A}_1$ is either of the form (tf, t) with $t > 0$ and $f \in \mathcal{A}$ or of the form $(f, 0)$ where f is in the recession cone (asymptotic cone) of $\mathcal{A}$. This cone is defined as $\cap_{t>0}\, t\,\mathcal{A}$. The utility function u_1 defined on $L^\infty(\Omega_1)$ does not have an easy expression as a function of u. But we have $u(f) \geq 0$ if and only if $u_1(f, 1) \geq 0$ and $u(f) = 0$ if and only if $u_1(f, 1) = 0$. Important is to note that when $\mathcal{A}$ is weak* closed then also $\mathcal{A}_1$ is weak* closed. This follows from the expression for the recession cone. So u_1 is Fatou when u is. Since we will only apply this trick for Fatou utility functions, we will suppose that u and hence u_1, has the Fatou property. The scenario set $\mathcal{S}_1$ for u_1, a subset of $L^1(\Omega_1)$, is represented as $(\alpha\mathbb{Q}, (1-\alpha)\Delta_p)$ where

(1) $0 \leq \alpha \leq 1$
(2) $\mathbb{Q} \in \mathcal{P}$
(3) for all $\xi \in \mathcal{A}$ and all $t > 0$, we have $\alpha\mathbb{E}_\mathbb{Q}[\xi]t + (1-\alpha)t \geq 0$. This is equivalent to $\alpha \leq \frac{1}{1+c(\mathbb{Q})}$. If $c(\mathbb{Q}) = \infty$ this means that the measure becomes Δ_p.

Another representation of the set $\mathcal{S}_1$ is through the Radon-Nikodym derivative with respect to $\mathbb{P}_1$. We find that

$$\mathcal{S}_1 = \left\{ (f, \beta) \mid 0 \leq f \in L^1(\Omega), \frac{2c(\mathbb{Q})}{c(\mathbb{Q}) + 1} \geq \beta \geq 0, \mathbb{E}[f] + \beta = 2 \right\},$$

where the measure $\mathbb{Q}$ is defined as $d\mathbb{Q} = \frac{f}{\mathbb{E}[f]}\, d\mathbb{P}$ for $\mathbb{E}[f] > 0$. If $f = 0$ we simply ask that $\beta = 2$ since in this case we get the Dirac measure Δ_p.

The Jouini-Schachermayer-Touzi Theorem

This section is devoted to the characterisation of a weak compactness theorem. The theorem is a generalisation of the beautiful result of James' on weakly compact sets, see [Di]. The original proof of [JST] followed the rather complicated proof of James. The proof below uses the homogenisation trick and allows to apply the original version of James' theorem. Let us recall this theorem

Theorem. *Let E be a Banach space and let C be a convex closed subset of E. A necessary and sufficient condition for C to be weakly compact in E is that every continuous linear function $e^* \in E^*$ attains its maximum on C.*

In one direction the statement is trivial: if C is weakly compact then every continous linear functional attains its maximum on C. From a topological viewpoint the converse is surprising. First a topological space for which every real-valued continuous function attains its maximum need not be compact, second we only need linear functions, a class that is not dense in the space of continous functions defined on C.

In the following theorem we use the same notation as in the homogenisation.

Theorem (Jouini-Schachermayer-Touzi,[JST]**).** *If u is a concave monetary utility function satisfying the Fatou property then are equivalent:*

