A COMPACTNESS PRINCIPLE FOR BOUNDED SEQUENCES OF MARTINGALES WITH APPLICATIONS

F. Delbaen
W. Schachermayer

Departement für Mathematik, Eidgenössische Technische Hochschule Zürich
Institut für Statistik, Universität Wien

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Abstract. For $\mathcal{H}^1$-bounded sequences of martingales, we introduce a technique, related to the Kadeč–Pełczynski-decomposition for L^1 sequences, that allows us to prove compactness theorems. Roughly speaking, a bounded sequence in $\mathcal{H}^1$ can be split into two sequences, one of which is weakly compact, the other forms the singular part. If the martingales are continuous then the singular part tends to zero in the semi-martingale topology. In the general case the singular parts give rise to a process of bounded variation. The technique allows to give a new proof of the optional decomposition theorem in Mathematical Finance.

1. Introduction

Without any doubt, one of the most fundamental results in analysis is the theorem of Heine–Borel:

1.1 Theorem. *From a bounded sequence* $(x_n)_{n\geq 1} \in \mathbb{R}^d$ *we can extract a convergent subsequence* $(x_{n_k})_{k\geq 1}$.

If we pass from $\mathbb{R}^d$ to infinite-dimensional Banach spaces X this result does not hold true any longer. But there are some substitutes which often are useful.

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The following theorem can be easily derived from the Hahn–Banach theorem and was wellknown to S. Banach and his contemporaries. (see [DRS93] for related theorems)

1.2 Theorem. *Given a bounded sequence $(x_n)_{n\geq 1}$ in a reflexive Banach space X (or, more generally, a relatively weakly compact sequence in a Banach space X) we may find a sequence $(y_n)_{n\geq 1}$ of convex combinations of $(x_n)_{n\geq 1}$,*

$$y_n \in \operatorname{conv}(x_n, x_{n+1}, \dots),$$

which converges with respect to the norm of X.

Note—and this is a "Leitmotiv" of the present paper—that, for sequences $(x_n)_{n\geq 1}$ in a vector space, passing to convex combinations usually does not cost more than passing to a subsequence. In most applications the main problem is to find a limit $x_0 \in X$ and typically it does not matter whether $x_0 = \lim_k x_{n_k}$ for a subsequence $(x_{n_k})_{k\geq 1}$ or $x_0 = \lim_n y_n$ for a sequence of convex combinations $y_n \in \operatorname{conv}(x_n, x_{n+1}, \dots)$.

If one passes to the case of non-reflexive Banach spaces there is—in general—no analogue to theorem 1.2 pertaining to any bounded sequence $(x_n)_{n\geq 1}$, the main obstacle being that the unit ball fails to be weakly compact. But sometimes there are Hausdorff topologies on the unit ball of a (non-reflexive) Banach space which have some kind of compactness properties. A noteworthy example is the Banach space $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and the topology of convergence in measure.

1.3 Theorem. *Given a bounded sequence $(f_n)_{n\geq 1} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ then there are convex combinations*

$$g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$$

such that $(g_n)_{n\geq 1}$ converges in measure to some $g_0 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

The preceding theorem is a somewhat vulgar version of Komlos' theorem [Ko67]. Note that Komlos' result is more subtle as it replaces the convex combinations $(g_n)_{n\geq 1}$ by the Cesaro-means of a properly chosen subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$.

But the above "vulgar version" of Komlos' theorem has the advantage that it extends to the case of $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ for reflexive Banach spaces E as we shall presently see (theorem 1.4 below), while Komlos' theorem does not. (J. Bourgain [Bou79] proved that the precise necessary and sufficient condition for the Komlos theorem to hold for E-valued functions is that $L^2(\Omega, \mathcal{F}, \mathbb{P}; E)$ has the Banach–Saks property; compare [G80] and [S81]).

Here is the vector-valued version of theorem 1.3:

1.4 Theorem. *If E is a reflexive Banach space and $(f_n)_{n\geq 1}$ a bounded sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$, we may find convex combinations*

$$g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$$

and $g_0 \in L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ *such that* $(g_n)_{n\geq 1}$ *converges to* f_0 *almost surely, i.e.,*

$$\lim_{n\to\infty} \|g_n(\omega) - g_0(\omega)\|_E = 0 \qquad \textit{for a.e. } \omega \in \Omega.$$

The preceding theorem seems to be of folklore type and to be known to specialists for a long time (compare also [DRS93]). We shall give a proof in section 2 below.

Let us have a closer look at what is really happening in theorems 1.3 and 1.4 above by following the lines of Kadeč and Pełczynski [KP65]. These authors have proved a remarkable decomposition theorem which essentially shows the following (see th. 2.1 below for a more precise statement): Given a bounded sequence $(f_n)_{n\geq 1}$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ we may find a subsequence $(f_{n_k})_{k\geq 1}$ which may be split into a "regular" and a "singular" part, $f_{n_k} = f^r_{n_k} + f^s_{n_k}$, such that $(f^r_{n_k})_{k\geq 1}$ is uniformly integrable and $(f^s_{n_k})_{k\geq 1}$ tends to zero almost surely.

Admitting this result, theorem 1.3 becomes rather obvious: As regards the "regular part" $(f^r_{n_k})_{k\geq 1}$ we can apply theorem 1.2 to find convex combinations converging with respect to the norm of L^1 and therefore in measure. As regards the "singular part" $(f^s_{n_k})_{k\geq 1}$ we do not have any problems as any sequence of convex combinations will also tend to zero almost surely.

A similar reasoning allows to deduce the vector-valued case (th. 1.4 above) from the Kadeč–Pełczynski decomposition result (see section 2 below).

After this general prelude we turn to the central theme of this paper. Let $(M_t)_{t\in\mathbb{R}_+}$ be an $\mathbb{R}^d$-valued càdlàg local martingale based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$ and $(H^n)_{n\geq 1}$ a sequence of M-integrable processes, i.e., predictable $\mathbb{R}^d$-valued stochastic processes such that the integral

$$(H^n \cdot M)_t = \int_0^t H^n_u \, dM_u$$

makes sense for every $t \in \mathbb{R}_+$, and suppose that the resulting processes $((H^n \cdot M)_t)_{t\in\mathbb{R}_+}$ are martingales. The theme of the present paper is: *Under what conditions can we pass to a limit* H^0 *?* More precisely: by passing to convex combinations of $(H^n)_{n\geq 1}$ (still denoted by H^n) we would like to insure that the sequence of martingales $H^n \cdot M$ converges to some martingale N which is of the form $N = H^0 \cdot M$.

Our motivation for this question comes from applications of stochastic calculus to Mathematical Finance where this question turned out to be of crucial relevance. For example, in the work of the present authors as well as in the recent work of D. Kramkov ([DS94], [K96]) the passage to the limit of a sequence of integrands is the heart of the matter. We shall come back to the applications of the results obtained in this paper to Mathematical Finance in section 5 below.

Let us review some known results in the context of the above question. The subsequent theorem 1.5, going back to the foundations of stochastic integration given by Kunita and Watanabe [KW67], is a straightforward consequence of the Hilbert space isometry of stochastic integrands and integrals (see, e.g., [P90], p. 153 for the real-valued and Jacod [Ja80] for the vector-valued case).

1.5 Theorem. *(Kunita–Watanabe) Let M be an $\mathbb{R}^d$-valued càdlàg local martingale, $(H^n)_{n\geq 1}$ be a sequence of M-integrable predictable stochastic processes such that each $(H^n \cdot M)$ is an L^2-bounded martingale and such that the sequence of random variables $((H^n \cdot M)_\infty)_{n\geq 1}$ converges to a random variable $f_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the norm of L^2.*

Then there is an M-integrable predictable stochastic process H^0 such that $H^0 \cdot M$ is an L^2-bounded martingale and such that $(H^0 \cdot M)_\infty = f_0$.

It is not hard to extend the above theorem to the case of L^p, for $1 < p \leq \infty$. But the extension to $p = 1$ is a much more delicate issue which has been settled by M. Yor [Y78], who proved the analogue of theorem 1.5 for the case of $\mathcal{H}^1$ and L^1.

1.6 Theorem. *(Yor) Let $(H^n)_{n\geq 1}$ be a sequence of M-integrable predictable stochastic processes such that each $(H^n \cdot M)$ is an $\mathcal{H}^1$-bounded (resp. a uniformly integrable) martingale and such that the sequence of random variables $((H^n \cdot M)_\infty)_{n\geq 1}$ converges to a random variable $f_0 \in \mathcal{H}^1(\Omega, \mathcal{F}, \mathbb{P})$ (resp. $f_0 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$) with respect to the $\mathcal{H}^1$-norm (resp. L^1-norm); (or even with respect to the $\sigma(\mathcal{H}^1$,BMO$)$ (resp. $\sigma(L^1, L^\infty)$) topology).*

Then there is an M-integrable predictable stochastic process H^0 such that $H^0 \cdot M$ is an $\mathcal{H}^1$-bounded (resp. uniformly integrable) martingale and such that $(H^0 \cdot M)_\infty = f_0$.

We refer to Jacod [Ja80], Theorème 4.63, p.143 for the $\mathcal{H}^1$-case. It essentially follows from Davis' inequality for $\mathcal{H}^1$-martingales. The L^1-case (see [Y78]) is more subtle. Using delicate stopping time arguments M. Yor succeeded in reducing the L^1 case to the $\mathcal{H}^1$ case. In section 4 we take the opportunity to translate Yor's proof into the setting of the present paper.

Let us also mention in this context a remarkable result of Memin ([M80], th. V.4) where the process M is only assumed to be a semi-martingale and not necessarily a local martingale and which also allows to pass to a limit $H^0 \cdot M$ of a Cauchy sequence $H^n \cdot M$ of M-integrals (w.r. to the semimartingale topology).

All these theorems are *closedness results* in the sense that, if $(H^n \cdot M)$ is a *Cauchy-sequence* with respect to some topology, then we may find H^0 such that $(H^0 \cdot M)$ equals the limit of $(H^n \cdot M)$.

The aim of our paper is to prove *compactness results* in the sense that, if $(H^n \cdot M)$ is a *bounded sequence* in the martingale space $\mathcal{H}^1$, then we may find a subsequence $(n_k)_{k\geq 1}$ as well as decompositions $H^{n_k} = {}^rK^k + {}^sK^k$ so that the sequence ${}^rK^k \cdot M$ is relatively weakly compact in $\mathcal{H}^1$ and such that the singular parts ${}^sK^k \cdot M$ hopefully tend to zero in some sense to be made precise. The regular parts ${}^rK^k \cdot M$ then allow to take convex combinations that converge in the norm of $\mathcal{H}^1$.

It turns out that for *continuous* local martingales M the situation is nicer (and easier) than for the general case of local martingales with jumps. We now state the main result of this paper, in its continuous and in its general version (theorem A and B below).

1.7 Theorem A. *Let $(M^n)_{n\geq 1}$ be an $\mathcal{H}^1$-bounded sequence of real-valued continuous local martingales.*

Then we can select a subsequence, which we still denote by $(M^n)_{n\geq 1}$, as well as an increasing sequence of stopping times $(T_n)_{n\geq 1}$, such that $\mathbb{P}[T_n < \infty]$ tends to zero and such that the sequence of stopped processes $\left((M^n)^{T_n}\right)_{n\geq 1}$ is relatively weakly compact in $\mathcal{H}^1$.

If all the martingales are of the form $M^n = H^n \cdot M$ for a fixed continuous local martingale taking values in $\mathbb{R}^d$, then the elements in the $\mathcal{H}^1$-closed convex hull of the sequence $\left((M^n)^{T_n}\right)_{n\geq 1}$ are also of the form $H \cdot M$.

As a consequence we obtain the existence of convex combinations

$$K^n \in \operatorname{conv}\{H^n, H^{n+1}, \dots\}$$

such that $K^n \mathbf{1}_{[\![0,T_n]\!]} \cdot M$ tends to a limit $H^0 \cdot M$ in $\mathcal{H}^1$. Also remark that the remaining “singular” parts $K^n \mathbf{1}_{]\!]T_n,\infty]\!]} \cdot M$ tend to zero in a stationary way, i.e. for almost each $\omega \in \Omega$ the set $\{t \mid \exists\, n \geq n_0, K^n_t \neq 0\}$ becomes empty for large enough n_0. As a result we immediately derive that the sequence $K^n \cdot M$ tends to $H^0 \cdot M$ in the semi-martingale topology.

If the local martingale M is not continuous the situation is more delicate. In this case we cannot obtain a limit of the form $H^0 \cdot M$ and also the decomposition is not just done by stopping the processes at well selected stopping times.

