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Representation of the penalty term of dynamic concave utilities

Freddy Delbaen · Shige Peng · Emanuela Rosazza Gianin

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Abstract In the context of a Brownian filtration and with a fixed finite time horizon, we provide a representation of the penalty term of general dynamic concave utilities (hence of dynamic convex risk measures) by applying the theory of g-expectations.

Keywords Dynamic concave utilities · Dynamic convex risk measures · Penalty functions · g-expectations · Backward stochastic differential equations

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F. Delbaen
Department of Mathematics, ETH, Zürich, Switzerland
e-mail: delbaen@math.ethz.ch

S. Peng
Institute of Mathematics, Shandong University, Jinan, China
e-mail: peng@sdu.edu.cn

E. Rosazza Gianin (✉)
Dipartimento di Metodi Quantitativi per le Scienze Economiche ed Aziendali, Università di Milano-Bicocca, Building U7, 4th floor, Via Bicocca degli Arcimboldi 8, 20126 Milan, Italy
e-mail: emanuela.rosazza1@unimib.it

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1 Introduction

Coherent risk measures were introduced by Artzner et al. [2] in finite sample spaces and later by Delbaen [16] and [17] in general probability spaces. The aim of this financial tool is to quantify the intertemporal riskiness which an investor would face at a maturity date T in order to decide if this risk could be acceptable for him or not. The family of coherent risk measures was extended later by Föllmer and Schied [24, 25] and Frittelli and Rosazza Gianin [26, 27] to the class of convex risk measures.

g-expectations were introduced by Peng [34] as solutions of a class of nonlinear backward stochastic differential equations (BSDEs for short), a class which was first studied by Pardoux and Peng [33]. Financial applications and particular cases were discussed in detail by El Karoui et al. [22].

As shown by Rosazza Gianin [39], the families of static risk measures and of g-expectations are not disjoint. Indeed, under suitable hypotheses on the functional g, g-expectations provide examples of coherent and/or convex static risk measures. Furthermore, by defining a "dynamic risk measure" as a "map" which quantifies, at any intermediate time t, the riskiness which will be faced at maturity T, a class of dynamic risk measures can be obtained by means of conditional g-expectations. In particular, any dynamic risk measure induced by a conditional g-expectation satisfies a "time-consistency property" (in line with the notion introduced by Koopmans [31] and Duffie and Epstein [21]) or, in the language of Artzner et al. [3], a "recursivity property." Further discussions on dynamic risk measures and on risk measures for processes can be found in Artzner et al. [3], Barrieu and El Karoui [5], Bion-Nadal [6, 7], Cheridito et al. [11, 12], Cheridito and Kupper [14], Detlefsen and Scandolo [20], Frittelli and Rosazza Gianin [27], and Klöppel and Schweizer [30], among many others.

The main aim of this paper is to represent the penalty term of general dynamic concave utilities (hence of dynamic convex risk measures) in the context of a Brownian filtration, a fixed finite time horizon T, and under the assumption of the existence of an equivalent probability measure with zero penalty. By applying the theory of g-expectations, we shall prove that the penalty term is of the form

$$c_{s,t}(Q) = E_Q\left[\int_s^t f(u, q_u)\,du \,\middle|\, \mathcal{F}_s\right]$$

(see the exact statement in Theorem 3.2).

The paper is organised as follows. Some well-known results on BSDEs and on risk measures are recalled in Sect. 2. Section 3 contains the main result of the paper, that is, the representation of the penalty term of suitable dynamic concave utilities. As we shall see later, this representation will be obtained by applying the theory of g-expectations.

2 Notation and preliminaries

Let $(B_t)_{t\geq 0}$ be a standard d-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, and let $\{\mathcal{F}_t\}_{t\geq 0}$ the augmented filtration generated by $(B_t)_{t\geq 0}$.

In the sequel, we identify a probability measure $Q \ll P$ with its Radon–Nikodým density $\frac{dQ}{dP}$. Furthermore, because of the choice of the Brownian setting, we also identify a probability measure Q equivalent to P with the predictable process $(q_t)_{t\in[0,T]}$ induced by the stochastic exponential, i.e., such that

$$E_P\left[\frac{dQ}{dP}\middle|\mathcal{F}_t\right] = \mathcal{E}(q.B)_t := \exp\left(-\frac{1}{2}\int_0^t \|q_s\|^2\,ds + \int_0^t q_s\,dB_s\right)$$

(see Proposition VIII.1.6 of Revuz and Yor [36]).

Consider now a function

$$g : \mathbb{R}^+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},$$
$$(t, \omega, y, z) \mapsto g(t, \omega, y, z)$$

satisfying at least the following assumptions (as in Coquet et al. [15], but without imposing a priori a time horizon T). To simplify the notation, we often write $g(t, y, z)$ instead of $g(t, \omega, y, z)$.

Basic assumptions on g:

(A) g is Lipschitz in (y, z), i.e., there exists a constant $\mu > 0$ such that we have, $(dt \times dP)$-a.s., for any $(y_0, z_0), (y_1, z_1) \in \mathbb{R} \times \mathbb{R}^d$,

$$\left|g(t, y_0, z_0) - g(t, y_1, z_1)\right| \le \mu\left(|y_0 - y_1| + \|z_0 - z_1\|\right).$$

(B) For all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\cdot, y, z)$ is a predictable process such that for any finite $T > 0$, we have $E[\int_0^T (g(t, \omega, y, z))^2\,dt] < +\infty$ for any $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$.

(C) $(dt \times dP)$-a.s., $\forall y \in \mathbb{R}$, $g(t, y, 0) = 0$.

Once the time horizon $T > 0$ is fixed, Pardoux and Peng [33] introduced the backward stochastic differential equation (BSDE, for short)

$$\begin{cases} -dY_t = g(t, Y_t, Z_t)\,dt - Z_t\,dB_t, \\ Y_T = \xi, \end{cases}$$

where ξ is a random variable in $L^2(\Omega, \mathcal{F}_T, P)$. Moreover, they showed (see also El Karoui et al. [22]) that there exists a unique solution $(Y_t, Z_t)_{t\in[0,T]}$ consisting of predictable stochastic processes (the former $\mathbb{R}$-valued, the latter $\mathbb{R}^d$-valued) such that $E[\int_0^T Y_t^2\,dt] < +\infty$ and $E[\int_0^T \|Z_t\|^2\,dt] < +\infty$.

Peng [34] defined the g-expectation of ξ as

$$\mathcal{E}_g(\xi) := Y_0$$

and the conditional g-expectation of ξ at time t as

$$\mathcal{E}_g(\xi|\mathcal{F}_t) := Y_t.$$

When $g(t, y, z) = \mu\|z\|$ (with $\mu > 0$), $\mathcal{E}_g$ will be denoted by $\mathcal{E}^\mu$.

In the sequel, we shall only consider essentially bounded random variables ξ, i.e., $\xi \in L^\infty(\Omega, \mathcal{F}_T, P)$.

Further assumptions on g:

(1_g) g does not depend on y.
(2_g) g is convex in z: $\forall \alpha \in [0,1], \forall z_0, z_1 \in \mathbb{R}^d$, $(dt \times dP)$-a.s.,
$g(t, \alpha z_0 + (1-\alpha) z_1) \leq \alpha g(t, z_0) + (1-\alpha) g(t, z_1)$.

In the sequel, we shall write "g *with the usual assumptions*" when g satisfies hypotheses (A)–(C) and (1_g)–(2_g).

Some sufficient conditions for a functional to be induced by a g-expectation have been provided by Coquet et al. [15]. Before recalling this result, we introduce what is needed.

Definition 2.1 (Coquet et al. [15]) A functional $\mathcal{E} : L^2(\mathcal{F}_T) \to \mathbb{R}$ is called an $\mathcal{F}$*-consistent expectation* if it satisfies the following properties:

(i) *constancy*: $\mathcal{E}(c) = c$ for any $c \in \mathbb{R}$.
(ii) *strict monotonicity*: if $\xi \geq \eta$, then $\mathcal{E}(\xi) \geq \mathcal{E}(\eta)$. Moreover, if $\xi \geq \eta$, then $\xi = \eta$ if and only if $\mathcal{E}(\xi) = \mathcal{E}(\eta)$.
(iii) *consistency*: for any $\xi \in L^2(\mathcal{F}_T)$ and $t \in [0,T]$, there exists a random variable $\mathcal{E}(\xi|\mathcal{F}_t) \in L^2(\mathcal{F}_t)$ such that for any $A \in \mathcal{F}_t$, it holds

$$\mathcal{E}(\xi 1_A) = \mathcal{E}\big(\mathcal{E}(\xi|\mathcal{F}_t) 1_A\big).$$

Again in the terminology of [15], $\mathcal{E}$ is said to satisfy *translation invariance* (or to be *monetary*) if for any $t \in [0,T]$,

$$\mathcal{E}(\xi + \eta|\mathcal{F}_t) = \mathcal{E}(\xi|\mathcal{F}_t) + \eta, \quad \forall \xi \in L^2(\mathcal{F}_T), \eta \in L^2(\mathcal{F}_t),$$

while it is said to be $\mathcal{E}^\mu$-dominated (for some $\mu > 0$) if

$$\mathcal{E}(\xi + \eta) - \mathcal{E}(\xi) \leq \mathcal{E}^\mu(\eta), \quad \forall \xi, \eta \in L^2(\mathcal{F}_T).$$

Theorem 2.2 (Coquet et al. [15], Theorem 7.1) *Let* $\mathcal{E}$ *be an* $\mathcal{F}$*-consistent expectation. If* $\mathcal{E}$ *satisfies translation invariance and if it is dominated by some* $\mathcal{E}^\mu$ *with* $\mu > 0$, *then it is induced by a conditional* g*-expectation, that is, there exists a function* g *satisfying* (A)–(C) *and* (1_g) *such that for any* $t \in [0,T]$,

$$\mathcal{E}(\xi|\mathcal{F}_t) = \mathcal{E}_g(\xi|\mathcal{F}_t), \quad \forall \xi \in L^2(\mathcal{F}_T).$$

Some relevant extensions of such a result can be found in Peng [35] and in Hu et al. [28], while some applications to risk measures can be found in Rosazza Gianin [39]. The last author, in particular, showed that g-expectations (respectively, conditional g-expectations) provide static (respectively, dynamic) risk measures. More precisely, the following result holds true. For definitions, representations, and details on (static) risk measures, interested readers can see Artzner et al. [2], Delbaen [16, 17], Föllmer and Schied [24, 25], and Frittelli and Rosazza Gianin [26], among many others.