(1) *For each $\xi \in L^\infty$ there is a $\mathbb{Q} \in \mathcal{P}$ such that $u(\xi) = \mathbb{E}_\mathbb{Q}[\xi] + c(\mathbb{Q})$.*
(2) *If $(\xi, t) \in L^\infty(\Omega_1)$ there is a $\mathbb{Q}_1 \in \mathcal{S}_1$ such that $u_1(\xi, t) = \int_{\Omega_1} (\xi, t)\, d\mathbb{Q}_1$.*
(3) *The set $\mathcal{S}_1$ is weakly compact in $L^1(\Omega_1)$.*
(4) *The homogenisation u_1 satisfies the Lebesgue property. This means that for uniformly bounded sequences $(\theta_n)_n$ in $L^\infty(\Omega_1)$, converging in probability to say θ, we have $u_1(\theta_n) \to u_1(\theta)$.*
(5) *If ξ_n is a uniformly bounded sequence in L^∞ converging in probability to a function ξ, then $u(\xi_n) \to u(\xi)$, i.e. u has the Lebesgue property.*
(6) *For each $k \in \mathbb{R}$ the set $\{\mathbb{Q} \mid c(\mathbb{Q}) \le k\}$ is weakly compact (or uniformly integrable and closed) in L^1, in particular $c(\mu) = +\infty$ for non countably additive elements of $\mathcal{P}^{\mathbf{ba}}$.*

In the sequel we will say that such utility functions satisfy the weak compactness property. For coherent utility functions with the Fatou property, the property means $\mathcal{S}^{\mathbf{ba}} = \mathcal{S}$ is a weakly compact convex set in L^1.

Remark. A simple reasoning shows that item 6 is implied by: "There is $k > 0$ such that $\{\mathbb{Q} \mid c(\mathbb{Q}) \le k\}$ is weakly compact. "

Proof. The proof is divided into different steps $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6 \Rightarrow 1$. We start with $1 \Rightarrow 2$. Let us consider an element (ξ, t) in $L^\infty(\Omega_1)$. We have to find an element of $\mathcal{S}_1$ so that the minimum is attained. It does not matter if we replace (ξ, t) by an element of the form $(\xi + \eta, t + \eta)$ where $\eta \in \mathbb{R}$. So we may suppose that $u_1(\xi, t) = 0$. There are two possibilities. If $t = 0$ this means that ξ is in the recession cone and hence $\mathbb{E}_\mathbb{Q}[\xi] \ge 0$ whenever $c(\mathbb{Q}) < \infty$. In this case we may take the measure Δ_p to realise the minimum. The other case is when $t > 0$. In this case we may multiply by $1/t$ to get an element $(\xi, 1)$ with $u_1(\xi, 1) = 0$. This implies $u(\xi) = 0$. By hypothesis there is an element $\mathbb{Q}$ such that $\mathbb{E}_\mathbb{Q}[\xi] + c(\mathbb{Q}) = 0$. The element $\mathbb{Q}_1 = \frac{1}{1+c(\mathbb{Q})}\mathbb{Q} + \frac{c(\mathbb{Q})}{1+c(\mathbb{Q})}\Delta_p$ then gives $\mathbb{E}_{\mathbb{Q}_1}[(\xi, 1)] = 0$. This is then a minimum since all other elements of $\mathcal{S}_1$ will give a larger expected value. The implication $2 \Rightarrow 3$ is the famous James' theorem. The implication $3 \Rightarrow 4$ is standard since weakly compact sets are uniformly integrable. The implication $4 \Rightarrow 5$ is easy. Indeed, we can take a bounded sequence ξ_n converging in probability to ξ. We may suppose (substract $u(\xi)$ if necessary, that $u(\xi) = 0$. Then we use that $(\xi_n, 1)$ converges in probability $(\mathbb{P}_1)$ to $(\xi, 1)$. This gives $u_1(\xi_n, 1)$ will tend to zero. But as seen above this means that $u(\xi_n)$ tends to zero as well. For the implication $5 \Rightarrow 6$ we only need to show that if A_n is a decreeasing sequence of sets with $\mathbb{P}[A_n] \to 0$, then $\sup\{\mathbb{Q}[A_n] \mid c(\mathbb{Q}) \le k\}$ tends to zero. Let us put $\xi_n = -\alpha \mathbf{1}_{A_n}$ where $\alpha > 0$. Let us apply item 5. We get that $u(\xi_n)$ tends to zero, or $\inf_\mathbb{Q} (\mathbb{E}_\mathbb{Q}[-\alpha \mathbf{1}_{A_n}] + c(\mathbb{Q}))$ tends to zero. This implies that $\liminf_{n\to\infty} \inf_{c(\mathbb{Q})\le k} (\mathbb{E}_\mathbb{Q}[-\alpha \mathbf{1}_{A_n}] + c(\mathbb{Q})) \ge 0$. Hence we have $\liminf_{n\to\infty} \inf_{c(\mathbb{Q})\le k} (\mathbb{E}_\mathbb{Q}[-\alpha \mathbf{1}_{A_n}] + k) \ge 0$. This is the same as