1.8 Theorem B. *Let M be an $\mathbb{R}^d$-valued local martingale and $(H^n)_{n\geq 1}$ be a sequence of M-integrable predictable processes such that $(H^n \cdot M)_{n\geq 1}$ is an $\mathcal{H}^1$ bounded sequence of martingales.*

Then there is a subsequence, for simplicity still denoted by $(H^n)_{n\geq 1}$, an increasing sequence of stopping times $(T_n)_{n\geq 1}$, a sequence of convex combinations $L^n = \sum_{k\geq n} \alpha^n_k H^k$ as well as a sequence of predictable sets $(E^n)_{n\geq 1}$ such that

(1) *$E^n \subset [\![0,T_n]\!]$ and T_n increases to ∞,*
(2) *the sequence $\left(H^n \mathbf{1}_{[\![0,T_n]\!]\cap(E^n)^c} \cdot M\right)_{n\geq 1}$ is weakly relatively compact in $\mathcal{H}^1$,*
(3) *$\sum_{n\geq 1} \mathbf{1}_{E^n} \leq d$,*
(4) *the convex combinations $\sum_{k\geq n} \alpha^k_n H^k \mathbf{1}_{[\![0,T_n]\!]\cap(E^n)^c} \cdot M$ converge in $\mathcal{H}^1$ to a stochastic integral of the form $H^0 \cdot M$, for some predictable process H^0,*
(5) *the convex combinations $V_n = \sum_{k\geq n} \alpha^k_n H^k \mathbf{1}_{]\!]T_n,\infty[\![\cup E^n} \cdot M$ converge to a càdlàg optional process Z of finite variation in the following sense: a.s. we have that $Z_t = \lim_{s \underset{>}{\to} t, s\in\mathbb{Q}} \lim_{n\to\infty} (V_n)_s$ for each $t \in \mathbb{R}_+$,*

(6) *the brackets* $[(H^0 - L^n) \cdot M, (H^0 - L^n) \cdot M]_\infty$ *tend to zero in probability.*

If, in addition, the set

$$\{\Delta(H^n \cdot M)_T^- \mid n \in \mathbb{N};\ T \text{ stopping time}\}$$

resp.

$$\{|\Delta(H^n \cdot M)_T| \mid n \in \mathbb{N};\ T \text{ stopping time}\}$$

is uniformly integrable, e.g. there is an integrable function $w \geq 0$ *such that*

$$\Delta(H^n \cdot M) \geq -w \quad \text{resp. } |\Delta(H^n \cdot M)| \leq w, \text{ a.s.}$$

then the process $(Z_t)_{t \in \mathbb{R}_+}$ *is decreasing (resp. vanishes identically).*

For general martingales, not necessarily of the form $H^n \cdot M$ for a fixed local martingale M, we can prove the following theorem:

1.9 Theorem C. *Let* $(M^n)_{n \geq 1}$ *be an* $\mathcal{H}^1$*-bounded sequence of* $\mathbb{R}^d$*-valued martingales. Then there is a subsequence, for simplicity still denoted by* $(M_n)_{n \geq 1}$ *and an increasing sequence of stopping times* $(T_n)_{n \geq 1}$ *with the following properties:*

(1) T_n *increases to* ∞,

(2) *the martingales* $N^n = (M^n)^{T_n} - \Delta M_{T_n} \mathbf{1}_{[\![T_n, \infty[\![} + C^n$ *form a relatively weakly compact sequence in* $\mathcal{H}^1$*. Here* C^n *denotes the compensator (dual predictable projection) of the process* $\Delta M_{T_n} \mathbf{1}_{[\![T_n, \infty[\![}$,

(3) *there are convex combinations* $\sum_{k \geq n} \alpha_n^k N^k$ *that converge to an* $\mathcal{H}^1$, *martingale* N^0 *in the norm of* $\mathcal{H}^1$

(4) *there is a càdlàg optional process of finite variation* Z *such that almost everywhere for each* $t \in \mathbb{R}$: $Z_t = \lim_{s \overset{>}{\to} t, s \in \mathbb{Q}} \lim_{n \to \infty} \sum_{k \geq n} \alpha_n^k C_s^k$.

If, in addition, the set

$$\{\Delta(M^n)_T^- : n \in \mathbb{N}; T \text{ stopping time}\}$$

resp.

$$\{|\Delta(M^n)_T| : n \in \mathbb{N}; T \text{ stopping time}\}$$

is uniformly integrable, e.g. there is an integrable function $w \geq 0$ *such that*

$$\Delta(M^n) \geq -w \quad \text{resp. } |\Delta(M^n)| \leq w, \text{ a.s.}$$

then the process $(Z_t)_{t \in \mathbb{R}_+}$ *is increasing (resp. vanishes identically).*

Let us comment on these theorems. Theorem A shows that in the continuous case we may cut off some "small" singular parts in order to obtain a relatively weakly compact sequence $((M^n)^{T_n})_{n \geq 1}$ in $\mathcal{H}^1$. By taking convex combinations we then obtain a sequence that converges in the norm of $\mathcal{H}^1$. The singular parts are small enough so that they do not influence the almost sure passage to the limit. Note that—in general—there is no hope to get rid of the singular parts. Indeed, a Banach space E such that for each bounded sequence $(x_n)_{n\geq 1} \in E$ there is a norm-convergent sequence $y_n \in \operatorname{conv}(x_n, x_{n+1}, \dots)$ is reflexive; and, of course, $\mathcal{H}^1$ is only reflexive if it is finite-dimensional.

The general situation of local martingales M (possibly with jumps) described in theorem B is more awkward. As regards the convex combinations $(\sum_{k\geq n} \alpha_n^k \times H^k \mathbf{1}_{[\![0,T_n]\!]\cap(E^n)^c} \cdot M)_{n\geq 1}$ we have convergence in $\mathcal{H}^1$ but for the "singular" parts $(V^n)_{n\geq 1}$ we cannot assert that they tend to zero. Nevertheless there is some control on these processes. We may assert that the processes $(V^n)_{n\geq 1}$ tend, in a certain pointwise sense, to a process $(Z_t)_{t\in\mathbb{R}_+}$ *of integrable variation*. We shall give an example (section 3 below) which illustrates that in general one cannot do better than that. But under special assumptions, e.g., one-sided or two-sided bounds on the jumps of the processes $(H^n \cdot M)$, one may deduce certain features of the process Z (e.g., Z being monotone or vanishing identically). It is precisely this latter conclusion which has applications in Mathematical Finance and allows to give an alternative proof of Kramkov's recent "optional decomposition theorem" [K95] (see theorem 5.1 below).

To finish the introduction we shall state the main application of theorem B. Note that the subsequent statement of theorem D does not use the concept of $\mathcal{H}^1(\mathbb{P})$-martingales (although the proof heavily relies on this concept) which makes it more applicable in general situations.

1.10 Theorem D. *Let M be an $\mathbb{R}^d$-valued local martingale and $w \geq 1$ an integrable function. Given a sequence $(H^n)_{n\geq 1}$ of M-integrable $\mathbb{R}^d$-valued predictable processes such that*

$$(H^n \cdot M)_t \geq -w, \qquad \text{for all } n, t,$$

then there are convex combinations

$$K^n \in \operatorname{conv}\{H^n, H^{n+1}, \dots\},$$

and there is a super-martingale $(V_t)_{t\in\mathbb{R}_+}$, $V_0 \leq 0$, such that

(i) $$\lim_{\substack{s \to t, s\in\mathbb{Q}_+ \\ >}} \lim_{n\to\infty} (K^n \cdot M)_s = V_t \qquad \text{for } t \in \mathbb{R}_+, \text{ a.s.},$$

and an M-integrable predictable process H^0 such that

(ii) $$((H^0 \cdot M)_t - V_t)_{t\in\mathbb{R}_+} \qquad \text{is increasing.}$$

In addition, $H^0 \cdot M$ is a local martingale and a super-martingale.

Loosely speaking, theorem D says that for a sequence $(H^n \cdot M)_{n\geq 1}$, obeying the crucial assumption of uniform lower boundedness with respect to an integrable weight function w, we may pass—by forming convex combinations—to a limiting supermartingale V in a pointwise sense and—more importantly—to a local martingale of the form $(H^0 \cdot M)$ which dominates V.

The paper is organized as follows: Section 2 introduces notation and fixes general hypotheses. We also give a proof of the Kadeč–Pełczynski decomposition and we recall basic facts about weak compactness in $\mathcal{H}^1$. We give additional (and probably new) information concerning the convergence of the maximal function and the convergence of the square function. Section 3 contains an example. In section 4, we give the proofs of theorems A, B, C and D. We also reprove M. Yor's theorem 1.6. In section 5 we reprove Kramkov's optional decomposition theorem.

2. Notations and Preliminaries

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, where the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfies the "usual assumptions" of completeness and right continuity. We also assume that $\mathcal{F}$ equals $\mathcal{F}_\infty$. In principle, the letter M will be reserved for a càdlàg $\mathbb{R}^d$-valued local martingale. We assume that $M_0 = 0$ to avoid irrelevant difficulties at $t = 0$.

We denote by $\mathcal{O}$ (resp. $\mathcal{P}$) the sigma-algebra of optional (resp. predictable) subsets of $\mathbb{R}_+ \times \Omega$. For the notion of an M-integrable $\mathbb{R}^d$-valued predictable process $H = (H_t)_{t\in\mathbb{R}_+}$ and the notion of the stochastic integral

$$H \cdot M = \int_0^\cdot H_u \, dM_u$$

we refer to [P90] and to [Ja80]. Most of the time we shall assume that the process $H \cdot M$ is a local martingale (for the delicacy of this issue compare [E78] and [AS92]) and, in fact, a uniformly integrable martingale.

For the definition of the bracket process $[M, M]$ of the real-valued local martingale M as well as for the σ-finite, nonnegative measure $d[M, M]$ on the σ-algebra $\mathcal{O}$ of optional subsets of $\Omega \times \mathbb{R}_+$, we also refer to [P90]. In the case $d > 1$ the bracket process $[M, M]$ is defined as a matrix with components $[M^i, M^j]$ where $M = (M^1, \ldots, M^d)$. The process $[M, M]$ takes values in the cone of nonnegative definite $d \times d$ matrices. This is precisely the Kunita-Watanabe inequality for the bracket process. One can select representations so that for almost each $\omega \in \Omega$ the measure $d[M, M]$ induces a σ-finite measure, denoted by $d[M, M]_\omega$, on the Borel-sets of $\mathbb{R}_+$ (and with values in the cone of $d \times d$ nonnegative definite matrices).

For an $\mathbb{R}^d$-valued local martingale $X, X_0 = 0$, we define the $\mathcal{H}^1$-norm by

$$\|X\|_{\mathcal{H}^1} = \| \left(\operatorname{tr}([X, X]_\infty)\right)^{\frac{1}{2}} \|_{L^1(\Omega,\mathcal{F},\mathbb{P})} = \mathbb{E}\left[\left(\int_0^\infty d\left(\operatorname{tr}([X, X]_t)\right)\right)^{\frac{1}{2}}\right] \leq \infty$$

where tr denotes the trace of a $d \times d$-Matrix and the L^1-norm by

$$\|X\|_{L^1} = \sup_T \mathbb{E}\left[|X_T|\right] \leq \infty,$$

where $|\,.\,|$ denotes a fixed norm on $\mathbb{R}^d$, where the sup is taken over all finite stopping times T and which, in the case of a uniformly integrable martingale X, equals

$$\|X\|_{L^1} = \mathbb{E}\left[|X_\infty|\right] < \infty.$$

The Davis' inequality for $\mathcal{H}^1$ martingales ([RY94], theorem IV.4.1, see also [M76]) states that there are universal constants, c_1 and c_2 (only depending on the dimension d), such that for each $\mathcal{H}^1$ martingale X we have:

$$c_1 \|X^*_\infty\|_{L^1} \leq \|X\|_{\mathcal{H}^1} \leq c_2 \|X^*_\infty\|_{L^1},$$

where $X^*_u = \sup_{t \leq u} |X_t|$ denotes the maximal function.

We denote by $\mathcal{H}^1 = \mathcal{H}^1(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ and $L^1 = L^1(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ the Banach spaces of real-valued uniformly integrable martingales with finite $\mathcal{H}^1$- or L^1-norm respectively. Note that the space $L^1(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ may be isometrically identified with the space of integrable random variables $L^1(\Omega, \mathcal{F}, \mathbb{P})$ by associating to a uniformly integrable martingale X the random variable X_∞.

Also note that for a local martingale of the form $H \cdot M$ we have the formula

$$\begin{aligned}\|H \cdot M\|_{\mathcal{H}^1} &= \|\,[H \cdot M, H \cdot M]^{\frac{1}{2}}_\infty\,\|_{L^1(\Omega,\mathcal{F},\mathbb{P})} \\ &= \mathbb{E}\left[\left(\int_0^\infty H'_t d[M,M]_t H_t\right)^{\frac{1}{2}}\right],\end{aligned}$$

where H' denotes the transpose of H.

We now state and prove the result of Kadeč–Pełczynski [KP65] in a form that will be useful in the rest of our paper.

2.1 Theorem. *(Kadeč–Pełczynski) If $(f_n)_{n \geq 1}$ is an L^1-bounded sequence in the positive cone $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$, and g is a nonnegative integrable function, then there is a subsequence $(n_k)_{k \geq 1}$ as well as an increasing sequence of strictly positive numbers $(\beta_k)_{k \geq 1}$ such that β_k tend to ∞ and $(f_{n_k} \wedge (\beta_k(g+1)))_{k \geq 1}$ is uniformly integrable. The sequence $(f_{n_k} \wedge (\beta_k(g+1)))_{k \geq 1}$ is then relatively weakly compact by the Dunford–Pettis theorem.*

Proof. We adapt the proof of [KP65]. Without loss of generality we may suppose that the sequence $(f_n)_{n \geq 1}$ is bounded by 1 in L^1-norm but not uniformly integrable, i.e.,

$$\mathbb{E}[f_n] \leq 1; \quad \delta(\beta) = \sup_n \mathbb{E}[f_n - f_n \wedge \beta(g+1)] < \infty; \quad 0 < \delta(\infty) = \inf_{\beta > 0} \delta(\beta)$$

(it is an easy exercise to show that $\delta(\infty) = 0$ implies uniform integrability). For $k = 1$ and $\beta_1 = 1$ we select n_1 so that $\mathbb{E}[f_{n_1} - f_{n_1} \wedge \beta_1(g+1)] > \delta(\infty)/2$. Having chosen $n_1, n_2, \ldots, n_{k-1}$ as well as ¿$\beta_1, \beta_2, \ldots, \beta_{k-1}$ we put $\beta_k = 2\beta_{k-1}$ and we select $n_k > n_{k-1}$ so that $\mathbb{E}[f_{n_k} - f_{n_k} \wedge \beta_k(g+1)] > (1 - 2^{-k})\delta(\infty)$. The sequence $(f_{n_k} \wedge \beta_k(g+1))_{k\geq 1}$ is now uniformly integrable. To see this, let us fix K and let $k(K)$ be defined as the smallest number k such that $\beta_k > K$. Clearly $k(K) \to \infty$ as K tends to ∞. For $l < k(K)$ we then have that $f_{n_l} \wedge \beta_l(g+1) = f_{n_l} \wedge \beta_l(g+1) \wedge K(g+1)$, whereas for $l \geq k(K)$ we have

$$\begin{aligned}
&\mathbb{E}[f_{n_l} \wedge \beta_l(g+1) - f_{n_l} \wedge \beta_l(g+1) \wedge K(g+1)] \\
&\quad = \mathbb{E}[f_{n_l} - f_{n_l} \wedge K(g+1)] - \mathbb{E}[f_{n_l} - f_{n_l} \wedge \beta_l(g+1)] \\
&\quad \leq \delta(K) - \left(\delta(\infty) - \frac{\delta(\infty)}{2^{k(K)}}\right) \\
&\quad \leq \delta(\infty) - \delta(K) + \frac{\delta(\infty)}{2^{k(K)}}
\end{aligned}$$

The latter expression clearly tends to 0 as $K \to \infty$.

q.e.d.