Proposition 2.3 (Rosazza Gianin [39], Proposition 11) *If g satisfies the usual assumptions (including convexity in z), then the risk measure ρ_g defined as*

$$\rho_g(X) := \mathcal{E}_g(-X)$$

is a convex risk measure satisfying monotonicity, constancy, and translation invariance.

Moreover, if g also satisfies positive homogeneity in z, then ρ_g is coherent.

In view of the result above, some sufficient conditions for a risk measure to be induced by a g-expectation have been found in [39] as an application of Theorem 2.2.

Note that, at least in the sublinear case and under some suitable assumptions, one can prove a one-to-one correspondence between the functional g and the m-stable set of generalized scenarios $\mathcal{S}$ of a suitable risk measure. Hence, one may find (as an application of the results of Delbaen [18] on m-stable sets) a one-to-one correspondence between time-consistent coherent risk measures and conditional g-expectations. See also Chen and Epstein [9].

In the sequel, we prefer to work with concave utilities instead of convex risk measures. Note that, given a risk measure ρ, the associated monetary utility functional (or, shortly, utility) is defined as $u := -\rho$.

3 Representation of the penalty term of dynamic concave utilities

In the sequel, we still work in a Brownian setting, and hence $\mathcal{F}_0$ is trivial. Let T be a fixed finite time horizon. Given two stopping times σ and τ such that $0 \le \sigma \le \tau \le T$, consider a *concave monetary utility* functional $u_{\sigma,\tau} : L^\infty(\mathcal{F}_\tau) \to L^\infty(\mathcal{F}_\sigma)$, i.e., a functional satisfying

(a) monotonicity: if $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$ and $\xi \le \eta$, then $u_{\sigma,\tau}(\xi) \le u_{\sigma,\tau}(\eta)$.
(b) translation invariance: $u_{\sigma,\tau}(\xi + \eta) = u_{\sigma,\tau}(\xi) + \eta$ for any $\xi \in L^\infty(\mathcal{F}_\tau)$ and $\eta \in L^\infty(\mathcal{F}_\sigma)$.
(c) concavity: $u_{\sigma,\tau}(\alpha\xi + (1-\alpha)\eta) \ge \alpha u_{\sigma,\tau}(\xi) + (1-\alpha)u_{\sigma,\tau}(\eta)$ for any $\alpha \in [0, 1]$ and $\xi, \eta \in L^\infty(\mathcal{F}_\tau)$.
(d) $u_{\sigma,\tau}(0) = 0$.

The family $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ is called a *dynamic concave utility*. In particular, we have $u_{0,T} : L^\infty(\mathcal{F}_T) \to \mathbb{R}$. The acceptance set $\mathcal{A}_{\sigma,\tau}$ induced by $u_{\sigma,\tau}$ is defined as $\mathcal{A}_{\sigma,\tau} := \{\xi \in L^\infty(\mathcal{F}_\tau) : u_{\sigma,\tau}(\xi) \ge 0\}$. To simplify the notation, we often write u_t instead of $u_{t,T}$.

On $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ we impose the following:

Assumption (e): $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ is continuous from above (or satisfies the Fatou property), i.e., for any decreasing sequence $(\xi_n)_{n\in\mathbb{N}}$ in $L^\infty(\mathcal{F}_\tau)$ such that $\lim_n \xi_n = \xi$, it holds true that $\lim_n u_{\sigma,\tau}(\xi_n) = u_{\sigma,\tau}(\xi)$.

Assumption (f): $(u_{\sigma,\tau})_{\sigma,\tau}$ is time-consistent, i.e., for all stopping times σ, τ, υ with $0 \le \sigma \le \tau \le \upsilon \le T$,

$$u_{\sigma,\upsilon}(\xi) = u_{\sigma,\tau}\big(u_{\tau,\upsilon}(\xi)\big), \quad \forall \xi \in L^\infty(\mathcal{F}_\upsilon).$$

Assumption (g): $(u_{\sigma,\tau})_{\sigma,\tau}$ satisfies

$$u_{\sigma,\tau}(\xi 1_A + \eta 1_{A^c}) = u_{\sigma,\tau}(\xi)1_A + u_{\sigma,\tau}(\eta)1_{A^c}, \quad \forall \xi, \eta \in L^\infty(\mathcal{F}_\tau), \forall A \in \mathcal{F}_\sigma. \quad (3.1)$$

Assumption (h): $c_t(P) := \text{ess.sup}_{\xi \in L^\infty(\mathcal{F}_T)}\{E_P[-\xi|\mathcal{F}_t] + u_t(\xi)\} = 0$ for any $t \in [0, T]$.

It is straightforward to check that the last condition is equivalent to $E_P[\xi|\mathcal{F}_t] \geq 0$ for any $\xi \in \mathcal{A}_{t,T}$. Furthermore, $c_0(P) = 0$ can be replaced by the hypothesis that there is a probability measure Q equivalent to P satisfying $c_0(Q) = 0$.

Note that up to a sign, dynamic concave utilities satisfying the assumptions above correspond to normalized time-consistent dynamic risk measures $(\rho_{\sigma,\tau})_{0\leq\sigma\leq\tau\leq T}$ studied, for instance, in Bion-Nadal [7] in a general setting. More precisely, it holds $u_{\sigma,\tau} = -\rho_{\sigma,\tau}$.

By Bion-Nadal [7] and Detlefsen and Scandolo [20], it is known that under the assumptions above and in the setting of a general filtration,

$$\begin{aligned} u_{s,t}(\xi) &= \underset{Q\sim P, Q=P \text{ on } \mathcal{F}_s}{\text{ess.inf}} \big\{E_Q[\xi|\mathcal{F}_s] + c_{s,t}(Q)\big\} \\ &= \underset{Q\in\mathcal{P}_{s,t}}{\text{ess.inf}}\big\{E_Q[\xi|\mathcal{F}_s] + c_{s,t}(Q)\big\} \end{aligned} \quad (3.2)$$

for any $0 \leq s \leq t \leq T$, where

$$c_{s,t}(Q) = \underset{\xi\in L^\infty(\mathcal{F}_t)}{\text{ess.sup}} \big\{E_Q[-\xi|\mathcal{F}_s] + u_{s,t}(\xi)\big\}$$

is the *minimal* penalty term associated to $u_{s,t}$, and

$$\mathcal{P}_{s,t} = \big\{Q \text{ on } (\Omega, \mathcal{F}_t) : Q \sim P,\, Q = P \text{ on } \mathcal{F}_s\big\}.$$

In particular,

$$\begin{aligned} u_t(\xi) &= \underset{Q\sim P, Q=P \text{ on } \mathcal{F}_t}{\text{ess.inf}} \big\{E_Q[\xi|\mathcal{F}_t] + c_{t,T}(Q)\big\}, \\ u_0(\xi) &= \inf_{Q\sim P}\big\{E_Q[\xi] + c_{0,T}(Q)\big\}, \end{aligned}$$

where $c_t(Q) := c_{t,T}(Q) = \text{ess.sup}_{\xi\in\mathcal{A}_t} E_Q[-\xi|\mathcal{F}_t] \geq 0$, and $\mathcal{A}_t = \mathcal{A}_{t,T}$ denotes the acceptance set induced by $u_t = u_{t,T}$. Note that

$$c_{0,T}(Q) = \sup_{\xi\in L^\infty(\mathcal{F}_T)} \big\{E_Q[-\xi] + u_{0,T}(\xi)\big\},$$

hence $c_{0,T}$ is lower semi-continuous and is the Fenchel–Legendre transform of $u_{0,T}$.

Furthermore, Bion-Nadal (see Theorem 3 in [7]) proved that $(\rho_{t,T})_{t\in[0,T]}$ (and hence $(u_{t,T})_{t\in[0,T]}$) admits a càdlàg modification. We shall prove in the Appendix that the same is true for $(c_{t,T})_{t\in[0,T]}$.

Note that in [7] and [20] the representation (3.2) was shown with $Q \ll P$ instead of $Q \sim P$. However, assumption (h) guarantees that the representation (3.2) also holds true (for a proof, see Klöppel and Schweizer [30] and, in discrete time, Cheridito et al. [13] and Föllmer and Penner [23]).

Remark 3.1 It is evident that if $(u_t)_{t\geq 0}$ is time-consistent if $u_t(0)=0$ and if it satisfies condition (3.1), then

$$u_0(\xi 1_A) = u_0\big(u_t(\xi 1_A)\big) = u_0\big(u_t(\xi) 1_A\big)$$

for any $t \in [0,T]$, $\xi \in L^\infty(\mathcal{F}_T)$, and $A \in \mathcal{F}_t$. It is therefore clear that if $(u_{\sigma,\tau})_{\sigma,\tau}$ is time-consistent, then everything is defined by u_0. The relevance of time-consistency of the dynamic concave utility is also underlined by the following results. On one hand, as shown by Delbaen [18] and Cheridito et al. [13], time-consistency is equivalent to the *decomposition property* of acceptable sets, that is,

$$\mathcal{A}_{\sigma,\upsilon} = \mathcal{A}_{\sigma,\tau} + \mathcal{A}_{\tau,\upsilon}$$

for all stopping times σ, τ, υ such that $0 \leq \sigma \leq \tau \leq \upsilon \leq T$. On the other hand, both the properties above are equivalent to the *cocycle property* of the minimal penalty term c, that is,

$$c_{\sigma,\upsilon}(Q) = c_{\sigma,\tau}(Q) + E_Q\big[c_{\tau,\upsilon}(Q)|\mathcal{F}_\sigma\big]$$

for all stopping times σ, τ, υ such that $0 \leq \sigma \leq \tau \leq \upsilon \leq T$ (see Bion-Nadal [7] for the definition and the proof).