$\limsup_n \sup_{c(\mathbb{Q})\le k} \mathbb{Q}[A_n] \le k/\alpha$. Since α can be taken arbitrary large, we get that $\lim_n \sup_{c(\mathbb{Q})\le k} \mathbb{Q}[A_n] = 0$. The implication $6 \Rightarrow 1$ is standard. Let us take $\xi \in L^\infty$. In the calculation of $u(\xi)$ we only need to take $\mathbb{Q}$ with $c(\mathbb{Q}) \le 2\|\xi\|_\infty$. Since this set is weakly compact and since c is lower semi-continuous, we will find an element realising the infimum.

A Consequence of Ekeland's variational principle and other family members of Bishop-Phelps

Differentiability properties of convex functions are well studied. We will use [Ph] as a basic reference. In our context there are two functions that are important: the utility function u and the penalty function c. The function u is defined on the whole space whereas the function c is only defined on a (subset of) $\mathcal{P}$ or $\mathcal{P}^{\mathbf{ba}}$. The subgradient of u at a point ξ is defined as the set of elements $\nu \in \mathbf{ba}$ such that $u(\eta) \le u(\xi) + \nu(\eta - \xi)$. It is also equal to the set

$$\partial u(\xi) = \{\mu \in \mathcal{P}^{\mathbf{ba}} \mid u(\xi) = \mu(\xi) + c(\mu)\}.$$

At the same time we find that the subgradient of c at a point $\mathbb{Q} \in \mathcal{P}$ is the set

$$\{\xi \in L^\infty \mid u(\xi) = \mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q})\}.$$

The Borwein-Rockafellar theorem (see [Ph]) now shows that the set

$$\{\xi \mid \text{ there is } \mathbb{Q} \in \mathcal{P} \text{ with } u(\xi) = \mathbb{E}_{\mathbb{Q}}[\xi] + c(\mathbb{Q})\}$$

is norm dense in L^∞. The function u is Gateaux differentiable at ξ if the set $\partial u(\xi)$ is a singleton.

A Consequence of Automatic Continuity

Theorem. *Suppose that u is a monetary concave utility function satisfying the Fatou property. If u is Gateaux differentiable at g, then its derivative is an element of $\mathcal{P}$, i.e. it is countably additive.*

Proof. Since u is Fatou, it is a Borel function for the weak* topology on L^∞. Suppose that $\mu \in \mathcal{P}^{\mathbf{ba}}$ is the derivative of u at the point g, then we have that μ is the pointwise limit of the *sequence*

$$\mu(f) = \lim_n n\left(u\left(g + \frac{1}{n}f\right) - u(g)\right).$$

The linear function μ is then a Borel measurable function for the weak* topology and hence, by the automatic continuity theorem, [Chr], it must be weak* continuous, i.e. induced by an element of L^1.

Corollary. *Under the extra hypothesis that u is also coherent, we have that the derivative $\mathbb{Q}$ is an extreme (even exposed) point of $\mathcal{S}^{\mathbf{ba}}$, but already lying in $\mathcal{S}$.*

The one-sided derivative

Because of concavity, monetary concave utility functions have a one-sided derivative at a point $g \in L^\infty$, defined as

$$\varphi_g(f) = \lim_{\epsilon \downarrow 0} \frac{u(g + \epsilon f) - u(g)}{\epsilon}.$$

If $g = 0$ we get

$$\varphi(f) = \lim_{\epsilon \downarrow 0} \frac{u(\epsilon f)}{\epsilon}.$$

Proposition. *The function φ is the smallest coherent utility function ψ such that $\psi \geq u$.*

Proof. This is easy since for each $\epsilon > 0$ we have $\psi(f) = \psi(\epsilon f)/\epsilon \geq u(\epsilon f)/\epsilon$. Taking limits gives $\psi(f) \geq \varphi(f)$.