Corollary. *If the sequence β_k is such that $f_{n_k} \wedge \beta_k(g+1)$ is uniformly integrable, then there also exists a sequence γ_k such that γ_k/β_k tends to infinity and such that the sequence $f_{n_k} \wedge \gamma_k(g+1)$ remains uniformly integrable.*

Proof. In order to show the existence of γ_k we proceed as follows. The sequence

$$h_k = \beta_k(g+1)\mathbf{1}_{\{f_{n_k} \geq \beta_k(g+1)\}}$$

tends to zero in $L^1(\mathbb{P})$, since the sequence $f_{n_k} \wedge \beta_k(g+1)$ is uniformly integrable and $\mathbb{P}[f_{n_k} \geq \beta_k(g+1)] \leq 1/\beta_k \to 0$. Let now α_k be a sequence that tends to infinity but so that $\alpha_k h_k$ still tends to 0 in $L^1(\mathbb{P})$. If we define $\gamma_k = \alpha_k \beta_k$ we have that

$$f_{n_k} \wedge \gamma_k(g+1) \leq f_{n_k} \wedge \beta_k(g+1) + \alpha_k h_k$$

and hence we obtain the uniform integrability of the sequence $f_{n_k} \wedge \gamma_k(g+1)$.

q.e.d.

Remark. In most applications of the Kadeč–Pełczynski decomposition theorem, we can take $g = 0$. However in section 4, we will need the easy generalisation to the case where g is a non-zero integrable nonnegative function. The general case can in fact be reduced to the case $g = 0$ by replacing the functions f_n by $f_n/(g+1)$ and by replacing the measure $\mathbb{P}$ by the probability measure $\mathbb{Q}$ defined as $d\mathbb{Q} = \frac{(g+1)}{\mathbb{E}[g+1]} d\mathbb{P}$.

Remark. We will in many cases drop indices like n_k and simply suppose that the original sequence $(f_n)_{n\geq 1}$ already satisfies the conclusions of the theorem. In most cases such passing to a subsequence is allowed and we will abuse this simplification as many times as possible.

Remark. The sequence of sets $\{f_n > \beta_n(g+1)\}$ is, of course, not necessarily a disjoint sequence. In case we need two by two disjoint sets we proceed as follows. By selecting a subsequence we may suppose that $\sum_{n>k} \mathbb{P}[f_n > \beta_n(g+1)] \le \varepsilon_k$, where the sequence of strictly positive numbers $(\varepsilon_k)_{k\ge1}$ is chosen in such a way that $\int_B f_k \, d\mathbb{P} < 2^{-k}$ whenever $\mathbb{P}[B] < \varepsilon_k$. It is now easily seen that the sequence of sets $(A_n)_{n\ge1}$ defined by $A_n = \{f_n > \beta_n(g+1)\} \setminus \bigcup_{k>n}\{f_k > \beta_k(g+1)\}$ will do the job.

As a first application of the Kadeč–Pełczynski decomposition we prove the vector-valued Komlos-type theorem stated in the introduction:

1.4 Theorem. *If E is a reflexive Banach space and $(f_n)_{n\ge1}$ a bounded sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ we may find convex combinations*

$$g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$$

and $g_0 \in L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ such that $(g_n)_{n\ge1}$ converges to g_0 almost surely, i.e.,

$$\lim_{n\to\infty} \|g_n(\omega) - g_0(\omega)\|_E = 0, \qquad \text{for a.e. } \omega \in \Omega.$$

Proof. By the remark made above there is a subsequence, still denoted by $(f_n)_{n\ge1}$ as well as a sequence $(A_n)_{n\ge1}$ of mutually disjoint sets such that the sequence $\|f_n\|\mathbf{1}_{A_n^c}$ is uniformly integrable. By a well known theorem on $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ of a *reflexive space E*, [DU77], see also [DRS93], the sequence $(f_n\mathbf{1}_{A_n^c})_{n\ge1}$ is therefore relatively weakly compact in $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$. Therefore (see theorem 1.2 above) there is a sequence of convex combinations $h_n \in \operatorname{conv}\{f_n\mathbf{1}_{A_n^c}, f_{n+1}\mathbf{1}_{A_{n+1}^c}, \dots\}$, $h_n = \sum_{k\ge n} \alpha_n^k f_k \mathbf{1}_{A_k^c}$ such that h_n converges to a function g_0 with respect to the norm of $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$. Since the sequence $f_n\mathbf{1}_{A_n}$ converges to zero a.s. we have that the sequence $g_n = \sum_{k\ge n} \alpha_n^k f_k$ converges to g_0 in probability. If needed one can take a further subsequence that converges a.s., i.e., $\|g_n(\omega) - g_0(\omega)\|_E$ tends to zero for almost each ω.

q.e.d.

The preceding theorem allows us to give an alternative proof of lemma 4.2 in Kramkov, [K95].

2.2 Lemma. *Let $(N^n)_{n\ge1}$ be a sequence of adapted càdlàg stochastic processes, $N_0^n = 0$, such that*

$$\mathbb{E}[\operatorname{var} N^n] \le 1, \qquad n \in \mathbb{N},$$

where $\operatorname{var} N^n$ denotes the total variation of the process N^n. Then there is a sequence $R^n \in \operatorname{conv}(N^n, N^{n+1} \dots)$ and an adapted càdlàg stochastic process $Z = (Z_t)_{t\in\mathbb{R}_+}$ such that

$$\mathbb{E}[\operatorname{var} Z] \le 1$$

and such that almost surely the measure dZ_t, defined on the Borel sets of $\mathbb{R}_+$, is the weak limit of the sequence dR_t^n. In particular we have that*

$$Z_t = \lim_{\substack{s \to t \\ >}} \limsup_{n\to\infty} R_s^n = \lim_{\substack{s \to t \\ >}} \liminf_{n\to\infty} R_s^n.$$

Proof. We start the proof with some generalities of functional analysis that will allow us to reduce the statement to the setting of theorem 1.4.

The space of finite measures $\mathcal{M}$ on the Borel sets of $\mathbb{R}_+$ is the dual of the space $\mathcal{C}_0$ of continuous functions on $\mathbb{R}_+ = [0, \infty[$, tending to zero at infinity. If $(f_k)_{k=1}$ is a dense sequence in the unit ball of $\mathcal{C}_0$, then for bounded sequences $(\mu_n)_{n\geq 1}$ in $\mathcal{M}$, the weak* convergence of the sequence μ_n is equivalent to the convergence, for each k, of $\int f_k \, d\mu_n$. The mapping $\Phi(\mu) = (2^{-k} \int f_k \, d\mu)_{k\geq 1}$ maps the space of measures into the space l^2. The image of a bounded weak* closed convex set is closed in l^2. Moreover on bounded subsets of $\mathcal{M}$, the weak* topology coincides with the norm topology of its image in l^2.

For each n the càdlàg process N^n of finite variation can now be seen as a function of Ω into $\mathcal{M}$, mapping the point ω onto the measure $dN_t^n(\omega)$. Using theorem 1.3, we may find convex combinations $P^n \in \text{conv}(N^n, N^{n+1}, \dots), P^n = \sum_{k\geq n} \alpha_n^k N^k$ such that the sequence $\sum_{k\geq n} \alpha_n^k \, \text{var}(N^k)$ converges a.s.. This implies that a.s. the sequence $P^n(\omega)$ takes its values in a bounded set of $\mathcal{M}$. Using theorem 1.4 on the sequence $(\Phi(P^n))_{n\geq 1}$ we find convex combinations $R^n = \sum_{k\geq n} \beta_n^k P^k$ of $\left(P^k\right)_{k\geq n}$ such that the sequence $\Phi(dR^n) = \Phi(\sum_{k\geq n} \beta_n^k dP_t^k)$ converges a.s.. Since a.s. the sequence of measures $dR^n(\omega)$ takes its values in a bounded set of $\mathcal{M}$, the sequence $dR_t^n(\omega)$ converges a.s. weak* to a measure $dZ_t(\omega)$. The last statement is an obvious consequence of the weak* convergence. It is also clear that Z is optional and that $\mathbb{E}[\text{var}(Z)] \leq 1$.

q.e.d.

Remark. If we want to obtain the process Z as a limit of a sequence of processes then we can proceed as follows. Using once more convex combinations together with a diagonalisation argument, we may suppose that R_s^n converges a.s. for each rational s. In this case we can write that a.s. $Z_t = \lim_{\substack{s \to t, s\in\mathbb{Q} \\ >}} \lim_{n\to\infty} R_s^n$. We will use such descriptions in section 4 and 5.

Remark. Even if the sequence N^n consists of predictable processes, the process Z need not be predictable. Take e.g. T a totally inaccessible stopping time and let N^n describe the point mass at $T + \frac{1}{n}$. Clearly this sequence tends, in the sense described above, to the process $\mathbf{1}_{[\![T,\infty[\![}$, i.e. the point mass concentrated at time T, a process which fails to be predictable. Also in general, there is no reason that the process Z should start at 0.

Remark. It might be useful to observe that if T is a stopping time such that Z is continuous at T, i.e. $\Delta Z_T = 0$, then a.s. $Z_T = \lim R_T^n$.

We next recall well known properties on weak compactness in $\mathcal{H}^1$. The results are due to Dellacherie, Meyer and Yor, (see [DMY78]).

2.3 Theorem. *For a family $(M^i)_{i\in I}$ of elements of $\mathcal{H}^1$ the following assertions are equivalent:*

(1) *the family is relatively weakly compact in $\mathcal{H}^1$*
(2) *the family of square functions $([M^i, M^i]_\infty^{1/2})_{i\in I}$ is uniformly integrable*
(3) *the family of maximal functions $((M^i)^*_\infty)_{i\in I}$ is uniformly integrable.*

q.e.d.

This theorem immediately implies the following:

2.4 Theorem. *If $(N^n)_{n\geq 1}$ is a relatively weakly compact sequence in $\mathcal{H}^1$, if $(H^n)_{n\geq 1}$ is a uniformly bounded sequence of predictable processes with $H^n \to 0$ pointwise on $\mathbb{R}_+ \times \Omega$, then $H^n \cdot N^n$ tends weakly to zero in $\mathcal{H}^1$.*

Proof. We may and do suppose that $|H^n| \leq 1$ and $\|N^n\|_{\mathcal{H}^1} \leq 1$ for each n. For each n and each $\varepsilon > 0$, we define E^n as the predictable set $E^n = \{|H^n| > \varepsilon\}$. We split the stochastic integrals $H^n \cdot N^n$ as $(\mathbf{1}_{E^n} H^n) \cdot N^n + \left(\mathbf{1}_{(E^n)^c} H^n\right) \cdot N^n$. We will show that the first terms form a sequence that converges to 0 weakly. Because obviously $\| \left(\mathbf{1}_{(E^n)^c} H^n\right) \cdot N^n\|_{\mathcal{H}^1} \leq \varepsilon$, the theorem follows.

From the previous theorem it follows that the sequence $(H^n \mathbf{1}_{E^n} \cdot N^n)_{n\geq 1}$ is already weakly relatively compact in $\mathcal{H}^1$. Clearly $\mathbf{1}_{E^n} \to 0$ pointwise. It follows that $F^n = \bigcup_{k\geq n} E^n$ decreases to zero as n tends to ∞. Let N be a weak limit point of the sequence $((H^k \mathbf{1}_{E^k}) \cdot N^k)_{k\geq 1}$. We have to show that $N = 0$. For each $k \geq n$ we have that $\mathbf{1}_{F^n} \cdot ((H^k \mathbf{1}_{E^k}) \cdot N^k) = (H^k \mathbf{1}_{E^k}) \cdot N^k$. From there it follows that $\mathbf{1}_{F^n} \cdot N = N$ and hence by taking limits as $n \to \infty$, we also have $N = \mathbf{1}_\emptyset \cdot N = 0$.

q.e.d.

Related to the Davis' inequality, is the following lemma, due to Garsia and Chou, (see [G73] p. 34–41 and [N75] p. 198 for the discrete time case); the continuous time case follows easily from the discrete case by an application of Fatou's lemma. The reader can also consult [M76] p. 351, (31.6) for a proof in the continuous time case.