In the sequel, we use the terminology of Rockafellar [37, 38] on convex functions. Our aim is now to prove the following result.

Theorem 3.2 *Let $(u_{\sigma,\tau})_{0\leq\sigma\leq\tau\leq T}$ be a dynamic concave utility satisfying assumptions* (a)–(h).

(i) *For all stopping times σ, τ such that $0 \leq \sigma \leq \tau \leq T$ and for any probability measure Q equivalent to P,*

$$c_{\sigma,\tau}(Q) = E_Q\left[\int_\sigma^\tau f(u,q_u)\,du \,\middle|\, \mathcal{F}_\sigma\right]$$

for some suitable function $f : [0,T] \times \Omega \times \mathbb{R}^d \to [0,+\infty]$ such that $f(t,\omega,\cdot)$ is proper, convex, and lower semi-continuous.

(ii) *For all stopping times σ, τ such that $0 \leq \sigma \leq \tau \leq T$ and $\xi \in L^\infty(\mathcal{F}_T)$, the dynamic concave utility in* (3.2) *can be represented as*

$$u_{\sigma,\tau}(\xi) = \operatorname*{ess.inf}_{Q\in\mathcal{P}_{\sigma,\tau}} E_Q\left[\xi + \int_\sigma^\tau f(u,q_u)\,du \,\middle|\, \mathcal{F}_\sigma\right].$$

Remark 3.3 For dynamic concave utilities satisfying assumptions (e), (g), (h), it follows from Theorem 1 of Bion-Nadal [7] that Theorem 3.2(i) is equivalent to time-consistency (assumption (f)) of $(u_{\sigma,\tau})_{0\leq\sigma\leq\tau\leq T}$.

Remark 3.4 In an incomplete market, the lower price $\inf_{Q\in\mathcal{M}} E_Q[\xi]$ (where $\mathcal{M}$ denotes the set of all risk-neutral probability measures) defines a utility satisfying all our properties, but it is not given by a g-expectation. See Delbaen [18] for details about how to get f.

The proof of Theorem 3.2 will be decomposed into several steps as outlined below. Set

$$u^n_{s,t}(\xi) = \operatorname*{ess.inf}_{Q\sim P, \|q\|\le n} \big\{E_Q[\xi|\mathcal{F}_s] + c_{s,t}(Q)\big\}. \tag{3.3}$$

Note that by definition of u^n and by assumption (h), for any $\xi \in L^\infty(\mathcal{F}_T)$, it holds $u^0_t(\xi) = E_P[\xi|\mathcal{F}_t]$ and $u^n_t(\xi) \le E_P[\xi|\mathcal{F}_t]$.

Remark 3.5 The reason why the truncated utility u^n has been defined as above is due to the fact that the set $\{Q \sim P : \|q\| \le n\}$ is weakly compact. This property will be useful in the proof of Proposition 3.6.

Proposition 3.6 *Suppose that the dynamic concave utility* $(u_{\sigma,\tau})_{0\le\sigma\le\tau\le T}$ *satisfies assumptions* (a)–(h). *Then*:

(i) u^n *is a dynamic concave utility satisfying assumptions* (e)–(g). *Moreover, the acceptance sets induced by* u^n *satisfy the decomposition property and*

$$c^n_{s,t}(Q) = \begin{cases} c_{s,t}(Q) & \text{if } \|q\| \le n, \\ +\infty & \text{otherwise} \end{cases}$$

satisfies the cocycle property and $c^n_{s,t}(P) = 0$.

(ii) u^n *is induced by a conditional* g_n*-expectation, i.e.*,

$$u^n_t(\xi) = -\mathcal{E}_{g_n}(-\xi|\mathcal{F}_t)$$

for some convex function g_n *satisfying the usual assumptions and such that* $g_n(\cdot,\cdot,z)$ *is predictable for any* $z \in \mathbb{R}^d$. *In other words,* u^n *satisfies the BSDE*

$$\begin{cases} du^n_t(\xi) = g_n(t, Z^n_t)\,dt - Z^n_t\,dB_t, \\ u^n_T(\xi) = \xi. \end{cases} \tag{3.4}$$

(iii) *For any probability measure* $Q \sim P$ *such that* $\|q\| \le n$, *it holds for any* $0 \le s \le t \le T$ *that*

$$c^n_{0,t}(Q) = E_Q\left[\int_0^t f_n(u, q_u)\,du\right],$$

$$c^n_{s,t}(Q) = E_Q\left[\int_s^t f_n(u, q_u)\,du \,\middle|\, \mathcal{F}_s\right],$$

where $f_n : [0,T] \times \Omega \times \mathbb{R}^d \to [0,+\infty]$ *is induced* (*by duality*) *by* g_n, *and* $f_n(t,\omega,\cdot)$ *is proper, convex, and lower semi-continuous.*

(iv) *The sequence of convex functions* g_n *is increasing in* n.

(v) *The sequence of* f_n *is decreasing in n and, for any* $n \ge 0$, $f_n(t,\omega,q) = +\infty$ *for* $\|q\| > n$. *Furthermore, once* (t,ω) *is fixed, for any* q, *either there exists* $n \ge 0$ *such that*

$$f_n(t,\omega,q) = f_m(t,\omega,q) = f(t,\omega,q) < +\infty, \quad m \ge n,$$

or for all $n \geq 0$,

$$f_n(t, \omega, q) = +\infty = f(t, \omega, q)$$

for some function $f : [0, T] \times \Omega \times \mathbb{R}^d \to [0, +\infty]$. Hence the function f is given by $f(t, \omega, x) = \inf_n f_n(t, \omega, x)$, and $f(t, \omega, \cdot)$ is proper, convex, and lower semi-continuous.

Remark 3.7 Notice that c^n is indeed the minimal penalty term associated to u^n since it is lower semi-continuous, convex, and such that $\inf_Q c^n(Q) = 0$.

Proof of Proposition 3.6 (i) From the representation (3.3) it follows that u^n is a dynamic concave utility which is continuous from above (see Detlefsen and Scandolo [20] and Klöppel and Schweizer [30]). Still from (3.3) one deduces that $u^n_{\sigma,\tau}(\xi 1_A) = u^n_{\sigma,\tau}(\xi) 1_A$ for any $\xi \in L^\infty(\mathcal{F}_T)$, $0 \leq \sigma \leq \tau \leq T$, and $A \in \mathcal{F}_\sigma$. Hence, by Proposition 2.9 of Detlefsen and Scandolo [20], also assumption (g) is satisfied.

The cocycle property of c^n is guaranteed by

$$c^n_{s,t}(Q) = \begin{cases} c_{s,t}(Q) & \text{if } \|q\| \leq n, \\ +\infty & \text{otherwise.} \end{cases}$$

Since for the probability measure P it holds $q^P \equiv 0$, $c^n_{s,t}(P) = c_{s,t}(P) = 0$. The time-consistency of u^n follows from the stability of the set $\{Q \sim P : \|q\| \leq n\}$ (in the sense of [6] and [18]), from the properties satisfied by c, and from Theorem 4.4 of Bion-Nadal [6]. The decomposition property of acceptance sets is due to Theorem 4.6 of Cheridito et al. [13] and, later, to Theorem 1 of Bion-Nadal [7].

(ii) Set $\pi^n_t(\xi) := -u^n_t(-\xi) = \text{ess.sup}_{Q \sim P, \|q\| \leq n}\{E_Q[\xi|\mathcal{F}_t] - c_t(Q)\}$. From (i), $(\pi^n_{\sigma,\tau})_{0 \leq \sigma \leq \tau \leq T}$ is time-consistent. Furthermore, it is easy to check that it satisfies monotonicity, translation invariance, and constancy (this last follows from the assumption $c_t(P) = 0$). Moreover, π^n_0 satisfies strict monotonicity. This property follows from the weak compactness of the set $\{Q \sim P : \|q\| \leq n\}$ (see Remark 3.5). In order to verify strict monotonicity, consider $\eta \geq \xi$ such that $P(\eta > \xi) > 0$. Since $\pi^n_0(\xi) = E_Q[\xi] - c_0(Q)$ for some $Q \sim P$ such that $\|q\| \leq n$, we have

$$\pi^n_0(\eta) \geq E_Q[\eta] - c_0(Q) > E_Q[\xi] - c_0(Q) = \pi^n_0(\xi).$$

Finally, we show that π^n_0 is dominated by some $\mathcal{E}^\mu$. For any $\xi, \eta \in L^\infty(\mathcal{F}_T)$,

$$\begin{aligned} &\pi^n_0(\xi + \eta) - \pi^n_0(\xi) \\ &= \sup_{Q:\|q\| \leq n} \left\{E_Q[\xi + \eta] - c_0(Q)\right\} - \sup_{Q:\|q\| \leq n} \left\{E_Q[\xi] - c_0(Q)\right\} \\ &\leq \sup_{Q:\|q\| \leq n} E_Q[\eta] = \mathcal{E}^n(\eta). \end{aligned}$$

The last equality follows from Lemma 3 of Chen and Peng [10] ($\mathbb{R}$ case), which may be extended to $\mathbb{R}^d$. By the arguments above and Remark 3.1, $(\pi^n_t)_{t \geq 0}$ satisfies the hypothesis of Theorem 2.2. Hence there exists a functional $g_n : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfying assumptions (A)–(C) and (1_g) and such that $\pi^n_t(\xi) = \mathcal{E}_{g_n}(\xi|\mathcal{F}_t)$.

It can be checked that $g_n(\cdot,\cdot,z)$ is predictable for any $z \in \mathbb{R}^d$ (see also Theorem 3.1 of Peng [35]). Furthermore, since π_t^n is a convex functional, by Theorem 3.2 of Jiang [29] it follows that $g_n(t,\omega,\cdot)$ has to be convex. Hence,

$$u_t^n(\xi) = -\mathcal{E}_{g_n}(-\xi|\mathcal{F}_t),$$
$$u_0^n(\xi) = -\mathcal{E}_{g_n}(-\xi)$$

for some function g_n satisfying the usual assumptions. It is therefore immediate to check that u^n satisfies the BSDE in (3.4). Moreover, for almost all (t,ω), it holds that the set $\{z \in \mathbb{R}^d : g_n(t,\omega,z) \leq \alpha\}$ is closed for any $\alpha \in \mathbb{R}$. The closedness of such a set (or, equivalently, the lower semi-continuity of $g_n(t,\omega,\cdot)$) is due to the fact that g_n is Lipschitz with constant n (see the arguments above and Theorem 2.2). Hence $g_n(t,\omega,\cdot)$ is convex, proper, and lower semi-continuous.