The acceptance cone for φ is easily obtained.

Proposition. *The acceptance cone of φ, $\mathcal{A}_\varphi$, is given by the $\|.\|_\infty$ closure of the union $\cup_n n\mathcal{A}$ (where as before $\mathcal{A} = \{f \mid u(f) \geq 0\}$).*

Proof. Suppose first that $f \in n\mathcal{A}$, then for $\epsilon \leq 1/n$ we have by convexity of $\mathcal{A}$ that $u(\epsilon f) \geq 0$. This shows that $\varphi(f) \geq 0$. It follows that $\cup_n n\mathcal{A} \subset \mathcal{A}_\varphi$. Since the latter set is norm closed we have that it also must contain the norm-closure of this union. If $\varphi(f) > 0$, we have that for ϵ small enough $u(\epsilon f) > 0$ and hence for n big enough we have $f \in n\mathcal{A}$. This shows the opposite inclusion.

The scenario set that defines the coherent utility function φ is given by the following theorem

Theorem. *With the notation introduced above we have*

$$\varphi(f) = \inf_{\mu \in \mathcal{S}^{\mathbf{ba}}} \mu(f),$$

where the set $\mathcal{S}^{\mathbf{ba}}$ is defined as $\mathcal{S}^{\mathbf{ba}} = \{\mu \in \mathcal{P}^{\mathbf{ba}} \mid c(\mu) = 0\}$.

Proof. Because of the previous proposition we have that $\mu \in \mathcal{S}^{\mathbf{ba}}$ if and only if $\mu(f) \geq 0$ for all $f \in \mathcal{A}$. This is equivalent to saying that $c(\mu) = 0$.

Corollary. *The one-sided derivative φ of u at 0 is Fatou if and only if $\{\mathbb{Q} \in \mathcal{P} \mid c(\mathbb{Q}) = 0\}$ is weak* dense in $\{\mu \mid c(\mu) = 0\}$.*

The previous corollary allows for easy constructions of non-Fatou coherent utility functions.

For the derivative at a point g we use the transformation $u_g(f) = u(g+f) - u(g)$. It follows that the derivative at a point g is given by

$$\varphi_g(f) = \inf\{\mu(f) \mid c(\mu) + \mu(g) = u(g)\}.$$

An Example

Let us fix a countable partition of Ω into a sequence of measurable sets A_n with $\mathbb{P}[A_n] > 0$. For $\mu \in \mathcal{P}^{\mathbf{ba}}$ we define

$$c(\mu) = \sum_n \mu[A_n]^2.$$

Proposition. *The function c is convex, takes values in [0,1],* $\min_{\mu \in \mathcal{P}^{\mathbf{ba}}} c(\mu) = 0$ *and is lower semi-continuous for the weak* topology on* $\mathcal{P}^{\mathbf{ba}}$. *The utility function u defined by c is Fatou.*

Proof. The first four statements are obvious since the mapping $\mu \to \mu[A_n]^2$ is convex and weak* continuous. c is therefore the increasing limit of a sequence of continuous convex functions and hence is lower semi-continuous and convex. The existence of elements in $\mathcal{P}^{\mathbf{ba}}$ so that for all n, $\mu(A_n) = 0$ is well known and can be proved using the Hahn-Banach theorem. Of course we have for $\mu \in \mathcal{P}^{\mathbf{ba}}$: $\sum_n \mu(A_n)^2 \le \sum_n \mu(A_n) \le 1$. The Fatou property is less trivial. As seen before we must show that for $\mu \in \mathcal{P}^{\mathbf{ba}}$ we can find a generalised sequence or net $\mathbb{Q}_\alpha$ in $\mathcal{P}$ so that $c(\mathbb{Q}_\alpha)$ tends to $c(\mu)$. For this it is sufficient to show the following. Given μ, given $\epsilon > 0$ and given a finite partition of Ω in non-zero sets $B_1, \dots, B_N$ we must find $\mathbb{Q} \in \mathcal{P}$ so that $c(\mathbb{Q}) \le c(\mu) + \epsilon$ and $\mathbb{Q}(B_j) = \mu(B_j)$ for $j = 1, \dots, N$. For a set B_j there are two possibilities: either there is s with $B_j \subset \cup_{n=1}^s A_n$ or there are infinitely many indices n with $\mathbb{P}[B_j \cap A_n] > 0$. Since all the sets A_n have a non-zero measure and since the family $(B_j)_j$ forms a partition of Ω the last alternative must occur for at least one index j. So let us renumber the sets B_j and let us select s so that