2.5 Lemma. *There is a constant c such that, for each $\mathcal{H}^1$-martingale X, we have*

$$\mathbb{E}\left[\frac{[X,X]_\infty}{X^*_\infty}\right] \leq c\|X\|_{\mathcal{H}^1}.$$

This inequality together with an interpolation technique yields:

2.6 Theorem. *There is a constant C such that for each $\mathcal{H}^1$-martingale X and for each $0 < p < 1$ we have:*

$$\|[X,X]_\infty^{1/2}\|_p \leq C\|X\|_{\mathcal{H}^1}^{1/2} \|X^*_\infty\|_{\frac{p}{2-p}}^{1/2}.$$

Proof. The following series of inequalities is an obvious application of the preceding lemma and Hölder's inequality for the exponents $\frac{2}{p}$ and $\frac{2}{2-p}$. The constant c is the same as in the preceding lemma.

$$\begin{aligned}\mathbb{E}\left[[X,X]_\infty^{\frac{p}{2}}\right] &= \mathbb{E}\left[(X_\infty^*)^{\frac{p}{2}}\left(\frac{[X,X]_\infty}{X_\infty^*}\right)^{\frac{p}{2}}\right]\\ &\le \left(\mathbb{E}\left[\frac{[X,X]_\infty}{X_\infty^*}\right]\right)^{\frac{p}{2}}\left(\mathbb{E}\left[(X_\infty^*)^{\frac{p}{2-p}}\right]\right)^{\frac{2-p}{2}}\\ &\le c^{\frac{p}{2}}\|X\|_{\mathcal{H}^1}^{\frac{p}{2}}\|X_\infty^*\|_{\frac{p}{2-p}}^{\frac{p}{2}}.\end{aligned}$$

Hence

$$\|[X,X]_\infty^{1/2}\|_p \le c^{1/2}\|X\|_{\mathcal{H}^1}^{1/2}\|X_\infty^*\|_{\frac{p}{2-p}}^{1/2}.$$

q.e.d.

2.7 Corollary. *If X^n is a sequence of $\mathcal{H}^1$-martingales such that $\|X^n\|_{\mathcal{H}^1}$ is bounded and such that $(X^n)_\infty^*$ tends to zero in probability, then $[X^n,X^n]_\infty$ tends to zero in probability. In fact, for each $p<1$, $(X^n)_\infty^*$ as well as $[X^n,X^n]_\infty^{1/2}$ tend to zero in the quasi-norm of $L^p(\Omega,\mathcal{F},\mathbb{P})$.*

Proof. Fix $0<p<1$. Obviously we have by the uniform integrability of the sequence $\left((X^n)_\infty^*\right)^{\frac{p}{2-p}}$, that $\|\,(X^n)_\infty^*\,\|_{\frac{p}{2-p}}$ converges to zero. It then follows from the theorem that also $[X^n,X^n]_\infty \to 0$ in probability.

q.e.d.

Remark. It is well known that, for $0\le p<1$, there is no connection between the convergence of the maximal function and the convergence of the bracket, [MZ38], [BG70], [M94]. But as the theorem shows, for *bounded* sets in $\mathcal{H}^1$ the situation is different. The convergence of the maximal function implies the convergence of the bracket. The result also follows from the result on convergence in law as stated in corollary 6.7, p. 342 in [JS87] . This was kindly pointed out to us by A. Shiryaev. The converse of our corollary 2.7 is not true as the example in the next section shows. In particular the relation between the maximal function and the bracket is not entirely symmetric in the present context.

Remark. In the case of *continuous* martingales there is also an inverse inequality of the type

$$\mathbb{E}\left[\frac{(X_\infty^*)^2}{[X,X]_\infty^{1/2}}\right] \le c\|X\|_{\mathcal{H}^1}.$$

The reader can consult [RY94], ex. 4.17 and 4.18 p. 160.

3. An example

3.1 Example. *There is a uniformly bounded martingale $M = (M_t)_{t\in[0,1]}$ and a sequence $(H^n)_{n\geq 1}$ of M-integrands satisfying*

$$\|H^n \cdot M\|_{\mathcal{H}^1} \leq 1, \qquad \text{for } n \in \mathbb{N},$$

and such that

(1) *for each $t \in [0,1]$ we have*

$$\lim_{n\to\infty} (H^n \cdot M)_t = -t/2 \qquad a.s.$$

(2) $[H^n \cdot M, H^n \cdot M]_\infty \to 0$ *in probability.*

Proof. Fix a collection $((\varepsilon_{n,k})_{k=1}^{2^{n-1}})_{n\geq 1}$ of independent random variables,

$$\varepsilon_{n,k} = \begin{cases} -2^{-n} & \text{with probability } (1-4^{-n}) \\ 2^n(1-4^{-n}) & \text{with probability } 4^{-n} \end{cases}$$

so that $\mathbb{E}[\varepsilon_{n,k}] = 0$. We construct a martingale M such that at times

$$t_{n,k} = \frac{2k-1}{2^n}, \qquad n \in \mathbb{N},\ k = 1, \dots, 2^{n-1},$$

M jumps by a suitable multiple of $\varepsilon_{n,k}$, e.g.

$$M_t = \sum_{(n,k): t_{n,k} \leq t} 8^{-n} \varepsilon_{n,k}, \qquad t \in [0,1],$$

so that M is a welldefined uniformly bounded martingale (with respect to its natural filtration).

Defining the integrands H^n by

$$H^n = \sum_{k=1}^{2^{n-1}} 8^n \chi_{\{t_{n,k}\}}, \qquad n \in \mathbb{N},$$

we obtain, for fixed $n \in \mathbb{N}$,

$$(H^n \cdot M)_t = \sum_{k: t_{n,k} \leq t} \varepsilon_{n,k},$$

so that $H^n \cdot M$ is constant on the intervals $[\frac{2k-1}{2^n}, \frac{2k+1}{2^n}[$ and, on a set of probability bigger that $1-2^{-n}$, $H \cdot M$ equals $-\frac{k}{2^n}$ on the intervals $[\frac{2k-1}{2^n}, \frac{2k+1}{2^n}[$. Also on a set of probability bigger than $1-2^{-n}$ we have that $[H^n \cdot M, H^n \cdot M]_1 = \sum_{k=1}^{2^{n-1}} 2^{-2n} = 2^{-n-1}$.

From the Borel–Cantelli lemma we infer that, for each $t \in [0,1]$, the random variables $(H^n \cdot M)_t$ converge almost surely to the constant function $-t/2$ and that $[H^n \cdot M, H^n \cdot M]_1$ tend to 0 a.s., which proves the final assertions of the above claim.

We still have to estimate the $\mathcal{H}^1$-norm of $H^n \cdot M$:

$$\begin{aligned} \|H^n \cdot M\|_{\mathcal{H}^1} &\leq \sum_{k\geq 1}^{2^{n-1}} \|\varepsilon_{n,k}\|_{L^1} \\ &= 2^{n-1}[2^{-n}(1-4^{-n}) + 2^n(1-4^{-n}) \cdot 4^{-n}] \leq 1. \end{aligned}$$

q.e.d.

3.2 Remark. What is the message of the above example? First note that passing to convex combinations $(K^n)_{n\geq 1}$ of $(H^n)_{n\geq 1}$ does not change the picture: we always end up with a sequence of martingales $(K^n \cdot M)_{n\geq 1}$ bounded in $\mathcal{H}^1$ and such that the pointwise limit equals $Z_t = -t/2$. Of course, the process Z is far from being a martingale.

Hence, in the setting of theorem B, we cannot expect (contrary to the setting of theorem A) that the sequence of martingales $(K^n \cdot M)_{n\geq 1}$ converges in some pointwise sense to a martingale. We have to allow that the singular parts ${}^sK^n \cdot M$ converge (pointwise a.s.) to some process Z; the crucial information about Z is that Z is of integrable variation and—in the case of jumps uniformly bounded from below as in the preceding example—decreasing.

4. A substitute of compactness for bounded subsets of $\mathcal{H}^1$.

This section is devoted to the proof of theorems A, B, C, D as well as Yor's Theorem 1.6.

Because of the technical character of this section, let us give an overview of its contents. We start with some generalities that allow the sequence of martingales to be replaced by a more suitable subsequence. This (obvious) preparation is done in the next paragraph. In subsection a, we then give the proof of Theorem A, i.e. the case of continuous martingales. Because of the continuity, stopping arguments can easily be used. We stop the martingales as soon as the maximal functions reach a level that is given by the Kadeč–Pełczynski decomposition theorem. Immediately after the proof of Theorem A, we give some corollaries as well as a negative result that shows that boundedness in $\mathcal{H}^1$ is needed instead of the weaker boundedness in L^1. We end the subsection a with a remark that shows that the proof of the continuous case can be adapted to the case where the set of jumps of all the martingales form a uniformly integrable family. Roughly speaking this case can be handled in the same way as the continuous case. Subsection b then gives the proof of Theorem C. We proceed in the same way as in the continuous case, i.e. we stop when the maximal function of the martingales reaches a certain level. Because this time we did not assume that the jumps are uniformly integrable we have to proceed with more care and eliminate their big parts (the "singular" parts in the Kadeč–Pełczynski decomposition). Subsection c then treats the case where all the martingales are stochastic integrals, $H^n \cdot M$, with respect to a given d-dimensional local martingale M. This part is the most technical one as we want the possible decompositions to be done on the level of the integrands H^n. We cannot proceed in the same way as in Theorem C, although the idea is more or less the same. Yor's theorem is then (re)proved in subsection d. Subsection e is devoted to the proof of Theorem D. The reader who does not want to go through all the technicalities can limit her first reading to subsections a, b, d and only read the statements of the theorems and lemmata in the other subsections c and e.

By $(M^n)_{n\geq 1}$ we denote a bounded sequence of martingales in $\mathcal{H}^1$. Without loss of generality we may suppose that $\|M^n\|_{\mathcal{H}^1} \leq 1$ for all n. By the Davis' inequality this implies the existence of a constant $c < \infty$ such that for all n: $\mathbb{E}[(M^n)^*] \leq c$. From the Kadeč–Pełczynski decomposition theorem we deduce the existence of a sequence $(\beta_n)_{n\geq 1}$, tending to ∞ and such that $(M^n)^* \wedge \beta_n$ is uniformly integrable. The reader should note that we replaced the original sequence by a subsequence. Passing to a subsequence once more also allows to suppose that $\sum \frac{1}{\beta_n} < \infty$. For each n we now define

$$\tau_n = \inf\{t \mid |M^n_t| > \beta_n\}.$$

Clearly $\mathbb{P}[\tau_n < \infty] \leq \frac{c}{\beta_n}$ for some constant c. If we let $T_n = \inf_{k\geq n} \tau_k$ we obtain an increasing sequence of stopping times $(T_n)_{n\geq 1}$ such that $\mathbb{P}[T_n < \infty] \leq \sum_{k\geq n} \frac{c}{\beta_k}$ and hence tends to zero. Let us now start with the case of continuous martingales.

a. Proof of theorem A. *The case when the martingales M^n are continuous.*

By the definition of the stopping times T_n, we obtain that $((M^n)^{T_n})^* \leq (M^n)^* \wedge \beta_n$ and hence the sequence $((M^n)^{T_n})_{n\geq 1}$ forms a relatively weakly compact sequence in $\mathcal{H}^1$. Also the maximal functions of the remaining parts $M^n - (M^n)^{T_n}$ tend to zero a.s.. As a consequence we obtain the existence of convex combinations $N^n = \sum_{k\geq n} \alpha^k_n (M^k)^{T_k}$ that converge in $\mathcal{H}^1$-norm to a continuous martingale M^0. We also have that $R^n = \sum_{k\geq n} \alpha^k_n M^k$ converge to M^0 in the semi-martingale topology and that $(M^0 - R^n)^*_\infty$ tends to zero in probability. From corollary 2.7 in section 2 we now easily derive that $[M^0 - R^n, M^0 - R^n]_\infty$ as well as $(M^0 - R^n)^*_\infty$ tend to zero in L^p, for each $p < 1$.

If all the martingales M^n are of the form $H^n \cdot M$ for a fixed continuous $\mathbb{R}^d$-valued local martingale M, then of course the element M^0 is of the same form. This follows from Yor's theorem 1.6, stating that the space of stochastic integrals with respect to M, is a closed subspace of $\mathcal{H}^1$. This concludes the proof of theorem A.

q.e.d.

4.1 Corollary. *If $(M^n)_{n\geq 1}$ is a sequence of continuous $\mathcal{H}^1$-martingales such that*

$$\sup_n \|M^n\|_{\mathcal{H}^1} < \infty \text{ and } M^n_\infty \to 0 \text{ in probability,}$$

then M^n tends to zero in the semi-martingale topology. As a consequence we have that $(M^n)^* \to 0$ in probability.

Proof. Of course we may take subsequences in order to prove the statement. So let us take a subsequence as well as stopping times as described in theorem A. The sequence $(M^n)^{T_n}$ is weakly relatively compact in $\mathcal{H}^1$ and since $M^n_{T_n}$ tendsto zero in probability (because $\mathbb{P}[T_n < \infty]$ tends to zero and M^n_∞ tends to zero in probability), we easily see that $M^n_{T_n}$ tends to zero in L^1. Doob's maximum inequality then implies that $((M^n)^{T_n})^*$ tends to zero in probability. It is then obvious that also $(M^n)^*$ tends to zero in probability.

Because $((M^n)^{T_n})^*$ tends to zero in probability and because this sequence is uniformly integrable, we deduce that the sequence $(M^n)^{T_n}$ tends to zero in $\mathcal{H}^1$. The sequence M^n therefore tends to zero in the semi-martingale topology.

q.e.d.