(iii) Set now

$$f_n(t,\omega,q) := \sup_{z\in\mathbb{R}^d} \{q\cdot z - g_n(t,\omega,z)\}. \tag{3.5}$$

Note that $f_n(t,\omega,q) \geq 0$ (take, for instance, $z=0$ in the definition of f_n) and, because of the assumption $c_0(P)=0$, $f_n(t,0)=0$. Since $g_n(t,\omega,z)$ is predictable by (ii),

$$f_n(t,\omega,q) = \sup_{z\in\mathbb{R}^d} \{q\cdot z - g_n(t,\omega,z)\} = \sup_{z\in\mathbb{Q}^d} \{q\cdot z - g_n(t,\omega,z)\}$$

is predictable for any $q \in \mathbb{R}^d$ as supremum of countably many predictable elements. Note that $\|q\| > n$ implies $f_n(t,\omega,q) = +\infty$ by (3.5). Since $g_n(t,\omega,\cdot)$ is convex, proper, and lower semi-continuous and $f_n(t,\omega,\cdot)$ is the convex conjugate of $g_n(t,\omega,\cdot)$, i.e., $f_n(q) = g_n^*(q)$, also f_n is convex, proper, and lower semi-continuous; see Rockafellar [37].

As a consequence of the dual representation of a g-expectation in Theorem 7.4 of Barrieu and El Karoui [5], we get

$$c_{0,T}^n(Q) = E_Q\left[\int_0^T f_n(u,q_u)\,du\right]$$

for any probability measure $Q \sim P$ such that $\|q\| \leq n$. It remains to show that $c_{s,t}^n(Q) = E_Q[\int_s^t f_n(u,q_u)\,du|\mathcal{F}_s]$ for any $0 \leq s \leq t \leq T$ and for any probability measure $Q \sim P$ such that $\|q\| \leq n$. Also this result can be deduced by Theorem 7.4 of Barrieu and El Karoui [5]. Nevertheless, since the proof will be useful later, we postpone it to Lemma 3.8.

(iv) It is easy to check that the sequences u_0^n and c_0^n are decreasing in $n \in \mathbb{N}$. By applying the converse comparison theorem on BSDEs (see Briand et al. [8]) and Lemma 2.1 of Jiang [29], we shall show that the sequence of convex functions g_n (which induce u^n) is increasing in n.

In order to prove the above assertion, we proceed in a similar way as in Jiang [29]. By the definition of u^n, $u_{0,T}^n(\xi) \geq u_{0,T}^{n+1}(\xi)$ and $u_{s,T}^n(\xi) \geq u_{s,T}^{n+1}(\xi)$ for any

$\xi \in L^\infty(\mathcal{F}_T)$. By (ii), we deduce therefore that, for any $\xi \in L^\infty(\mathcal{F}_T)$,

$$\begin{aligned} \mathcal{E}_{g_n}(\xi) &\le \mathcal{E}_{g_{n+1}}(\xi), \\ \mathcal{E}_{g_n}(\xi|\mathcal{F}_s) &\le \mathcal{E}_{g_{n+1}}(\xi|\mathcal{F}_s). \end{aligned} \tag{3.6}$$

Denote now by $\mathcal{E}_g^{s,t}$ the conditional g-expectation at time s with final time t. To apply Lemma 2.1 of Jiang [29], we need to verify that

$$\mathcal{E}_{g_n}^{s,t}(\xi) \le \mathcal{E}_{g_{n+1}}^{s,t}(\xi), \quad \forall s,t \in [0,T] \text{ with } s \le t, \forall \xi \in L^\infty(\mathcal{F}_t). \tag{3.7}$$

Condition (3.7) has already been established for $(s,t) = (0,T)$ and $(s,t) = (s,T)$. Consider now the case $(s,t) = (0,t)$. Since $\mathcal{E}_g^{s,t}(\eta) = \mathcal{E}_g^{s,T}(\eta)$ for any $\eta \in L^\infty(\mathcal{F}_t)$ (see Peng [34] for details), from (3.6) we deduce that $\mathcal{E}_{g_n}^{0,t}(\xi) \le \mathcal{E}_{g_{n+1}}^{0,t}(\xi)$ for any $0 \le t \le T$ and $\xi \in L^\infty(\mathcal{F}_t)$. For general (s,t), inequality (3.6) can be checked as above. Indeed, for any $\xi \in L^\infty(\mathcal{F}_t)$, we have $\mathcal{E}_{g_n}^{s,t}(\xi) = \mathcal{E}_{g_n}^{s,T}(\xi) \le \mathcal{E}_{g_{n+1}}^{s,T}(\xi) = \mathcal{E}_{g_{n+1}}^{s,t}(\xi)$.

Set now

$$\mathcal{S}^z(g) := \left\{ t \in [0,T) : g(t,z) = L^1\text{-}\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \mathcal{E}_g^{t,t+\varepsilon}\big(z(B_{t+\varepsilon} - B_t)\big) \right\}.$$

From Lemma 2.1 of Jiang [29] it follows that

$$m\big([0,T) \setminus \mathcal{S}^z(g_i)\big) = 0, \quad \forall z \in \mathbb{R}^d,$$

for $i = n, n+1$, where m denotes the Lebesgue measure on $[0,T]$. By the arguments above it follows that, for any $z \in \mathbb{R}^d$,

$$\text{if } t \in \mathcal{S}^z(g_n) \cap \mathcal{S}^z(g_{n+1}) \neq \emptyset, \quad \text{then } g_n(t,z) \le g_{n+1}(t,z) \quad P\text{-a.s.}$$

and

$$m\big([0,T) \setminus \big(\mathcal{S}^z(g_n) \cap \mathcal{S}^z(g_{n+1})\big)\big) = m\big(\big([0,T) \setminus \mathcal{S}^z(g_n)\big) \cup \big([0,T) \setminus \mathcal{S}^z(g_{n+1})\big)\big) = 0.$$

Hence, by proceeding as in Jiang [29] it can be checked that, for any $z \in \mathbb{R}^d$,

$$g_n(t,z) \le g_{n+1}(t,z) \quad (dt \times dP)\text{-a.s.}$$

The positivity of any g_n is due to the fact that $u_t^0(\xi) = E_P[\xi|\mathcal{F}_t] = -\mathcal{E}_{g_0}(-\xi|\mathcal{F}_t)$ where $g_0 \equiv 0$. By the same arguments as above, therefore, $g_n \ge g_0 \equiv 0$.

(v) From (iii) and (iv) it follows that the sequence of f_n is decreasing in n. Consider again the measurable space $([0,T] \times \Omega, \mathcal{P}, m \times P)$, where $\mathcal{P}$ denotes the predictable σ-algebra, and m denotes Lebesgue measure on $[0,T]$. Denote by $\overline{\mathcal{P}}$ the completion of $\mathcal{P}$. Take $N > 0$ and, for any $\varepsilon > 0$, set

$$\begin{aligned} E &= E_{N,\varepsilon} \\ &:= \big\{(t,\omega,q) \in [0,T] \times \Omega \times \mathbb{R}^d : \|q\| \le n,\, f_{n+1}(t,\omega,q) + \varepsilon < f_n(t,\omega,q) \le N\big\}, \end{aligned}$$

and denote by $\pi(E)$ its projection on $[0,T]\times\Omega$. Note that $E\in\overline{\mathcal{P}}\otimes\mathcal{B}(\mathbb{R}^d)$. By the measurable selection theorem (see Aumann [4] and Aliprantis and Border [1], Sect. 17.4), $\pi(E)\in\overline{\mathcal{P}}$, and there exists a $\overline{\mathcal{P}}$-measurable $\overline{q}:\pi(E)\to\mathbb{R}^d$ such that $(t,\omega,\overline{q}(t,\omega))\in E$ for $(m\times P)$-a.e. $(t,\omega)\in\pi(E)$. Set now $\overline{q}=0$ on $\pi(E)^c$. To such a $\overline{q}$, it is therefore possible to associate a $q:[0,T]\times\Omega\to\mathbb{R}^d$ which is $\mathcal{P}$-measurable and equal to $\overline{q}$ $(m\times P)$-almost everywhere.

Let Q be the probability measure associated to q as above. By definition, $\|q\|\leq n$. Hence, $c^n_{0,T}(Q)=c^{n+1}_{0,T}(Q)=c_{0,T}(Q)<+\infty$. Furthermore, by the definition of E it follows that

$$\begin{aligned}
c^n_{0,T}(Q)&=E_Q\left[\int_0^T f_n(u,q_u)\,du\right]\\
&=E_Q\left[\int_0^T f_n(u,q_u)1_{\pi(E)}\,du\right]+E_Q\left[\int_0^T f_n(u,q_u)1_{\pi(E)^c}\,du\right]\\
&=E_Q\left[\int_0^T f_n(u,q_u)1_{\pi(E)}\,du\right]\\
&\geq E_Q\left[\int_0^T \left[f_{n+1}(u,q_u)+\varepsilon\right]1_{\pi(E)}\,du\right]\\
&=E_Q\left[\int_0^T f_{n+1}(u,q_u)1_{\pi(E)}\,du\right]+\varepsilon(m\times Q)\big(\pi(E)\big)\\
&=c^{n+1}_{0,T}(Q)+\varepsilon(m\times Q)\big(\pi(E)\big).
\end{aligned}$$

If $(m\times Q)(\pi(E))>0$, then $c^{n+1}_{0,T}(Q)+\tilde{\varepsilon}<c^n_{0,T}(Q)<+\infty$, which is a contradiction. Hence, $(m\times Q)(\pi(E))=0$, i.e.,

$$(m\times Q)\big(\big\{(t,\omega):N\geq f_n(t,\omega,q_t)>f_{n+1}(t,\omega,q_t)+\varepsilon\big\}\big)=0.$$

By letting N tend to $+\infty$, from the arguments above and since $Q\sim P$, it follows that if $f_n<+\infty$ on $\{x:\|x\|\leq n\}$, then $f_n=f_{n+1}$ $(m\times dP)$-a.s., and hence $f_n=f_{n+1}=f$ $(m\times dP)$-a.s. for some functional f. In other words,

$$f_n(t,\omega,x)=f_{n+1}(t,\omega,x)=f(t,\omega,x)\quad (m\times dP)\text{-a.s. for } \|x\|\leq n.$$

Furthermore, we may conclude that, once (t,ω) is fixed, for any q, either (1) there exists $n\geq 0$ such that $f_n(t,\omega,q)<+\infty$ (hence $f_m(t,\omega,q)=f(t,\omega,q)<+\infty$ for any $m\geq n$ and $m\geq\|q\|$) or (2) for all $n\geq 0$, it holds $f_n(t,\omega,q)=+\infty=f(t,\omega,q)$. Hence,

$$f(t,\omega,x)=\inf_{n\geq 0} f_n(t,\omega,x).$$

By the properties of the sequence f_n, it follows that $f(\cdot,\cdot,0)=0$.