(1) for $j \le N' \le N$ there are infinitely many indices with $\mathbb{P}[A_n \cap B_j] > 0$,
(2) for $N' < j \le N$ (if any) we have that $B_j \subset \cup_{n=1}^s A_n$.

Fix now an integer $L \ge 1$ so that $1/L \le \epsilon$. We will define the measure $\mathbb{Q}$ by its Radon-Nikodym density. For $j \le N'$ we find indices as follows, we take L indices $s < n_1^1 < n_2^1 \dots < n_L^1$ so that $\mathbb{P}[A_{n_k^1} \cap B_1] > 0$. We then take indices $n_L^1 < n_1^2 < n_2^2 < \dots < n_L^2$ with $\mathbb{P}[A_{n_k^2} \cap B_2] > 0$ and so on. We can now define the density of $\mathbb{Q}$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \sum_{j=1}^N \sum_{k=1}^s \frac{\mu(B_j \cap A_k)}{\mathbb{P}[B_j \cap A_k]} \mathbf{1}_{B_j \cap A_k} + \sum_{j=1}^N \sum_{p=1}^L \frac{\mu(B_j \cap (\cup_{n>s} A_n))}{L \ \mathbb{P}[B_j \cap A_{n_p^j}]} \mathbf{1}_{B_j \cap A_{n_p^j}}.$$

The reader can convince himself that there is no reason to drop the terms with denominator zero. For all $j \le N$ we have that $\mathbb{Q}[B_j] = \mu(B_j)$. Furthermore we have that for $n \le s : \mathbb{Q}[A_n] = \mu(A_n)$. For indices $n > s$ there is at most one of the N sets $B_j \cap A_n$ that is chosen. So we get for $n > s$:

$$\mathbb{Q}[A_{n_p^j}] = \frac{1}{L} \mu(B_i \cap (\cup_{n>s} A_n)) \text{ and for other indices } n \text{ we get } = 0.$$

Finally we find

$$\begin{aligned} c(\mathbb{Q}) &= \sum_n \mathbb{Q}[A_n]^2 = \sum_{n\le s} \mathbb{Q}[A_n]^2 + \sum_{n>s} \mathbb{Q}[A_n]^2 \\ &= \sum_{n\le s} \mu(A_n)^2 + \sum_{n>s} \mathbb{Q}[A_n]^2 \\ &\le c(\mu) + \sum_{j=1}^N \sum_{p=1}^L \frac{1}{L^2}\mu(B_j \cap (\cup_{n>s} A_n))^2 \\ &\le c(\mu) + \frac{1}{L}\sum_{j=1}^N \mu(B_j)^2 \\ &\le c(\mu) + \epsilon. \end{aligned}$$

Corollary. *The scenario set $\mathcal{S}^{\mathbf{ba}}$ for φ consists of purely finitely additive measures $\mathcal{S}^{\mathbf{ba}} = \{\mu \in \mathcal{P}^{\mathbf{ba}} \mid$ for all n : $\mu(A_n) = 0\}$. Consequently the coherent utility function φ is not Fatou.*

Remark. The coherent utility function φ is weak* Borel measurable since it is the limit of a sequence of Borel measurable functions. The acceptance cone $\mathcal{A}_\varphi$ is norm closed, is Borel measurable but is not weak* closed. With some little effort one can show that $\mathcal{A}_\varphi$ is a weak* $F_{\sigma\delta}$. This shows – the already known fact – that the automatic continuity theorems do not apply to concave or convex functions.