Remark. The above corollary, together with Theorem 2.6, show that M^n tends to zero in $\mathcal{H}^p$ (i.e. $(M^n)^*$ tends to zero in L^p) and in h^p (i.e. $[M^n, M^n]^{1/2}_\infty$ tends to zero in L^p) for each $p < 1$. For continuous local martingales however, $\mathcal{H}^p$ and h^p are the same.

Remark. That we actually need that the sequence M^n is bounded in $\mathcal{H}^1$, and not just in L^1, is illustrated in the following "negative" result.

4.2 Lemma. *Suppose that $(M^n)_{n\geq 1}$ is a sequence of continuous, nonnegative, uniformly integrable martingales such that $M^n_0 = 1$ and such that $M^n_\infty \to 0$ in probability. Then $\|M^n\|_{\mathcal{H}^1} \to \infty$.*

Proof. For $\beta > 1$ we define $\sigma_n = \inf\{t \mid M^n_t > \beta\}$. Since

$$1 = \mathbb{E}\left[M^n_{\sigma_n}\right] = \beta\mathbb{P}[\sigma < \infty] + \int_{\{(M^n)^* \leq \beta\}} M^n_\infty,$$

we easily see that $\lim_{n\to\infty} \mathbb{P}[\sigma_n < \infty] = 1/\beta$. It follows from the Davis' inequality that $\lim_n \|M^n\|_{\mathcal{H}^1} \geq c \lim_n \int_0^\infty \mathbb{P}[\sigma_n > \beta]\, d\beta = \infty$.

q.e.d.

Remark. There are two cases where theorem A can easily be generalised to the setting of $\mathcal{H}^1$-martingales with jumps. Let us describe these two cases separately. The first case is when the set

$$\{\Delta M^n_\sigma \mid n \geq 1, \;\; \sigma \text{ a stopping time}\}$$

is uniformly integrable. Indeed, using the same definition of the stopping times T_n we arrive at the estimate

$$(M^n)^*_{T_n} \leq (M^n)^* \wedge \beta_n + \left|\Delta M^n_{T_n}\right|.$$

Because of the hypothesis on the uniform integrability of the jumps and by the selection of the sequence β_n we may conclude that the sequence $((M^n)^{T_n})_{n\geq 1}$ is relatively weakly compact in $\mathcal{H}^1$. The corollary generalises in the same way.

The other generalisation is when the set

$$\{M^n_\infty \mid n \geq 1\}$$

is uniformly integrable. In this case the set

$$\{M^n_\sigma \mid n \geq 1, \; \sigma \text{ a stopping time}\}$$

is, as easily seen, also uniformly integrable. The maximal function of the stopped martingale $(M^n)^{T_n}$ is bounded by

$$\left((M^n)^{T_n}\right)^* \leq \max\left((M^n)^* \wedge \beta_n, |M^n_{T_n}|\right).$$

It is then clear that they form a uniformly integrable sequence. It is this situation that arises in the proof of M. Yor's theorem.

b. Proof of theorem C. *The case of an $\mathcal{H}^1$-bounded sequence M^n of càdlàg martingales.*

We again turn to the general situation. In this case we cannot conclude that the stopped martingales $(M^n)^{T_n}$ form a relatively weakly compact set in $\mathcal{H}^1$. Indeed the size of the jumps at times T_n might be too big. In order to remedy this situation we will compensate these jumps in order to obtain martingales that have "smaller" jumps at these stopping times T_n. For each n we denote by C^n the dual predictable projection of the process $(\Delta M^n)_{T_n} \mathbf{1}_{[\![T_n,\infty[\![}$. The process C^n is predictable and has integrable variation

$$\mathbb{E}[\operatorname{var} C^n] \leq \mathbb{E}[|(\Delta M^n)_{T_n}|] \leq 2c.$$

The Kadeč–Pełczynski decomposition 2.1 above yields the existence of a sequence η_n tending to ∞, $\sum_{n\geq 1} \frac{1}{\eta_n} < \infty$ and such that $(\operatorname{var} C^n) \wedge \eta_n$ forms a uniformly integrable sequence (again we replaced the original sequence by a subsequence). For each n we now define the *predictable* stopping time σ_n as

$$\sigma_n = \inf\{t \mid \operatorname{var} C^n_t \geq \eta_n\}.$$

Because the process C^n stops at time T_n we necessarily have that $\sigma_n \leq T_n$ on the set $\{\sigma_n < \infty\}$.

We remark that when X is a martingale and when ν is a predictable stopping time, then the process stopped at $\nu-$ and defined by $X^{\nu-}_t = X_t$ for $t < \nu$ and $X^{\nu-}_t = X_{\nu-}$ for $t \geq \nu$, is still a martingale.

Let us now turn our attention to the sequence of martingales

$$N^n = \left((M^n)^{T_n} - \left((\Delta M^n)_{T_n} \mathbf{1}_{[\![T_n,\infty[\![} - C^n\right)\right)^{\sigma_n -}.$$

The processes N^n can be rewritten as

$$N^n = \left((M^n)^{T_n}\right)^{\sigma_n} - \left(\Delta\left(M^n\right)\right)_{\sigma_n} \mathbf{1}_{[\![\sigma_n,\infty[\![} - (\Delta M^n)_{T_n} \mathbf{1}_{\{\sigma_n=\infty\}}\mathbf{1}_{[\![T_n,\infty[\![} + (C^n)^{\sigma_n -},$$

or which is the same:

$$N^n = (M^n)^{T_n\wedge\sigma_n} - \left(\Delta\left(M^n\right)\right)_{T_n\wedge\sigma_n} \mathbf{1}_{[\![T_n\wedge\sigma_n,\infty[\![} + (C^n)^{\sigma_n -}.$$

The maximal functions satisfy

$$(N^n)^* \leq (M^n)^* \wedge \beta_n + (\operatorname{var} C^n) \wedge \eta_n$$

and hence form a uniformly integrable sequence. It follows that the sequence N^n is a relatively weakly c compact sequence in $\mathcal{H}^1$. Using the appropriate convex combinations will then yield a limit M^0 in $\mathcal{H}^1$.

The problem is that the difference between M^n and N^n does not tend to zero in any reasonable sense as shown by example 3.1 above. Let us therefore analyse this difference:

$$M^n - N^n = M^n - (M^n)^{T_n\wedge\sigma_n} + (\Delta M^n)_{T_n\wedge\sigma_n}\mathbf{1}_{[\![T_n\wedge\sigma_n,\infty[\![} - (C^n)^{\sigma_n -}.$$

The maximal function of the first part

$$\left(M^n - ((M^n)^{T_n\wedge\sigma_n})\right)^*,$$

tends to zero a.s. because of $\mathbb{P}[T_n < \infty]$ and $\mathbb{P}[\sigma_n < \infty]$ both tending to zero. The same argument yields that the maximal function of the second part

$$\left((\Delta M^n)_{T_n\wedge\sigma_n}\mathbf{1}_{[\![T_n\wedge\sigma_n,\infty[\![}\right)^*$$

also tends to zero. The remaining part is $(-C^n)^{\sigma_n -}$. Applying theorem 2.2 then yields convex combinations that converge in the sense of theorem 2.2 to a càdlàg process of finite variation Z.

Summing up, we can find convex coefficients $(\alpha_n^k)_{k\geq n}$ such that the martingales $\sum_{k\geq n} \alpha_n^k N^n$ will converge in $\mathcal{H}^1$-norm to a martingale M^0 and such that, at the same time, $\sum_{k\geq n} \alpha_n^k C^n$ converge to a process of finite variation Z, in the sense described in lemma 2.2.

In the case where the jumps ΔM^n are bounded below by an integrable function w, or more generally when the set

$$\left\{ \Delta(M^n)^-_\zeta \mid n \geq 1 \ \zeta \text{ stopping time} \right\}$$

is uniformly integrable, we do not have to compensate the negative part of these jumps. So we replace $(\Delta M^n)_{T^n}$ by the more appropriate $((\Delta M^n)_{T^n})^+$. In this case their compensators C^n are increasing and therefore the process Z is decreasing.

The case where the jumps form a uniformly integrable family is treated in the remark after the proof of theorem A. The proof of theorem C is therefore completed.

q.e.d.

c. Proof of theorem B. *The case where all martingales are of the form $M^n = H^n \cdot M$.*

This situation requires, as we will see, some extra work. We start the construction as in the previous case but this time we work with the square functions, i.e., the brackets instead of the maximal functions.

Without loss of generality we may suppose that M is an $\mathcal{H}^1$- martingale. Indeed let $(\mu_n)_{n\geq 1}$be a sequence of stopping times that localises the local martingale M in such a way that the stopped martingales M^{μ_n} are all in $\mathcal{H}^1$. Take now a sequence of strictly positive numbers a_n such that $\sum_n a_n \|M^{\mu_n}\|_{\mathcal{H}^1} < \infty$, put $\mu_0 = 0$ and replace M by the $\mathcal{H}^1$-martingale:

$$\sum_{n\geq 1} a_n \left(M^{\mu_n} - M^{\mu_{n-1}} \right).$$

The integrands have then to be replaced by the integrands

$$\sum_{k\geq 1} \frac{1}{a_k} H^n \mathbf{1}_{]\!]\mu_{k-1},\mu_k]\!]}.$$

In conclusion, we may assume w.l.g., that M is in $\mathcal{H}^1$.

Also without loss of generality we may suppose that the predictable integrands are bounded. Indeed for each n we can take κ_n big enough so that

$$\| \left(H^n \mathbf{1}_{\{|H^n|\geq \kappa_n\}} \right) \cdot M \|_{\mathcal{H}^1} < 2^{-n}.$$

It is now clear that it is sufficient to prove the theorem for the sequence of integrands $H\mathbf{1}_{\{\|H^n\|\le\kappa_n\}}$. So we suppose that for each n we have $|H^n|\le\kappa_n$.

We apply the Kadeč–Pełczynski construction of theorem 2.1 with the function $g=(\mathrm{tr}([M,M]_\infty))^{1/2}$. Without changing the notation we pass to a subsequence and we obtain a sequence of numbers β_n, tending to ∞, such that the sequence

$$[H^n\cdot M,H^n\cdot M]_\infty^{1/2}\wedge\beta_n((\mathrm{tr}([M,M]_\infty))^{1/2}+1)$$

is uniformly integrable.

The sequence of stopping times T_n is now defined as:

$$T_n=\inf\{\,t\mid [H^n\cdot M,H^n\cdot M]_t^{1/2}\ge\beta_n((\mathrm{tr}([M,M])_t)^{1/2}+1)\}.$$

In the general case the sequence of jumps $\Delta\left(H^n\cdot M^n\right)_{T_n}$ is not uniformly integrable and so we have to eliminate the big parts of these jumps. But this time we want to stay in the framework of stochastic integrals with respect to M. The idea is, roughly speaking, to cut out of the stochastic interval $[\![0,T_n]\!]$, the predictable support of the stopping time T_n. Of course we then have to show that these supports form a sequence of sets that tends to the empty set. This requires some extra arguments.

Since $|\Delta(H^n\cdot M)_{T_n}|\le[H^n\cdot M,H^n\cdot M]_\infty^{1/2}$ we obtain that the sequence

$$|\Delta(H^n\cdot M)_{T_n}|\wedge\beta_n((\mathrm{tr}([M,M]_\infty))^{1/2}+1)$$

is uniformly integrable. As in the proof of the Kadeč–Pełczynski theorem we then find a sequence $\gamma_n\ge\beta_n$ such that $\gamma_n/\beta_n\to\infty$ and such that the sequence

$$|\Delta(H^n\cdot M)_{T_n}|\wedge\gamma_n(\mathrm{tr}([M,M]_\infty)^{1/2}+1)$$

is still uniformly integrable. As a consequence also the sequences

$$|\Delta(H^n\cdot M)_{T_n}|\wedge\beta_n(\mathrm{tr}([M,M]_{T_n})^{1/2}+1)$$

and

$$|\Delta(H^n\cdot M)_{T_n}|\wedge\gamma_n(\mathrm{tr}([M,M]_{T_n})^{1/2}+1)$$

are uniformly integrable.

By passing to a subsequence we may suppose that

(1) the sequences β_n, γ_n are increasing,
(2) $\sum_{n\ge1}\frac{1}{\beta_n}<\infty$ and hence $\sum\mathbb{P}[T_n<\infty]<\infty$,
(3) $\gamma_n/\beta_n\to\infty$,
(4) for each n we have

$$\frac{\kappa_n\beta_{n+1}(d+1)^2}{\gamma_{n+1}}\le\frac{1}{(d+1)^2},$$

which can be achieved by choosing inductively a subsequence, since γ_n/β_n becomes arbitrarily large.