It remains to prove that $f(t,\omega,\cdot)$ is proper, convex, and lower semi-continuous. The properness of $f(t,\omega,\cdot)$ is trivial. As $f(t,\omega,x)=\lim_n f_n(t,\omega,x)=\inf_n f_n(t,\omega,x)$ for almost all (t,ω) and any f_n is predictable and convex in x, it is easy to check that f also is predictable and convex in x. Furthermore, for almost all (t,ω), the set $\{q\in\mathbb{R}^d: f(t,\omega,q)\leq\alpha\}$ is closed for any $\alpha\in\mathbb{R}$. Indeed, take a sequence $\{q^k\}_{k\geq 0}$

such that $q^k \to q$ and $f(t,\omega,q^k) \le \alpha$. There exists $N \in \mathbb{N}$ such that $\|q^k\| \le N$ for all k. Hence $f(t,\omega,q^k) = f_N(t,\omega,q^k) \le \alpha$. Since $f_N(t,\omega,\cdot)$ is lower semi-continuous,

$$f(t,\omega,q) = f_N(t,\omega,q) \le \underline{\lim}_k f_N\big(t,\omega,q^k\big) \le \alpha.$$

Hence $f(t,\omega,\cdot)$ also is lower semi-continuous. □

Lemma 3.8 *If $c^n_{0,T}(Q) = E_Q[\int_0^T f_n(u,q_u)\,du]$ holds for any probability measure $Q \sim P$ such that $\|q\| \le n$, then $c^n_{s,t}(Q) = E_Q[\int_s^t f_n(u,q_u)\,du|\mathcal{F}_s]$ also holds for any $0 \le s \le t \le T$ and for any probability measure $Q \sim P$ such that $\|q\| \le n$.*

Proof Let Q be a probability measure equivalent to P and such that $\|q\| \le n$. Consider the case where $s = 0$ and take the probability measure $\overline{Q}$ corresponding to $\overline{q}$ given by

$$\overline{q}_u = \begin{cases} q_u & \text{if } 0 \le u \le t, \\ 0 & \text{if } t < u \le T, \end{cases}$$

obtained by pasting Q and P. It is clear that $\|\overline{q}\| \le n$. From the cocycle property of c^n established in Proposition 3.6(i) it follows that

$$c^n_{0,T}(\overline{Q}) = c^n_{0,t}(\overline{Q}) + E_{\overline{Q}}\big[c^n_{t,T}(\overline{Q})\big] = c^n_{0,t}(\overline{Q}) + E_{\overline{Q}}\big[c^n_{t,T}(P)\big] = c^n_{0,t}(\overline{Q}).$$

From the arguments above it follows that

$$\begin{aligned} c^n_{0,t}(Q) = c^n_{0,t}(\overline{Q}) = c^n_{0,T}(\overline{Q}) &= E_{\overline{Q}}\left[\int_0^T f_n(u,\overline{q}_u)\,du\right] \\ &= E_{\overline{Q}}\left[\int_0^t f_n(u,\overline{q}_u)\,du\right] = E_Q\left[\int_0^t f_n(u,q_u)\,du\right]. \end{aligned}$$

We now come back to the general case. Consider the probability measure Q^* obtained by pasting Q and P via

$$q^*_u = \begin{cases} 0 & \text{if } 0 \le u \le s, \\ q_u 1_A + 0\, 1_{A^c} & \text{if } s < u \le T, \end{cases}$$

with $A \in \mathcal{F}_s$. On the one hand, we deduce that $c^n_{0,s}(Q^*) = c^n_{0,s}(P) = 0$, while for any $s < t \le T$,

$$\begin{aligned} c^n_{0,t}(Q^*) &= E_{Q^*}\left[\int_0^t f_n(u,q^*_u)\,du\right] \\ &= E_{Q^*}\left[1_A \int_s^t f_n(u,q_u)\,du\right] \\ &= E_P\left[E_Q\left[1_A \int_s^t f_n(u,q_u)\,du\middle|\mathcal{F}_s\right]\right] \\ &= E_P\left[1_A E_Q\left[\int_s^t f_n(u,q_u)\,du\middle|\mathcal{F}_s\right]\right]. \end{aligned}$$

On the other hand, from the cocycle property, $E_{Q^*}[c^n_{s,t}(Q^*)] = c^n_{0,t}(Q^*) - c^n_{0,s}(Q^*)$, and hence

$$\begin{aligned} c^n_{0,t}(Q^*) &= c^n_{0,t}(Q^*) - c^n_{0,s}(Q^*) = E_{Q^*}\big[c^n_{s,t}(Q^*)\big] \\ &= E_{Q^*}\big[E_{Q^*}\big[c^n_{s,t}(Q^*)\big|\mathcal{F}_s\big]\big] \\ &= E_P\big[1_A E_Q\big[c^n_{s,t}(Q)\big|\mathcal{F}_s\big]\big]. \end{aligned}$$

Since the set A is arbitrary, we deduce that for any $A \in \mathcal{F}_s$,

$$E_P\left[1_A E_Q\left[\int_s^t f_n(u, q_u)\, du \middle| \mathcal{F}_s\right]\right] = E_P\big[1_A E_Q\big[c^n_{s,t}(Q)|\mathcal{F}_s\big]\big],$$

and hence

$$c^n_{s,t}(Q) = E_Q\big[c^n_{s,t}(Q)|\mathcal{F}_s\big] = E_Q\left[\int_s^t f_n(u, q_u)\, du \middle| \mathcal{F}_s\right]. \qquad \square$$

Lemma 3.9 *For any probability measure Q equivalent to P, it holds true that*

$$c_{0,T}(Q) \le E_Q\left[\int_0^T f(u, q_u)\, du\right],$$

$$c_{t,T}(Q) \le E_Q\left[\int_t^T f(u, q_u)\, du \middle| \mathcal{F}_t\right].$$

Proof We start by proving the inequality for $c_{0,T}(Q)$.

(1) If $\int_0^T f(u, q_u)\, du$ is bounded, we consider the probability measure Q^n corresponding to $q^n := q 1_{\|q\| \le n}$. Since $\int_0^T f(u, q_u)\, du$ is bounded, from Proposition 3.6(iii) it follows that

$$\begin{aligned} \lim_{n\to+\infty} c_{0,T}\big(Q^n\big) &= \lim_n E_{Q^n}\left[\int_0^T f_n\big(u, q^n_u\big)\, du\right] \\ &= \lim_n E_{Q^n}\left[\int_0^T f(u, q_u) 1_{\|q\|\le n}\, du\right] \\ &= \lim_n E_Q\left[\frac{dQ^n}{dQ} \int_0^T f(u, q_u) 1_{\|q\|\le n}\, du\right] \\ &= E_Q\left[\int_0^T f(u, q_u)\, du\right] < +\infty. \end{aligned}$$

Since $\frac{dQ^n}{dP} \to \frac{dQ}{dP}$ in L^1, the lower semi-continuity of $c_{0,T}(Q)$ implies that

$$c_{0,T}(Q) \le \liminf_n c_{0,T}\big(Q^n\big) \le E_Q\left[\int_0^T f(u, q_u)\, du\right].$$

(2) If $\int_0^T f(u,q_u)\,du \in L^1(Q)$, we set $\sigma_n := \inf\{t \geq 0 : \int_0^t f(u,q_u)\,du \geq n\}$ for any $n \in \mathbb{N}$. Then σ_n is a stopping time, and $\sigma_n \uparrow T$. Denote by Q^{σ_n} the probability measure corresponding to $\frac{dQ^{\sigma_n}}{dP} = \mathcal{E}(q \cdot B)^{\sigma_n}$. It is easy to check that $\frac{dQ^{\sigma_n}}{dP} \to \frac{dQ}{dP}$ in L^1. Furthermore,

$$E_{Q^{\sigma_n}}\left[\int_0^{\sigma_n} f\left(u, q_u^{\sigma_n}\right) du\right] = E_Q\left[\int_0^{\sigma_n} f(u,q_u)\,du\right] \to E_Q\left[\int_0^T f(u,q_u)\,du\right],$$

where the equality above is due to the fact that q and q^{σ_n} coincide on the stochastic interval $[\![0,\sigma_n]\!]$. By applying the arguments above, we obtain

$$\begin{aligned} c_{0,T}(Q) &\leq \liminf_n c_{0,T}\left(Q^{\sigma_n}\right) \leq \liminf_n E_{Q^{\sigma_n}}\left[\int_0^{\sigma_n} f\left(u, q_u^{\sigma_n}\right) du\right] \\ &\leq E_Q\left[\int_0^T f(u,q_u)\,du\right]. \end{aligned}$$

(3) In general, if $\int_0^T f(u,q_u)\,du \notin L^1(Q)$, then $E_Q[\int_0^T f(u,q_u)\,du] = +\infty$. Hence $c_{0,T}(Q) \leq E_Q[\int_0^T f(u,q_u)\,du]$.