Proposition. *For all s we have $u(\mathbf{1}_{\cup_{n=1}^s A_n}) = 0$ also for each $2 \ge \epsilon \ge 0$ we have $u(-\mathbf{1}_{A_n}) = -\frac{\epsilon^2}{4}$. For $\epsilon \ge 2$ we have $u(-\epsilon\mathbf{1}_{A_n}) = -\epsilon + 1$.*

Proof. The first statement is easy since it is sufficient to take an element $\mu \in \mathcal{P}^{\mathbf{ba}}$ with $\mu(A_n) = 0$ for all n. For the other equality, we may without loss of generality suppose that $n = 1$. Let us write

$$u(-\epsilon\mathbf{1}_{A_1}) \le \mu(-\epsilon\mathbf{1}_{A_1}) + \sum_n \mu(A_n)^2 = \mu(-\epsilon\mathbf{1}_{A_1}) + \mu(A_1)^2 + \sum_{n\ge 2}\mu(A_n)^2,$$

and observe that the minimum is taken for elements μ satisfying $\mu(A_1) = \epsilon/2$ and $\mu(A_k) = 0$ for $k \ge 2$. For $\epsilon \ge 2$ we find that $u(-\epsilon\mathbf{1}_{A_1}) = -\epsilon + 1$ and the minimum is taken for elements μ with $\mu(A_1) = 1$.

Remark. The last line of the proof shows that for some points g, the set $\{\mathbb{Q} \mid \mathbb{Q} \in \mathcal{P}, \mathbb{E}_{\mathbb{Q}}[g] + c(\mathbb{Q}) = u(g)\}$ is weak* dense in $\{\mu \mid \mu \in \mathcal{P}^{\mathbf{ba}}, \mu(g) + c(\mu) = u(g)\}$. Indeed this is the case for $\alpha\mathbf{1}_{A_n}$ with $\alpha \le -2$. This means that the one-sided derivative at these points g is Fatou.

Proposition. *For $f \in L^\infty$ we have*

$$\varphi(f) = \liminf_n \operatorname{ess.inf}(\mathbf{1}_{A_n} f) = \liminf_n \operatorname{ess.inf}(\mathbf{1}_{\cup_{m\ge n} A_m} f).$$

Proof. The proof follows from the description of $\mathcal{S}^{\mathbf{ba}}$. We leave the details to the reader.

Remark. The previous corollary shows that the function φ is related to the function $\liminf$ on the space l^∞. This function is also Borel measurable for the weak* topology. Its set of representing mesures is the convex weak* closed hull of the Banach limits. None of these representing measures is Borel measurable.

The example of an incomplete financial market

In this section we take a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$. We suppose that the filtration is continuous meaning that all martingales are continuous, equivalently that all stopping times are predictable. On this space we consider a $d-$dimensional *continuous* price process $S : [0,1] \times \Omega \to \mathbb{R}^d$. We assume that the price process S satisfies the $NFLVR$ property (see [DS]). More precisely and to make the notation easier we will suppose that the process S is a bounded martingale for $\mathbb{P}$. The market generated by S is supposed to be incomplete which in this case means that the set $\mathbb{M}^a = \{\mathbb{Q} \ll \mathbb{P} \mid S \text{ ia a } \mathbb{Q} - \text{martingale}\}$ is strictly bigger than $\{\mathbb{P}\}$. From [D1] we know that the set $\mathbb{M}^a$ is then a set without extreme points. The utility function u is defined as the bid price

$$u(\xi) = \inf\{\mathbb{E}_{\mathbb{Q}}[\xi] \mid \mathbb{Q} \in \mathbb{M}^a\}.$$

This is a Fatou coherent utility function. If u were Gateaux differentiable at a point ξ, then by the automatic continuity theorem we have that its derivative would be an extreme point of $\mathbb{M}^a$. Since this set has no extreme points we get

Theorem. *The function u is nowhere Gateaux differentiable.*

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