We now turn the sequence of stopping times T_n into a sequence of stopping times having mutually disjoint graphs. This is done exactly as in part a above. Since $\mathbb{P}[T_n < \infty]$ tends to zero, we may, taking a subsequence if necessary, suppose that

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{j\le n}[H^j \cdot M, H^j \cdot M]_\infty^{1/2} \mathbf{1}_{\cup_{k>n}\{T_k<\infty\}}\Big] = 0.$$

We now replace each stopping time T_n by the stopping time τ_n defined by

$$\tau_n = \begin{cases} T_n & \text{if } T_n < T_k \text{ for all } k > n, \\ \infty & \text{otherwise.} \end{cases}$$

For each n let $\tilde{T}_n$ be defined as

$$\tilde{T}_n = \begin{cases} \tau_n & \text{if } |\Delta(H^n \cdot M)_{\tau_n}| > \gamma_n((\mathrm{tr}([M,M]))_{\tau_n}^{1/2} + 1), \\ \infty & \text{otherwise.} \end{cases}$$

For each n let $\tilde{F}^n$ be the compensator of the process $\mathbf{1}_{[\![\tilde{T}_n,\infty[\![}$. We now analyse the supports of the measures $d\tilde{F}^n$. The measure $d\lambda = \sum_{n\ge1} \frac{1}{2^n} d\tilde{F}_n$ will serve as a control measure. The measure λ satisfies $\mathbb{E}[\lambda_\infty] < \infty$ by the conditions above. Let ϕ^n be a predictable Radon–Nikodym derivative $\phi^n = \frac{d\tilde{F}^n}{d\lambda}$. It is clear that for each n we have $E^n = \{\phi^n \neq 0\} \subset [\![0, T_n]\!]$. The idea is to show the following assertion:

Claim. $\sum_{n\ge1} \mathbf{1}_{E^n} \le d$, *$d\lambda$ a.s. Hence there are predictable sets, still denoted by* E^n, *such that* $\sum_{n\ge1} \mathbf{1}_{E^n} \le d$ *everywhere and such that* $E^n = \{\phi^n \neq 0\}$, *$d\lambda$ a.s.*

We will give the proof at the end of this section.

For each n we decompose the integrands $H^n = K^n + V^n + W^n$ where:

$$\begin{aligned} K^n &= \mathbf{1}_{[\![0,\tilde{T}_n]\!]} \mathbf{1}_{(E^n)^c} H^n\,, \\ V^n &= \mathbf{1}_{E^n} H^n\,, \\ W^n &= \mathbf{1}_{]\!]\tilde{T}_n,\infty[\![} H^n. \end{aligned}$$

Since $\mathbb{P}[\tilde{T}_n < \infty]$ tends to zero, the maximal functions $(W^n \cdot M)_\infty^*$ tend to zero in probability.

We now show that the sequence $K^n \cdot M$ is relatively weakly compact in $\mathcal{H}^1$. The brackets satisfy

$$\begin{aligned} &[K^n \cdot M, K^n \cdot M]_\infty^{1/2} \\ &\qquad \le [H^n \cdot M, H^n \cdot M]_\infty^{1/2} \wedge \gamma_n([M,M]_\infty^{1/2} + 1) + [H^n \cdot M, H^n \cdot M]_\infty^{1/2} \mathbf{1}_{\{\tilde{T}_n \neq T_n\}}. \end{aligned}$$

The first term defines a uniformly integrable sequence, the second term defines a sequence tending to zero in L^1. It follows that the sequence $[K^n \cdot M, K^n \cdot M]_\infty^{1/2}$ is uniformly integrable and hence the sequence $K^n \cdot M$ is relatively weakly compact in $\mathcal{H}^1$.

There are convex combinations $(\alpha_n^k)_{k\geq n}$ such that $(\sum_k \alpha_n^k K^k) \cdot M$ converges in $\mathcal{H}^1$ to a martingale which is necessarily of the form $H^0 \cdot M$. We may of course suppose that these convex combinations are disjointly supported, i.e. there are indices $0 = n_0 < n_1 < n_2 \ldots$ such that α_j^k is 0 for $k \leq n_{j-1}$ and $k > n_j$. We remark that if we take convex combinations of $(\sum_k \alpha_n^k K^k) \cdot M$, then these combinations still tend to $H^0 \cdot M$ in $\mathcal{H}^1$. We will use this remark in order to improve the convergence of the remaining parts of H^n.

Let us define $L^n = \sum_k \alpha_n^k H^k$. Clearly $\|L^n \cdot M\|_{\mathcal{H}^1} \leq 1$ for each n. From theorem 1.3, it follows that there are convex combinations $(\eta_n^k)_{k\geq n}$, disjointly supported, such that $\sum_k \eta_n^k [L^k \cdot M, L^k \cdot M]_\infty^{1/2}$ converges a.s.. Hence we have that $\sup_n \sum_k \eta_n^k [L^k \cdot M, L^k \cdot M]_\infty^{1/2} < \infty$ a.s. We also may suppose that $\max_k \eta_n^k \to 0$ as $n \to \infty$. From Minkowski's inequality for the bracket it follows that also $\sup_n [(\sum_k \eta_n^k L^k) \cdot M, (\sum_k \eta_n^k L^k) \cdot M]^{1/2} < \infty$ a.s. Because the convex combinations were disjointly supported we also obtain a.s. and for $R^n = \sum_k \eta_n^k \sum_j \alpha_k^j V^j$:

$$\sup_n [R^n \cdot M, R^n \cdot M]_\infty^{1/2} \leq \sup_n \left[\left(\sum_k \eta_n^k L^k\right) \cdot M, \left(\sum_k \eta_n^k L^k\right) \cdot M\right]_\infty^{1/2} < \infty.$$

From the fact that the convex combinations were disjointly supported and from $\sum_n \mathbf{1}_{E^n} \leq d$, we conclude that for each point $(t, \omega) \in I\!\!R_+ \times \Omega$, only d vectors $R^n(t, \omega)$ can be nonzero. Let us put $P^n = \sum_{s=2^n+1}^{s=2^{n+1}} 2^{-n} R^s$. It follows that a.s.

$$\begin{aligned}
\int P^n \, d[M,M] P^n &\leq d \int \Bigg(\sum_{s=2^n+1}^{s=2^{n+1}} 2^{-2n} R^s \, d[M,M] R^s \Bigg) \\
&\leq d \, 2^{-n} \sum_{s=2^n+1}^{s=2^{n+1}} 2^{-n} [R^s \cdot M, R^s \cdot M]_\infty \\
&\leq d \, 2^{-n} \sup_s [R^s \cdot M, R^s \cdot M]_\infty \\
&\to 0.
\end{aligned}$$

If we now put $U^n = \sum_{k=2^n+1}^{k=2^{n+1}} 2^{-n} \sum_k \eta_l^k \sum_l \alpha_k^l H^l$, we arrive at convex combinations $U^n = \sum \lambda_n^l H^l$ such that

(1) the convex combinations λ_n^k are disjointly supported,
(2) $\left(\sum_k \lambda_n^k K^k\right) \cdot M \to H^0 \cdot M$ in $\mathcal{H}^1$,
(3) $\left[\left(\sum_k \lambda_n^k V^k\right) \cdot M, \left(\sum_k \lambda_n^k V^k\right) \cdot M\right]_\infty \to 0$ in probability,
(4) $\left[\left(\sum_k \lambda_n^k W^k\right) \cdot M, \left(\sum_k \lambda_n^k W^k\right) \cdot M\right]_\infty \to 0$ in probability, and even
(5) $\left(\left(\sum_k \lambda_n^k W^k\right) \cdot M\right)^* \to 0$ in probability.

As a consequence we obtain that $\left[\left(U^n - H^0\right)\cdot M, \left(U^n - H^0\right)\cdot M\right]_\infty \to 0$ in probability, and hence in $L^p(\Omega,\mathcal{F},\mathbb{P})$ for each $p<1$.

We remark that these properties will remain valid if we take once more convex combinations of the predictable processes U^n. The stochastic integrals $\left(\sum_k \lambda_n^k V^k\right)\cdot M$ need not converge in the semi-martingale topology as the example in section 3 shows. But exactly as in the case b we will show that after taking once more convex combinations, they converge in a pointwise sense, to a process of finite variation.

We consider the martingales $\left(\sum_k \lambda_n^k V^k\right)\cdot M$. For each n let D^n be the compensator of $(\sum_k \lambda_n^k \Delta(H^k\cdot M)_{T_k}\mathbf{1}_{[\![\tilde{T}_k,\infty[\![}$. This is a predictable process of integrable variation . Moreover $\mathbb{E}[\operatorname{var} D^n] \le \sum_k \lambda_n^k \mathbb{E}[|\Delta(H^k\cdot M)_{T_k}|] \le 2\sum_k \lambda_n^k \|H^k\cdot M\|_{\mathcal{H}^1} \le 2$. We now apply the Kadeč–Pełczynski decomposition technique to the sequence $\operatorname{var} D^n$ and we obtain, if necessary by passing to a subsequence, a sequence of numbers $\sum_n \frac{1}{\xi^n} < \infty$ such that $\operatorname{var} D^n \wedge \xi^n$ is uniformly integrable. Again we define predictable stopping times $S_n = \inf\{\, t \mid \operatorname{var}(D^n)^t \ge \xi^n\}$. We stop the processes at time (S_n-) since this will not destroy the martingale properties. More precisely we decompose $\sum_k \lambda_n^k V^k\cdot M$ as follows:

$$
\begin{aligned}
&\sum_k \lambda_n^k V^k\cdot M\\
&\quad=\left(\sum_k \lambda_n^k V^k\cdot M - \left(\sum_k \lambda_n^k \Delta(H^k\cdot M)_{T_k}\mathbf{1}_{[\![\tilde{T}_k,\infty[\![} - \tilde{D}^n\right)\right)^{S_n-} \qquad \text{first term}\\
&\qquad+\left(\sum_k \lambda_n^k \Delta(H^k\cdot M)_{T_k}\mathbf{1}_{[\![\tilde{T}_k,\infty[\![} - \tilde{D}^n\right)^{S_n-} \qquad \text{second term}\\
&\qquad+\left(\left(\sum_k \lambda_n^k V^k\cdot M\right) - \left(\sum_k \lambda_n^k V^k\cdot M\right)^{S_n-}\right) \qquad \text{third term.}
\end{aligned}
$$

Since $\left([D^n,D^n]^{S_n-}\right)^{1/2} \le 2\left(\operatorname{var} D^n\right)^{S_n-} \le (\operatorname{var} D^n)\wedge \xi^n$, we obtain that the first term defines a relatively weakly compact sequence in $\mathcal{H}^1$. Indeed, for each n we have $[\![\tilde{T}_n]\!] \subset E^n \subset [\![0,T_n]\!]$ and hence:

$$
\begin{aligned}
&[\text{first term}, \text{first term}]_\infty^{1/2}\\
&\quad\le \sum \lambda_n^k [V^k\cdot M, V^k\cdot M]_{\tilde{T}_n-}^{1/2} + [D^n,D^n]_{S_n-}^{1/2}\\
&\quad\le [H^n\cdot M, H^n\cdot M]^{1/2}\wedge \beta_n([M,M]_\infty+1)\\
&\qquad + [H^n\cdot M, H^n\cdot M]^{1/2}\mathbf{1}_{\{T_n\ne\tilde{T}_n\}} + [D^n,D^n]_\infty\wedge\xi^n.
\end{aligned}
$$

It follows that the first term defines a weakly relatively compact sequence in $\mathcal{H}^1$. But the first term is supported by the set $\bigcup_{k\ge n} E^k$, which tends to the empty set if $n\to\infty$. From theorem 2.4, it then follows that the sequence defined by the first term tends to zero weakly. The appropriate convex combinations will therefore tend to 0 in the norm of $\mathcal{H}^1$.

The second term splits in

$$\sum_k \lambda_n^k \Delta(H^k \cdot M)_{T_k} \mathbf{1}_{[\![\tilde{T}_k,\infty[\![},$$

whose maximal functions tend to zero a.s. and the processes $(D^n)^{S_n-}$. On the latter we can apply theorem 2.2, which results in convex combinations that tend to a process of finite variation. The third term has a maximal function that tends to zero since

$$\sum_n \mathbb{P}\left[\cup_{k\geq n}\left(\{T_k<\infty\}\cup\{S_n<\infty\}\right)\right]<\infty.$$

Modulo the proof of the claim above, the proof of theorem B is complete. So let us now prove the claim.

It is sufficient to show that for an arbitrary selection of $d+1$ indices $n_1 < \cdots < n_{d+1}$ we necessarily have that $E = \bigcap_{k\ldots d+1} E^{n_k} = \emptyset$, $d\lambda$ a.s. For each k we look at the compensator of the processes

$$\left(\Delta(H^{n_k}\cdot M)_{T_{n_k}}\right)^+ \mathbf{1}_{[\![\tilde{T}_{n_k},\infty[\![} \quad \text{resp.} \quad \left(\Delta(H^{n_k}\cdot M)_{T_{n_k}}\right)^- \mathbf{1}_{[\![\tilde{T}_{n_k},\infty[\![}.$$

Let ${}^+E^{n_k}$ (resp. ${}^-E^{n_k}$) be the supports of the compensators of these processes. For each of the 2^{d+1} sign combinations $\varepsilon_k = +/-$ we look at the set $\bigcap_{k=1}^{d+1} {}^{\varepsilon_k}E^{n_k}$. If the set E is nonempty, then at least one of these 2^{d+1} sets would be nonempty and without loss of generality we may and do suppose that this is the case for $\varepsilon_k = +$ for each k.

For each k we now introduce the compensator $\tilde{C}^k$ of the process

$$\left(\left(\operatorname{tr}([M,M]_{T_{n_k}})\right)^{1/2}+1\right)\mathbf{1}_{\{\Delta(H^{n_k}\cdot M)_{T_{n_k}}>0\}}\mathbf{1}_{[\![\tilde{T}_{n_k},\infty[\![}.$$

The processes H^{n_k} are d-dimensional processes and hence for each (t,ω) we find that the vectors $H_t^{n_k}(\omega)$ are linearly dependent. Using the theory of linear systems and more precisely the construction of solutions with determinants we obtain $d+1$ predictable processes $(\alpha^k)_{k=1\ldots d+1}$ such that

(1) for each (t,ω) at least one of the numbers $\alpha^k(t,\omega)$ is nonzero,
(2) $\sum_k \alpha^k H^{n_k} = 0$,
(3) the processes α^k are all bounded by 1.