The inequality $c_{t,T}(Q) \leq E_Q[\int_t^T f(u,q_u)\,du|\mathcal{F}_t]$ can be checked by proceeding as in the proof of Lemma 3.8. □

Lemma 3.10 *Let Q be a probability measure equivalent to P with $c_{0,T}(Q) < +\infty$. If $\{\tau_n\}_{n\geq 0}$ is a sequence of stopping times with $P(\tau_n < T) \to 0$, then $c_0(Q^{\tau_n}) \uparrow c_0(Q)$, where Q^{τ_n} is defined by $\frac{dQ^{\tau_n}}{dP} = E_P[\frac{dQ}{dP}|\mathcal{F}_{\tau_n}]$.*

Proof On the one hand, by the cocycle property and by the definition of Q^{τ_n} it follows that

$$c_{0,T}(Q) = c_{0,\tau_n}(Q) + E_Q\left[c_{\tau_n,T}(Q)\right] \geq c_{0,\tau_n}(Q) = c_{0,T}\left(Q^{\tau_n}\right).$$

On the other hand, by the lower semi-continuity of c_0 and by $\frac{dQ^{\tau_n}}{dP} \to \frac{dQ}{dP}$ in L^1, it holds $c_{0,T}(Q) \leq \liminf_n c_{0,T}(Q^{\tau_n})$. So $\lim_n c_{0,T}(Q^{\tau_n}) = c_{0,T}(Q)$. □

Lemma 3.11 *Consider a general setting where the filtration satisfies the usual hypotheses but is not necessarily a Brownian filtration. Let Q be a probability measure equivalent to P such that $c_{0,T}(Q) < +\infty$ and $(c_t(Q))_{t\in[0,T]}$ is right-continuous. Then there exists a unique increasing, predictable process $(A_t)_{t\in[0,T]}$ (depending on Q) such that $A_0 = 0$ and*

$$c_t(Q) = E_Q[A_T - A_t|\mathcal{F}_t], \quad \forall t \in [0,T], \tag{3.8}$$

i.e., $c(Q)$ is a Q-potential.

Proof By Theorem VII.8 of Dellacherie and Meyer [19], (3.8) holds true if $(c_t(Q))_{t\in[0,T]}$ is a positive Q-supermartingale of class (D), i.e., if $(c_\sigma(Q))_{\sigma\in S}$ is uniformly integrable, where S is the family of all stopping times smaller than or equal to T. The process $(c_t(Q))_{t\in[0,T]}$ is clearly adapted and positive and, by hypothesis and the cocycle property, $c_t(Q) \in L^1(Q)$ for any $t \in [0,T]$. By the cocycle property we deduce that for any $0 \le s \le t \le T$,

$$E_Q\big[c_{t,T}(Q)\big|\mathcal{F}_s\big] = c_{s,T}(Q) - c_{s,t}(Q) \le c_{s,T}(Q),$$

i.e., $(c_t(Q))_{t\in[0,T]}$ is a Q-supermartingale. Furthermore, $c_T(Q) = 0$. It remains to show that $(c_t(Q))_{t\in[0,T]}$ is of class (D). This proof is postponed to the Appendix. □

Remark 3.12 Since in our setting $(c_t(Q))_{t\in[0,T]}$ is càdlàg (see the Appendix for the proof), as a particular case of the previous lemma, it follows that (3.8) holds for a càdlàg $(A_t)_{t\in[0,T]}$.

Note that (3.8) implies that $c_{t,u}(Q) = E_Q[A_u - A_t|\mathcal{F}_t]$ for $0 \le t \le u \le T$. Furthermore, the assumption $c_t(P) = 0$ implies that for $Q = P$, we have $A = A^P = 0$.

Lemma 3.13 *Let σ, τ be two stopping times such that $0 \le \sigma \le \tau \le T$, and Q^1, Q^2 two probability measures equivalent to P. Denote by A^1, A^2 the corresponding increasing processes as in (3.8). Let Q be the probability measure induced by*

$$q = \begin{cases} q^1 & \text{on } H^1 =]\!]0,\sigma]\!] \cup]\!]\tau, T]\!], \\ q^2 & \text{on } H^2 =]\!]\sigma, \tau]\!], \end{cases}$$

and denote by A the corresponding process as in (3.8). Then

$$dA = dA^1|_{H^1} + dA^2|_{H^2} = 1_{H^1}\, dA^1 + 1_{H^2}\, dA^2.$$

Proof Fix $t \in [0,T]$. For $t \ge \tau$, we have $c_t(Q) = c_t(Q^1) = E_{Q^1}[A^1_T - A^1_t|\mathcal{F}_t]$. For $\sigma \le t < \tau$, we deduce from the cocycle property that

$$\begin{aligned} c_t(Q) &= c_{t,\tau}(Q) + E_Q\big[c_{\tau,T}(Q)\big|\mathcal{F}_t\big] \\ &= E_{Q^2}\big[A^2_\tau - A^2_t\big|\mathcal{F}_t\big] + E_{Q^2}\big[E_{Q^1}\big[A^1_T - A^1_\tau\big|\mathcal{F}_\tau\big]\big|\mathcal{F}_t\big] \\ &= E_Q\big[A^2_\tau - A^2_t + A^1_T - A^1_\tau\big|\mathcal{F}_t\big] \\ &= E_Q\bigg[\int_{(t,T]}\big(1_{H^1}\, dA^1 + 1_{H^2}\, dA^2\big)\bigg|\mathcal{F}_t\bigg]. \end{aligned}$$

For $t \le \sigma$, we deduce from the cocycle property and the case above that

$$\begin{aligned} c_t(Q) &= c_{t,\sigma}(Q) + E_Q\big[c_{\sigma,T}(Q)\big|\mathcal{F}_t\big] \\ &= E_{Q^1}\big[A^1_\sigma - A^1_t\big|\mathcal{F}_t\big] + E_{Q^1}\bigg[E_Q\bigg[\int_{]\!]\sigma,T]\!]}\big(1_{H^1}\, dA^1 + 1_{H^2}\, dA^2\big)\bigg|\mathcal{F}_\sigma\bigg]\bigg|\mathcal{F}_t\bigg] \\ &= E_Q\bigg[\int_{(t,T]}\big(1_{H^1}\, dA^1 + 1_{H^2}\, dA^2\big)\bigg|\mathcal{F}_t\bigg]. \end{aligned}$$

Since $A_t := \int_{(0,t]}(1_{H^1}\,dA^1 + 1_{H^2}\,dA^2)$ is càdlàg, predictable, and increasing, we see that $(A_t)_{t\in[0,T]}$ is the process associated to Q in the sense of (3.8). □

Corollary 3.14 *Let $\sigma_1, \sigma_2, \ldots, \sigma_n, \tau_1, \tau_2, \ldots, \tau_n$ be stopping times such that $0 \le \sigma_1 \le \tau_1 \le \sigma_2 \le \tau_2 \le \cdots \le \sigma_n \le \tau_n \le T$, and let Q be a probability measure equivalent to P and whose corresponding increasing process is denoted by A. Set*

$$H := \rrbracket\sigma_1, \tau_1\rrbracket \cup \rrbracket\sigma_2, \tau_2\rrbracket \cup \cdots \cup \rrbracket\sigma_n, \tau_n\rrbracket. \tag{3.9}$$

Let Q^H be the probability measure induced by $q^H = q1_H$ and denote by A^H the corresponding process as in (3.8). Then

$$dA^H = 1_H\,dA.$$

Proof The proof of this result is a repeated application of Lemma 3.13 (with $Q^1 = P$ and $Q^2 = Q$). □

Lemma 3.15 *Let Q be a probability measure equivalent to P, and A the associated increasing process. Then there exists a sequence $(\tau^n)_{n\in\mathbb{N}}$ of stopping times such that*

(i) $\frac{dQ^{\rrbracket 0,\tau^n\rrbracket}}{dP} \to \frac{dQ}{dP}$ *in L^1, where $Q^{\rrbracket 0,\tau^n\rrbracket}$ denotes the probability measure induced by $q^{\rrbracket 0,\tau^n\rrbracket} = q1_{\rrbracket 0,\tau^n\rrbracket}$.*
(ii) $c_{0,T}(Q^{\rrbracket 0,\tau^n\rrbracket}) \uparrow c_{0,T}(Q)$.
(iii) A_{τ^n} *is bounded for each n.*

Proof For any $n \in \mathbb{N}$, set $\sigma^n := \inf\{t \ge 0 : A_t \ge n\}$. Hence σ^n is a predictable stopping time. For any fixed n, take now a sequence $(\tau^{n,m})_{m\in\mathbb{N}}$ such that $\tau^{n,m}$ is increasing (in m), $\tau^{n,m} < \sigma^n$ on $\{\sigma^n > 0\}$, and $\tau^{n,m} \uparrow \sigma^n$. By the definition of σ^n, from $\tau^{n,m} < \sigma^n$ it follows that $A_{\tau^{n,m}} \le n$. For any $\varepsilon > 0$ small enough, take now n and consequently m big enough to have $\|\frac{dQ^{\rrbracket 0,\tau^{n,m}\rrbracket}}{dP} - \frac{dQ}{dP}\|_1 \le \varepsilon$. For such indices, set $\tau^{(n)} := \tau^{n,m}$. Take now $\tau^n := \max_{k\le n}\tau^{(k)}$. It can be checked that $(\tau^n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times and that $A_{\tau^n} \le n$ (since also $\tau^n < \sigma^n$). Furthermore, since $\sigma^n = T$ for sufficiently big n and $\tau^n \uparrow T$, property (i) follows. Property (ii) can be checked as usual (see, for instance, the proof of Lemma 3.10). □

Lemma 3.16 *Let Q be a probability measure equivalent to P, and A the associated increasing process. Suppose that A is bounded. Let H be a predictable set. Suppose that $\mathcal{E}(q1_H \cdot B)$ is a uniformly integrable martingale. Set $\frac{dQ^H}{dP} := \mathcal{E}(q1_H \cdot B)_T$ and denote by A^H the associated increasing process. Then*

$$dA^H \le dA,$$

and hence $A^H_T \le A_T$.

Proof First of all, we recall that the sets of the form (3.9) form an algebra $\mathcal{A}$ and that the σ-algebra $\mathcal{P}$ of predictable sets is generated by $\mathcal{A}$. Consider now any predictable set $H \in \mathcal{P}$ satisfying the hypothesis. If $H \in \mathcal{A}$, we already know that $dA^H = 1_H\,dA$

from Corollary 3.14. For the general case, consider two stopping times σ, τ such that $0 \le \sigma \le \tau \le T$ and take a sequence $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that

$$E_Q\Big[\int_0^T |1_{H^n} - 1_H|\,dA\Big] \to 0, \qquad E\Big[\int_0^T |1_{H^n} - 1_H|\,dt\Big] \to 0. \tag{3.10}$$

Denote by Q^{H^n} the probability measure induced by $q^n = q1_{H^n}$ and by A^{H^n} the associated increasing process. Again from Corollary 3.14 it follows that $dA^{H^n} = 1_{H^n}\,dA$ since $H^n \in \mathcal{A}$. By (3.10) we have that $\frac{dQ^{H^n}}{dP} \to \frac{dQ^H}{dP}$ in L^1. By the lower semi-continuity of c and by (3.8), we get

$$\begin{aligned}E_{Q^H}\big[A_\tau^H - A_\sigma^H \big|\mathcal{F}_\sigma\big] &= c_{\sigma,\tau}\big(Q^H\big)\\ &\le \liminf_n c_{\sigma,\tau}\big(Q^{H^n}\big)\\ &= \liminf_n E_{Q^{H^n}}\big[A_\tau^{H^n} - A_\sigma^{H^n}\big|\mathcal{F}_\sigma\big].\end{aligned}$$

Because $\int_{]\!]\sigma,\tau]\!]} 1_{H^n}\,dA \to \int_{]\!]\sigma,\tau]\!]} 1_H\,dA$, $\int_{]\!]\sigma,\tau]\!]} 1_{H^n}\,dA$ is uniformly bounded, and $\mathcal{E}(1_{]\!]\sigma,\tau]\!]\cap H^n} q\cdot B) \to \mathcal{E}(1_{]\!]\sigma,\tau]\!]\cap H} q\cdot B)$ in L^1, we obtain

$$E_{Q^{H^n}}\Big[\int_{]\!]\sigma,\tau]\!]} 1_{H^n}\,dA\Big|\mathcal{F}_\sigma\Big] \to E_{Q^H}\Big[\int_{]\!]\sigma,\tau]\!]} 1_H\,dA\Big|\mathcal{F}_\sigma\Big]. \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\begin{aligned}E_{Q^H}\big[A_\tau^H - A_\sigma^H \big|\mathcal{F}_\sigma\big] &= E_{Q^H}\Big[\int_{]\!]\sigma,\tau]\!]} dA^H\Big|\mathcal{F}_\sigma\Big]\\ &\le \liminf_n E_{Q^{H^n}}\big[A_\tau^{H^n} - A_\sigma^{H^n}\big|\mathcal{F}_\sigma\big]\\ &= E_{Q^H}\Big[\int_{]\!]\sigma,\tau]\!]} 1_H\,dA\Big|\mathcal{F}_\sigma\Big],\end{aligned}$$

and hence $E_{Q^H}[\int_{]\!]\sigma,\tau]\!]} dA^H|\mathcal{F}_\sigma] \le E_{Q^H}[\int_{]\!]\sigma,\tau]\!]} 1_H\,dA|\mathcal{F}_\sigma]$. The same inequality holds if we replace $]\!]\sigma,\tau]\!]$ with any element $K \in \mathcal{A}$ (it is sufficient to sum over intervals of the same form as in (3.9)), that is,

$$E_{Q^H}\Big[\int_0^T 1_K\,dA^H\Big|\mathcal{F}_\sigma\Big] \le E_{Q^H}\Big[\int_0^T 1_K 1_H\,dA\Big|\mathcal{F}_\sigma\Big]. \tag{3.12}$$

Moreover, by passing to the limit we obtain that inequality (3.12) holds true for any $K \in \mathcal{P}$. So we get $dA^H \le 1_H\,dA$ as stochastic measures on $(0,T]$, and hence $A_T^H \le A_T$. □

Lemma 3.17 *Let Q be a probability measure equivalent to P and suppose that the corresponding increasing process A is bounded. If H^n is predictable, $H^n \uparrow 1_{(0,T]\times\Omega}$, and Q^{H^n} is the probability measure induced by $q^{H^n} = q 1_{H^n}$, then*

$$c_{0,T}\big(Q^{H^n}\big) \to c_{0,T}(Q).$$

Proof We already know by Lemma 3.16 that

$$dA^{H^n} \leq 1_{H^n}\, dA. \tag{3.13}$$

From $\frac{dQ^{H^n}}{dP} \to \frac{dQ}{dP}$ in L^1, inequality (3.13), and the lower semi-continuity of $c_{0,T}$ we get

$$\begin{aligned} c_{0,T}(Q) \leq \liminf_n c_{0,T}\big(Q^{H^n}\big) &= \liminf_n E_{Q^{H^n}}\big[A_T^{H^n}\big] \\ &\leq \liminf_n E_{Q^{H^n}}\bigg[\int_{(0,T]} 1_{H^n}\, dA\bigg]. \end{aligned}$$

Since $\int_{(0,T]} 1_{H^n} dA$ is bounded and $\frac{dQ^{H^n}}{dP} \to \frac{dQ}{dP}$ in L^1, we have that

$$\begin{aligned} c_{0,T}(Q) &\leq \liminf_n c_{0,T}\big(Q^{H^n}\big) \leq \liminf_n E_{Q^{H^n}}\bigg[\int_{(0,T]} 1_{H^n}\, dA\bigg] \\ &= E_Q\bigg[\int_{(0,T]} dA\bigg] = c_{0,T}(Q), \end{aligned}$$

and hence $c_{0,T}(Q^{H^n}) \to c_{0,T}(Q)$. □

Theorem 3.18 *Let Q be a probability measure equivalent to P, and A the associated increasing process. Then there exists a sequence $(Q^n)_{n\in\mathbb{N}}$ of probability measures with q^n bounded such that $\frac{dQ^n}{dP} \to \frac{dQ}{dP}$ in L^1 and $c_{0,T}(Q^n) \to c_{0,T}(Q)$.*

Proof From the arguments above and by stopping arguments, we may suppose that A is bounded. For any $n \in \mathbb{N}$, take $H^n := 1_{\|q\|\leq n}$ and as Q^n the probability measure induced by $q^n = q 1_{H^n}$. Hence H^n is predictable, and $H^n \uparrow 1_{(0,T]\times\Omega}$, so that it satisfies the hypothesis of Lemma 3.17. It follows that $\frac{dQ^n}{dP} \to \frac{dQ}{dP}$ in L^1 and by Lemma 3.17 that $c_{0,T}(Q^n) \to c_{0,T}(Q)$. □

We are now ready to prove the representation of the penalty term c in terms of f in Theorem 3.2.

Proof of Theorem 3.2 Since (ii) is a straightforward consequence of (i) and of the representation in (3.2), it remains to show that

$$c_{0,T}(Q) = E_Q\bigg[\int_0^T f(t, q_t)\, dt\bigg]. \tag{3.14}$$

By Lemma 3.9, we already know that $c_{0,T}(Q) \le E_Q[\int_0^T f(u,q_u)\,du]$ for any probability measure $Q \sim P$.

Suppose first that $\int_0^T f(t,\omega,q_t)\,dt \in L^1(Q)$. For any $n \in \mathbb{N}$, define

$$\sigma_n := \inf\left\{t \ge 0 : \int_0^t f(u,q_u) \ge n\right\}.$$

Then $(\sigma_n)_{n\ge 0}$ is a sequence of stopping times such that $\sigma_n \uparrow T$. Take now a sequence $(Q^m)_{m\in\mathbb{N}}$ of probability measures as in Theorem 3.18. Then

$$\begin{aligned}
c_{0,T}(Q) &\le E_Q\left[\int_0^T f(u,q_u)\,du\right] \\
&= \lim_n E_Q\left[\int_0^{\sigma_n} f(u,q_u)\,du\right] \\
&\le \sup_n \lim_m E_{Q^m}\left[\int_0^{\sigma_n} f(u,q_u)1_{\|q\|\le m}\,du\right] \\
&\le \lim_m \sup_n E_{Q^m}\left[\int_0^{\sigma_n} f(u,q_u)1_{\|q\|\le m}\,du\right] \\
&= \lim_m E_{Q^m}\left[\int_0^{T} f(u,q_u)1_{\|q\|\le m}\,du\right] \\
&= \lim_m c^m_{0,T}\big(Q^m\big) = \lim_m c_{0,T}\big(Q^m\big) = c_{0,T}(Q),
\end{aligned}$$

where the last equality is due to Theorem 3.18. Equality (3.14) has therefore been established for $\int_0^T f(t,q_t)\,dt \in L^1(Q)$.

If $\int_0^T f(t,\omega,q_t)\,dt \notin L^1(Q)$, Fatou's lemma gives

$$\begin{aligned}
c_{0,T}(Q) &\le E_Q\left[\int_0^T f(t,q_t)\,dt\right] \\
&\le \liminf_m E_{Q^m}\left[\int_0^T f(t,q_t)1_{\|q\|\le m}\,dt\right] \\
&= \liminf_m c^m_{0,T}\big(Q^m\big) \\
&= \liminf_m c_{0,T}\big(Q^m\big) = c_{0,T}(Q),
\end{aligned}$$

and hence $c_{0,T}(Q) = E_Q[\int_0^T f(t,q_t)\,dt] = +\infty$. The representation of $c_{s,t}(Q)$ (and hence of $c_{\sigma,\tau}(Q)$) can be deduced as usual. □

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Appendix

Let Q be a probability measure equivalent to P and such that $c_{0,T}(Q) < +\infty$. In the following, we prove that $(c_{t,T}(Q))_{t\in[0,T]}$ is of class (D) and that it admits a càdlàg modification.