We emphasize that these coefficients are obtained in a constructible way and that we do not need a measurable selection theorem!

We now look at the compensator of the processes

$$\Delta\left(H^{n_k}\cdot M\right)_{T_{n_l}} \mathbf{1}_{\{\Delta(H^{n_k}\cdot M)_{T_{n_k}}>0\}}\mathbf{1}_{[\![\tilde{T}_{n_k},\infty[\![}.$$

This compensator is of the form $g^{l,k} d\tilde{C}^l$ for a predictable process $g^{l,k}$. Because of the construction of the coefficients, we obtain that for each $l \le d+1$:

$$\sum g^{l,k} \alpha_n^k = 0.$$

The next step is to show on the set $\bigcap_{k=1}^{d+1} {}^{+}E^{n_k}$, the matrix $\left(g^{l,k}\right)_{l,k \le d+1}$ is non-singular. This will then give the desired contradiction, because the above linear system would only admit the solution $\alpha^k = 0$ for all $k \le d+1$. Because of the definition of the stopping times T_{n_k} we immediately obtain that $g^{k,k} \ge \gamma_{n_k}$. For the non diagonal elements we distinguish the cases $l < k$ and $l > k$. For $l < k$ we use the fact that on $\tilde{T}_{n_l} < \infty$, we have that $T_{n_l} < T_{n_k}$. It follows that $\left|\Delta(H^{n_k} \cdot M)_{T_{n_l}}\right| \le 2\beta_{n_k}\left(\left(\operatorname{tr}([M,M]_{T_{n_l}})\right)^{1/2} + 1\right)$ and hence $|g^{l,k}| \le \beta_{n_k}$. If $l > k$ then $\left|\Delta(H^{n_k} \cdot M)_{T_{n_l}}\right| \le \kappa_{n_k}\left(\left(\operatorname{tr}([M,M]_{T_{n_l}})\right)^{1/2} + 1\right)$ and hence $|g^{l,k}| \le \kappa_{n_k}$. We now multiply the last column of the matrix $g^{l,k}$ with the fraction $\frac{1}{\beta_{n_{d+1}}(d+1)^2}$ and then we multiply the last row by $\frac{\beta_{n_{d+1}}(d+1)^2}{\gamma_{n_{d+1}}}$. The result is that the diagonal element at place $(d+1, d+1)$ is equal to 1 and that the other elements of the last row and the last column are bounded in absolute value by $\frac{1}{(d+1)^2}$. We continue in the same way by multiplying the column d by $\frac{1}{\beta_{n_d}(d+1)^2}$ and the row d by $\frac{\beta_{n_d}(d+1)^2}{\gamma_{n_d}}$. The result is that the element at place (d,d) is 1 and that the other elements on row d and column d are bounded by $\frac{1}{(d+1)^2}$. We note that the elements at place $(d, d+1)$ and $(d+1, d)$ are further decreased by this procedure so that the bound $\frac{1}{(d+1)^2}$ will remain valid. We continue in this way and we finally obtain a matrix with 1 on the diagonal and with the off diagonal elements bounded by $\frac{1}{(d+1)^2}$. By the classical theorem, due to Hadamard [G66], p. 454 Satz 1, such a matrix with dominant diagonal is non-singular. The proof of the claim is now completed and so are the proofs of the theorems A,B and C.

q.e.d.

d. A proof of M. Yor's theorem 1.6 for the L^1-convergent case.

We now show how the ideas of the proof given in [Y78] fit in the general framework described above. We will use the generalisation of theorem A to processes with jumps (see the remarks following the proof of theorem A). In the next theorem we suppose that M is a d-dimensional local martingale.

1.6 Theorem. *Let $(H^n)_{n\ge1}$ be a sequence of M-integrable predictable stochastic processes such that each $(H^n \cdot M)$ is a uniformly integrable martingale and such that the sequence of random variables $((H^n \cdot M)_\infty)_{n\ge1}$ converges to a random variable $f_0 \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the L^1-norm; (or even only with respect to the $\sigma(L^1, L^\infty)$) topology). Then there is an M-integrable predictable stochastic process H^0 such that $H^0 \cdot M$ is a uniformly integrable martingale and such that $(H^0 \cdot M)_\infty = f_0$.*

Proof. If f_n converges only weakly to f_0 then we take convex combinations in order to obtain a strongly convergent sequence. We therefore restrict the proof to the case where f_n converges in L^1-norm to f_0. By selecting a subsequence we may suppose that $\|f_n\|_{L^1} \le 1$ for each n and that $\|f_n - f_0\|_{L^1} \le 4^{-n}$. Let N be the càdlàg martingale defined as $N_t = \mathbb{E}[f_0 \mid \mathcal{F}_t]$. From the maximal inequality for L^1-martingales it then follows that:

$$\mathbb{P}\Big[\sup_t |(H^n \cdot M)_t - N_t| \ge 2^{-n}\Big] \le 2^{-n}.$$

The Borel–Cantelli lemma then implies that

$$\sup_t \sup_n |(H^n \cdot M)_t| < \infty \qquad \text{a.s..}$$

For each natural number k we then define the stopping time T_k as:

$$T_k = \inf\{t \mid \text{ there is } n \text{ such that } |(H^n \cdot M)_t| \ge k\}.$$

Because of the uniform boundedness in t and n we obtain that the sequence T_k satisfies $\mathbb{P}[T_k < \infty] \to 0$. Also the sequence T_k is clearly increasing. For each k and each n we have that

$$\|(H^n \cdot M)^{T_k}\|_{\mathcal{H}^1} \le k + \|(H^n \cdot M)_{T_k}\|_{L^1}.$$

Since the sequence $f_n = (H^n \cdot M)_\infty$ is uniformly integrable (it is even norm convergent), we have that also the sequence of conditional expectations, $((H^n \cdot M)_{T_k})_{n\ge1}$ is uniformly integrable and hence the sequence $((H^n \cdot M)^{T_k})_{n\ge1}$ is weakly relatively compact in $\mathcal{H}^1$. Taking the appropriate linear combinations will give a limit in $\mathcal{H}^1$ of the form $K^k \cdot M$ with K^k supported by $[\![0, T_k]\!]$ and satisfying $(K^k \cdot M) = N^{T_k}$. We now take a sequence $(k_m)_{m\ge1}$ such that $\|N_{T_{k_m}} - f_0\| \le 2^{-m}$. If we define

$$H^0 = K^{k_1} + \sum_{m\ge2} K^{k_m} \mathbf{1}_{]\!]T_{k_{m-1}}, T_{k_m}]\!]},$$

we find that $H^0 \cdot M$ is uniformly integrable and that $(H^0 \cdot M)_\infty = f_0$.

q.e.d.

e. The proof of theorem D.

The basic ingredient is theorem C. Exactly as in M. Yor's theorem we do not have—a priori—a sequence that is bounded in $\mathcal{H}^1$. The lower bound w only permits to obtain a bound for the L^1 norms and we need again stopping time arguments. This is possible because of a uniform bound over the time interval, exactly as in the previous part. The uniformity is obtained as in [DS94] lemma 4.5.

4.10 Definition. *We say that an M-integrable predictable process H is w-admissible for some nonnegative integrable function w if $H \cdot M \geq -w$, i.e. the process stays above the level $-w$.*

Remark. the concept of a-admissible integrands, where $a > 0$ is a deterministic number, was used in the paper [DS94] where a short history of this concept is given. The above definition generalizes the admissibility as used in [DS94] in the sense that it replaces a constant function by a fixed nonnegative integrable function w. The concept was also used by the second named author in [S94], Proposition 4.5.

Exactly as in [DS94] we introduce the cone

$$C_{1,w} = \{ f \mid \text{ there is a } w\text{-admissible integrand } H \text{ such that } f \leq (H \cdot M)_\infty \} .$$

4.11 Theorem D. *Let M be a $\mathbb{R}^d$-valued local martingale and $w \geq 1$ an integrable function. Given a sequence $(H^n)_{n\geq 1}$ of M-integrable $\mathbb{R}^d$-valued predictable processes such that*

$$(H^n \cdot M)_t \geq -w, \qquad \text{for all } n, t,$$

there are convex combinations

$$K^n \in \operatorname{conv}\{H^n, H^{n+1}, \dots\},$$

and there is a super-martingale $(V_t)_{t\in\mathbb{R}_+}$, $V_0 \leq 0$, such that

$$\lim_{s \overset{\geq}{\to} t, s\in\mathbb{Q}_+} \lim_{n\to\infty} (K^n \cdot M)_s = V_t \qquad \text{for } t \in \mathbb{R}_+, \text{ a.s.},$$

and an M-integrable predictable process H^0 such that

$$((H^0 \cdot M)_t - V_t)_{t\in\mathbb{R}_+} \qquad \text{is increasing.}$$

In addition, $H^0 \cdot M$ is a local martingale and a super-martingale.

Before proving theorem D we shall deduce a corollary which is similar in spirit to ([DS94], th. 4.2) and which we will need in section 5 below. For a semi-martingale S we denote by $\mathcal{M}^e(S)$ the set of all probability measures $\mathbb{Q}$ on $\mathcal{F}$ equivalent to $\mathbb{P}$, such that S is a local martingale under $\mathbb{Q}$.

4.12 Corollary. *Let S be an $\mathbb{R}^d$-valued semi-martingale such that $\mathcal{M}^e(S) \neq \emptyset$ and $w \geq 1$ a weight function such that there is some $Q \in \mathcal{M}^e(S)$ with $\mathbb{E}_Q[w] < \infty$. Then the convex cone $C_{1,w}$ is closed in $L^0(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the topology of convergence in measure.*

Proof of corollary 4.12. As the assertion of the corollary is invariant under equivalent changes of measure we may assume that the original measure $\mathbb{P}$ is an element of $\mathcal{M}^e(S)$ for which $\mathbb{E}_{\mathbb{P}}[w] < \infty$, i.e., we are in the situation of theorem C above.

As in the proof of theorem B we also may assume that S is in $\mathcal{H}^1(\mathbb{P})$ and therefore a $\mathbb{P}$-uniformly integrable martingale.

Let

$$f_n = (H^n \cdot S)_\infty - h_n$$

be a sequence in $C_{1,w}$, where $(H^n)_{n\geq 1}$ is a sequence of w-admissible integrands and $h_n \geq 0$. Assuming that $(f_n)_{n\geq 1}$ tends to a random variable f_0 in measure we have to show that $f_0 \in C_{1,w}$.

It will be convenient to replace the time index set $[0,\infty[$ by $[0,\infty]$ by closing S and $H^n \cdot S$ at infinity, which clearly may be done as the martingale $(S_t)_{t\in \mathbb{R}_+}$ as well as the negative parts of the supermartingales $((H^n \cdot S)_t)_{t\in\mathbb{R}_+}$ are $\mathbb{P}$-uniformly integrable. Identifying the closed interval $[0,\infty]$ with the closed interval $[0,1]$, and identifying the processes S and $H^n \cdot S$ with process which remain constant after time $t=1$, we deduce from theorem D that we may find $K^n \in \mathrm{conv}(H^n, H^{n+1}, \dots)$, a w-admissible integrand H^0 and a process $(V_t)_{t\in\mathbb{R}_+}$ such that

(i) $$\lim_{s \overset{>}{\to} t, s\in\mathbb{Q}_+} \lim_{n\to\infty} (K^n \cdot S)_s = V_t, \qquad \text{a.s. for } t \in \mathbb{R}_+$$

and

$$\lim_{n\to\infty} (K^n \cdot S)_\infty = V_\infty,$$

(ii) $$((H^0 \cdot S)_t - V_t)_{t\in\mathbb{R}_+\cup\{\infty\}} \quad \text{is increasing.}$$

In particular $((K^n \cdot S)_\infty)_{n\geq 1}$ converges almost surely to the random variable U_∞ which is dominated by $(H^0 \cdot S)_\infty$.

As $(f_n)_{n\geq 1}$ was assumed to converge in measure to f_0 we deduce that $f_0 \leq (H^0 \cdot S)_\infty$, i.e. $f_0 \in C_{1,w}$.

q.e.d.

To pave the way for the proof of theorem D we start with some lemmas.

4.13 Lemma. *Under the assumptions of theorem D there is a sequence of convex combinations*

$$K^n \in \mathrm{conv}\{H^n, H^{n+1}, \dots\},$$

and a sequence $(L^n)_{n\geq 1}$ of w-admissible integrands and there are càdlàg supermartingales $V = (V_t)_{t\in\mathbb{R}_+}$ and $W = (W_t)_{t\in\mathbb{R}_+}$ with $W - V$ increasing such that

$$V_t = \lim_{s \overset{>}{\to} t, s\in\mathbb{Q}_+} \lim_{n\to\infty} (K^n \cdot M)_s \qquad \textit{for } t \in \mathbb{R}_+, \textit{a.s.}$$

$$W_t = \lim_{s \overset{>}{\to} t, s\in\mathbb{Q}_+} \lim_{n\to\infty} (L^n \cdot M)_s \qquad \textit{for } t \in \mathbb{R}_+, \textit{a.s.}$$

and such that W satisfies the following maximality condition: For any sequence $(\tilde{L}^n)_{n\geq 1}$ of w-admissible integrands such that

$$\tilde{W}_t = \lim_{s \overset{>}{\to} t, s\in\mathbb{Q}_+} \lim_{n\to\infty} (\tilde{L}^n \cdot M)_s$$

and $\tilde{W} - W$ increasing we have that $\tilde{W} = W$.