The following corollary of Lemma 3.10 will be useful later.

Corollary A.1 $\sup\{E_Q[c_{\tau,T}(Q)] : \tau$ *stopping time with* $P(\tau < T) \le \frac{1}{n}\} \to 0$ *as* $n \to \infty$.

The next result is a straightforward consequence of the cocycle property of c.

Lemma A.2 *Denote by S the family of all stopping times smaller than or equal to T. The family $(c_{\sigma,T}(Q))_{\sigma\in S}$ satisfies the following property: Given any pair of stopping times σ, τ such that $0 \le \sigma \le \tau \le T$, we have $c_{\sigma,T}(Q) \ge E_Q[c_{\tau,T}(Q)|\mathcal{F}_\sigma]$.*

Lemma A.3 *The family $(c_{\sigma,T}(Q))_{\sigma\in S}$ is Q-uniformly integrable.*

Proof We have to prove that

$$\lim_{n\to+\infty} \sup_{\sigma\in S} \int_{c_{\sigma,T}(Q)>n} c_{\sigma,T}(Q)\,dQ = 0. \tag{A.1}$$

Consider an arbitrary stopping time $\sigma \in S$ and set

$$\sigma^{(n)} = \begin{cases} \sigma & \text{if } c_{\sigma,T}(Q) > n, \\ T & \text{if } c_{\sigma,T}(Q) \le n. \end{cases}$$

By the cocycle property we get

$$\begin{aligned} c_0(Q) &= c_0\big(Q^{\sigma^{(n)}}\big) + E_Q\big[c_{\sigma^{(n)},T}(Q)\big] \\ &\ge E_Q\big[c_{\sigma^{(n)},T}(Q)\big] \\ &= \int_{c_{\sigma,T}(Q)>n} c_{\sigma,T}(Q)\,dQ \ge nP\big(c_{\sigma,T}(Q) > n\big). \end{aligned}$$

Hence $P(c_{\sigma,T}(Q) > n) \le \frac{c_0(Q)}{n}$ uniformly in σ, so that we get

$$\begin{aligned} 0 &\le \sup_{\sigma\in S} \int_{c_{\sigma,T}(Q)>n} c_{\sigma,T}(Q)\,dQ \\ &\le \sup\left\{E_Q\big[c_{\tau,T}(Q)\big] : \tau \text{ stopping time with } P(\tau < T) \le \frac{c_0(Q)}{n}\right\}. \end{aligned}$$

Since the last term tends to 0 as $n \to +\infty$ by Corollary A.1, (A.1) follows. □

Lemma A.4 *Let $\varepsilon > 0$ be such that $E_Q[-\xi] > c_0(Q) - \varepsilon$ with $\xi \in \mathcal{A}_{0,T}$. Then for any pair of stopping times σ, τ such that $0 \le \sigma \le \tau \le T$, it holds that*

$$E_Q\big[c_{\sigma,\tau}(Q)\big] \le E_Q\big[u_\sigma(\xi) - u_\tau(\xi)\big] + \varepsilon.$$

Proof By the translation invariance of $(u_{t,T})_{t\in[0,T]}$ it follows that $u_{\tau,T}(\xi - u_{\tau,T}(\xi)) = 0$, and hence $\xi - u_\tau(\xi) \in \mathcal{A}_{\tau,T}$. Furthermore, the time-consistency and translation invariance of u and $\xi \in \mathcal{A}_{0,T}$ imply that $u_\tau(\xi) - u_\sigma(\xi) \in \mathcal{A}_{\sigma,T}$ and that $u_\sigma(\xi) \in \mathcal{A}_{0,\sigma}$. The cocycle property or, equivalently, the decomposition property $\mathcal{A}_{0,T} = \mathcal{A}_{0,\sigma} + \mathcal{A}_{\sigma,\tau} + \mathcal{A}_{\tau,T}$ implies that

$$\begin{aligned} c_0(Q) &= E_Q\big[c_{0,\sigma}(Q)\big] + E_Q\big[c_{\sigma,\tau}(Q)\big] + E_Q\big[c_{\tau,T}(Q)\big] \\ &\ge E_Q\big[-u_\sigma(\xi)\big] + E_Q\big[u_\sigma(\xi) - u_\tau(\xi)\big] + E_Q\big[u_\tau(\xi) - \xi\big] \\ &\ge E_Q[-\xi] \ge c_0(Q) - \varepsilon, \end{aligned}$$

where the first inequality follows from $c_{t,T}(Q) = \operatorname{ess.sup}_{\xi\in\mathcal{A}_{t,T}} E_Q[-\xi|\mathcal{F}_t]$. By proceeding as above we get

$$\begin{aligned} c_0(Q) &\ge E_Q\big[-u_\sigma(\xi)\big] + E_Q\big[u_\sigma(\xi) - u_\tau(\xi)\big] + E_Q\big[u_\tau(\xi) - \xi\big] \\ &\ge c_0(Q) - \varepsilon \\ &\ge E_Q\big[-u_\sigma(\xi)\big] + E_Q\big[c_{\sigma,\tau}(Q)\big] + E_Q\big[u_\tau(\xi) - \xi\big] - \varepsilon, \end{aligned}$$

and hence $E_Q[c_{\sigma,\tau}(Q)] \le E_Q[u_\sigma(\xi) - u_\tau(\xi)] + \varepsilon$. □

Lemma A.5 *Let $\sigma \in S$. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times such that $\sigma_n \downarrow \sigma$, then $E_Q[c_{\sigma,\sigma_n}(Q)] \to 0$.*

Proof Suppose by contradiction that $E_Q[c_{\sigma,\sigma_n}(Q)]$ does not tend to 0 as $n \to +\infty$. Then there exists $\varepsilon > 0$ such that $E_Q[c_{\sigma,\sigma_n}(Q)] \ge \varepsilon > 0$ for any $n \in \mathbb{N}$. Take now $\xi \in \mathcal{A}_{0,T}$ such that $E_Q[-\xi] \ge c_0(Q) - \frac{\varepsilon}{2}$. Hence, by Lemma A.4,

$$E_Q\big[u_\sigma(\xi) - u_{\sigma_n}(\xi)\big] \ge E_Q\big[c_{\sigma,\sigma_n}(Q)\big] - \frac{\varepsilon}{2} \ge \frac{\varepsilon}{2}$$

for any $n \in \mathbb{N}$. This leads to a contradiction since $(u_{t,T})_{t\in[0,T]}$ admits a càdlàg version with $u_{\sigma_n}(\xi) \to u_\sigma(\xi)$ in $L^1(Q)$; see Lemma 4 of Bion-Nadal [7]. □

By the cocycle property it is easy to deduce the following result from the one above.

Corollary A.6 *Let $\sigma \in S$. If $\{\sigma_n\}_{n\in\mathbb{N}}$ is a sequence of stopping times such that $\sigma_n \downarrow \sigma$, then $E_Q[c_{\sigma_n,T}(Q)] \to E_Q[c_{\sigma,T}(Q)]$.*

Lemma A.7 $(c_{t,T}(Q))_{t\in[0,T]}$ *admits a càdlàg modification. Furthermore, if $(\overline{c}_t)_{t\in[0,T]}$ denotes this modification, for any stopping time $\sigma \in S$, it holds $c_{\sigma,T}(Q) = \overline{c}_\sigma$ a.s.*

We remark that this ends the proof of the statement in the beginning of the Appendix.

Proof of Lemma A.7 We already know that $(c_{t,T}(Q))_{t\in[0,T]}$ is a positive Q-supermartingale (see the proof of Lemma 3.11) and that for any sequence $\{t_n\}_{n\in\mathbb{N}}$ in $[0,T]$ and such that $t_n \downarrow t$, it holds $E_Q[c_{t_n,T}(Q)] \to E_Q[c_{t,T}(Q)]$ by Corollary A.6. By Theorem VII.4 of Dellacherie and Meyer [19] it follows that $(c_{t,T}(Q))_{t\in[0,T]}$ admits a càdlàg modification. This implies that for any stopping time $\sigma \in S$ taking rational values, it holds $\overline{c}_\sigma = c_{\sigma,T}(Q)$ a.s. For a general stopping time $\sigma \in S$, there exists a sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ of finite stopping times taking rational values and such that $\sigma_n \downarrow \sigma$. Hence,

$$\lim_{n\to+\infty} c_{\sigma_n,T}(Q) = \lim_{n\to+\infty} \overline{c}_{\sigma_n} = \overline{c}_\sigma \quad \text{a.s.}, \tag{A.2}$$

where the last equality follows from the fact that $(\overline{c}_t)_{t\in[0,T]}$ is càdlàg.

It remains to prove that $c_{\sigma,T}(Q) = \lim_{n\to+\infty} c_{\sigma_n,T}(Q)$. This proof is quite standard, and we only include it for completeness. By the cocycle property it follows that $(c_{\sigma_n,T}(Q), \mathcal{F}_{\sigma_n})_{n\in\mathbb{N}}$ is a positive reverse Q-supermartingale (see Neveu [32]). By Proposition V-3-11 of Neveu [32], $c_{\sigma_n,T}(Q)$ converges as $n \to +\infty$ to a positive $\mathcal{F}_\sigma$-measurable random variable η, and $E_Q[c_{\sigma_n,T}(Q)|\mathcal{F}_\sigma] \to \eta$ a.s. Since $E_Q[c_{\sigma_n,T}(Q)|\mathcal{F}_\sigma] \le c_{\sigma,T}(Q)$, we get $\eta \le c_{\sigma,T}(Q)$. Furthermore, by the Q-uniform integrability of $(c_{\sigma_n,T}(Q))_{n\in\mathbb{N}}$ (see Lemma A.3) we get

$$E_Q\big[c_{\sigma,T}(Q)\big] = \lim_n E_Q\big[c_{\sigma_n,T}(Q)\big] = E_Q[\eta],$$

where the first equality is due to Corollary A.6. By the arguments above it follows that $\eta = c_{\sigma_n}(Q)$ a.s., hence the assertion. □

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