Proof. By theorem 1.3 we may find $K^n \in \operatorname{conv}\{H^n, H^{n+1}, \dots\}$ such that, for every $t \in \mathbb{Q}_+$, the sequence $((K^n \cdot M)_t)_{n\geq 1}$ converges a.s. to a random variable $\hat{V}_t$. As w is assumed to be integrable we obtain that the process $(\hat{V}_t)_{t\in\mathbb{Q}_+}$ is a supermartingale and therefore its càdlàg regularisation,

$$V_t = \lim_{s \overset{>}{\to} t, s\in\mathbb{Q}_+} \hat{V}_s, \qquad t \in \mathbb{R}_+$$

is an a.s. welldefined càdlàg supermartingale.

Let $\mathcal{W}$ denote the family of all càdlàg super-martingales $W = (W_t)_{t\in\mathbb{R}_+}$ such that $W - V$ is increasing and such that there is a sequence $(L^n)_{n\geq 1}$ of w-admissible integrands such that

$$W_t = \lim_{s \overset{>}{\to} t, s\in\mathbb{Q}_+} \lim_{n\to\infty} (L^n \cdot M)_s \qquad \text{for } t \in \mathbb{R}_+$$

is a.s. well-defined.

Introducing—similarly as in [K 96]—the order $W^1 \geq W^2$ on $\mathcal{W}$, if $W^1 - W_2$ is increasing, we may find a maximal element $W \in \mathcal{W}$, with an associated sequence $(L^n)_{n\geq 1}$ of w-admissible integrands.

Indeed, let $(W^\alpha)_{\alpha\in I}$ be a maximal chain in $\mathcal{W}$ with associated sequences of integrands $(L^{\alpha,n})_{n\geq 1}$; then $(W^\alpha_\infty)_{\alpha\in I}$ is an increasing and bounded family of elements of $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and therefore there is an increasing sequence $(\alpha_j)_{j\geq 1}$ such that $(W^{\alpha_j}_\infty)_{j\geq 1}$ increases to the essential supremum of $(W^\alpha_\infty)_{\alpha\in I}$. The càdlàg supermartingale $W = \lim_j W^{\alpha_j}$ is welldefined and we may find a sequence $(L^{\alpha_j,n_j})_{j\geq 1}$, which we reliable by $(L^n)_{n\geq 1}$, so that

$$W_t = \lim_{s \overset{>}{\to} t, s\in\mathbb{Q}_+} \lim_{n\to\infty} (L^n \cdot M)_s.$$

Clearly W satisfies the required maximality condition

q.e.d.

4.14 Lemma. *Under the assumptions of the preceding lemma 4.13 we have that for $T \in \mathbb{R}_+$, the maximal functions*

$$((L^n \cdot M) - (L^m \cdot M))^*_T = \sup_{t\leq T} |(L^n \cdot M)_t - (L^m \cdot M)_t|$$

tend to zero in measure as $n, m \to \infty$.

Proof. The proof of the lemma will use—just as in ([DS94] lemma 4.5) and [K96]—the "buy low – sell high" argument motivated by the economic interpretation of L^n as trading strategies (see [DS94], remark 4.6).

Assuming that the assertion of the lemma is wrong there is $T \in \mathbb{R}_+$, $\alpha > 0$ and sequences $(n_k, m_k)_{k \geq 1}$ tending to ∞ such that

$$\mathbb{P}\Big[\sup_{t \leq T}((L^{n_k} - L^{m_k}) \cdot M)_t > \alpha\Big] \geq \alpha.$$

Defining the stopping times

$$T_k = \inf\{t \leq T : ((L^{n_k} - L^{m_k}) \cdot M)_t \geq \alpha\}$$

we have $\mathbb{P}[T_k \leq T] \geq \alpha$.

Define $\hat{L}^k$ as

$$\hat{L}^k = L^{n_k} \mathbf{1}_{[\![0,T_k]\!]} + L^{m_k} \mathbf{1}_{]\!]T_k,\infty[\![}$$

so that $\hat{L}^k$ is a w-admissible predictable integrand.

Denote by d_k the function indicating the difference between $L^{n_k} \cdot M$ and $L^{m_k} \cdot M$ at time T_k, if $T_k < \infty$, i.e.,

$$d_k = ((L^{n_k} - L^{m_k}) \cdot M)_{T_k} \mathbf{1}_{\{T_k < \infty\}}.$$

Note that, for $t \in \mathbb{R}_+$,

$$(\hat{L}^k \cdot M)_t = (L^{n_k} \cdot M)_t \mathbf{1}_{\{t \leq T_k\}} + ((L^{m_k} \cdot M)_t + d^k) \mathbf{1}_{\{t > T_k\}}$$

By passing to convex combinations $\sum_{j=k}^{\infty} \alpha_j \hat{L}^j$ of $\hat{L}^k$ we therefore get that, for each $t \in \mathbb{Q}_+$,

$$\left(\sum_{j=k}^{\infty} \alpha_j \hat{L}^j \cdot M\right)_t = \left(\sum_{j=k}^{\infty} \alpha_j L^{n_j} \cdot M\right)_t \mathbf{1}_{\{t \leq T_k\}} + \left(\sum_{j=k}^{\infty} \alpha_j L^{m_j} \cdot M\right)_t \mathbf{1}_{\{t > T_k\}} + D_t^k$$

where $(D_t^k)_{k \geq 1} = (\sum_{j=k}^{\infty} \alpha_j d^j \mathbf{1}_{\{t > T_k\}})_{k \geq 1}$ is a sequence of random variables which converges almost surely to a random variable D_t so that $(D_t)_{t \in \mathbb{Q}_+}$ is an increasing adapted process which satisfies $\mathbb{P}[D_T > 0] > 0$ by ([DS94], lemma A1.1).

Hence $(\hat{L}^k)_{k \geq 1}$ is a sequence of w-admissible integrands such that, for all $t \in \mathbb{Q}_+$, $(\hat{L}^k \cdot M)_t$ converges almost surely to $\hat{W}_t = W_t + D_t$, and $\mathbb{P}[D_T > 0] > 0$, a contradiction to the maximality of W finishing the proof.

q.e.d.

4.15 Lemma. *Under the assumptions of theorem D and lemma 4.13 there is a subsequence of the sequence* $(L^n)_{n\geq 1}$*, still denoted by* $(L^n)_{n\geq 1}$*, and an increasing sequence* $(T_j)_{j\geq 1}$ *of stopping times,* $T_j \leq j$ *and* $\mathbb{P}[T_j = j] \geq 1 - 2^{-j}$*, such that, for each* j*, the sequence of processes* $((L^n \cdot M)^{T_j -})_{n\geq 1}$ *is uniformly bounded and the sequence* $((L^n \cdot M)^{T_j})_{n\geq 1}$ *is a bounded sequence of martingales in* $\mathcal{H}^1(\mathbb{P})$*.*

Proof. First note that, fixing $j \in \mathbb{N}$, $C > 0$ and defining the stopping times

$$U_n = \inf\{t : |(L^n \cdot M)_t| \geq C\} \wedge j,$$

the sequence $((L^n \cdot M)^{U_n})_{n\geq 1}$ is bounded in $\mathcal{H}^1(\mathbb{P})$. Indeed, this is a sequence of super-martingales by [AS 92], hence

$$\mathbb{E}\left[|(L^n \cdot M)_{U_n}|\right] \leq 2\mathbb{E}\left[((L^n \cdot M)_{U_n})_-\right] \leq 2(C + \mathbb{E}[w]),$$

whence

$$\mathbb{E}\left[|\Delta(L^n \cdot M)_{U_n}|\right] \leq 2(C + \mathbb{E}[w]) + C.$$

As the maximal function $(L^n \cdot M)^*_{U_n}$ is bounded by $C + |\Delta(L^n \cdot M)_{U_n}|$ we obtain a uniform bound on the L^1-norms of the maximal functions $((L^n \cdot M)^*_{U_n})_{n\geq 1}$, showing that $((L^n \cdot M)^{U_n})_{n\geq 1}$ is a uniformly bounded sequence in $\mathcal{H}^1(\mathbb{P})$.

If we choose $C > 0$ sufficiently big we can make $\mathbb{P}[U_n < j]$ small, uniformly in n; but the sequence of stopping times $(U_n)_{n\geq 1}$ still depends on n and we have to replace it by just one stopping time T_j which works for all $(L^n{}_k)_{k\geq 1}$ for some subsequence $(n_k)_{k\geq 1}$; to do so, let us be a little more formal.

Assume that $T_0 = 0, T_1, \ldots, T_{j-1}$ have been defined as well as a subsequence, still denoted by $(L^n)_{n\geq 1}$, such that the claim is verified for $1, \ldots, j-1$; we shall construct T_j. Applying lemma 4.14 to $T = j$ we may find a subsequence $(n_k)_{k\geq 1}$ such that, for each k,

$$\mathbb{P}\left[((L^{n_{k+1}} \cdot M) - (L^{n_k} \cdot M))^*_j \geq 2^{-k}\right] < 2^{-(k+j+2)}.$$

Now find a number $C_j \in \mathbb{R}_+$ large enough such that

$$\mathbb{P}\left[(L^{n_1} \cdot M)^*_j \geq C_j\right] < 2^{-(j+1)}$$

and define the stopping time T_j by

$$T_j = \inf\left\{t : \sup_k |(L^{n_k} \cdot M)_t| \geq C_j + 1\right\} \wedge j$$

so that $T_j \leq j$ and

$$\mathbb{P}[T_j = j] \geq 1 - 2^{-j}.$$

Clearly $|(L^{n_k} \cdot M)_t| \leq C_j + 1$ for $t < T_j$, whence $((L^n{}_k \cdot M)^{T_j -})_{k \geq 1}$ is uniformly bounded.

We have that $T_j \leq U_{n_k}$ for each k, where U_{n_k} is the stopping time defined above (with $C = C_j + 1$). Hence we deduce from the $\mathcal{H}^1(\mathbb{P})$-boundedness of $((L^{n_k} \cdot M)^{U_{n_k}})_{k \geq 1}$ the $\mathcal{H}^1(\mathbb{P})$-boundedness of $(L^{n_k} \cdot M)^{T_j}$. This completes the inductive step and finishes the proof of lemma 4.15.

q.e.d.

Proof of 4.10 theorem D. Given a sequence $(H^n)_{n \geq 1}$ of w-admissible integrands choose the sequences $K^n \in \text{conv}(H^n, H^{n+1}, \dots)$ and L^n of w-admissible integrands and the super-martingales V and W as in lemma 4.13. Also fix an increasing sequence $(T_j)_{j \geq 1}$ of stopping times as in lemma 4.15.

We shall argue locally on the stochastic intervals $]\!]T_{j-1}, T_j]\!]$. Fix $j \in \mathbb{N}$ and let

$$L^{n,j} = L^n \mathbf{1}_{]\!]T_{j-1}, T_j]\!]}.$$

By lemma 4.15 there is a constant $C_j > 0$ such that $(L^{n,j})_{n \geq 1}$ is a sequence of $(w + C_j)$-admissible integrands and such that $(L^{n,j} \cdot M)_{n \geq 1}$ is a sequence of martingales bounded in $\mathcal{H}^1(\mathbb{P})$ and such that the jumps of each $L^{n,j} \cdot M$ are bounded downward by $w - 2C_j$. Hence—by passing to convex combinations, if necessary—we may apply theorem B to split $L^{n,j}$ into two disjointly supported integrands $L^{n,j} = {}^r L^{n,j} + {}^s L^{n,j}$ and we may find an integrand $H^{0,j}$ supported by $]\!]T_{j-1}, T_j]\!]$ such that

(i) $$\lim_{n \to \infty} \|({}^r L^{n,j} - H^{0,j}) \cdot M\|_{\mathcal{H}^1(\mathbb{P})} = 0$$

(ii) $$(Z_j)_t = \lim_{q \overset{\geq}{\to} t, q \in \mathbb{Q}_+} \lim_{n \to \infty} ({}^s L^{n,j} \cdot M)_q$$

where Z_j is a well-defined adapted càdlàg increasing process.

Finally we paste things together by defining $H^0 = \sum_{j \geq 1}^{\infty} H^{0,j}$ and $Z = \sum_{j \geq 1}^{\infty} Z_j$. By lemma 4.13 we have that

$$W_t = \lim_{s \overset{\geq}{\to} t, s \in \mathbb{Q}_+} (L^n \cdot M)_s$$

is a well-defined super-martingale. As

$$Z = (H^0 \cdot M) - W$$

is an increasing process and as $(H^0 \cdot M)$ is a local martingale and a super-martingale by [AS92] we deduce from the maximality of W that $H^0 \cdot M$ is in fact equal to W. Hence $(H^0 \cdot M) - V$ is increasing and the proof of theorem D is finished.

q.e.d.

5. Application.

In this section we apply the above theorems to give a proof of the "Optional Decomposition Theorem" due to N. El Karoui, M.-C. Quenez [KQ95], D. Kramkov [K96], Föllmer–Kabanov [FK95], Kramkov [K96a] and Föllmer–Kramkov [FKr96]. We refer the reader to these papers for the precise statements and for the different techniques used in the proofs.

We generalise the usual setting in finance in the following way. The process S will denote an $\mathbb{R}^d$-valued semi-martingale. In finance theory, usually the idea is to look for measures $\mathbb{Q}$ such that under $\mathbb{Q}$ the process S becomes a local martingale. In the case of processes with jumps this is too restrictive and the idea is to look for measures $\mathbb{Q}$ such that S becomes a *sigma-martingale.* A process S is called a $\mathbb{Q}$ sigma-martingale if there is a strictly positive, predictable process ϕ such that the stochastic integral $\phi \cdot S$ exists and is a $\mathbb{Q}$ martingale. We remark that it is clear that we may require the process $\phi \cdot S$ to be an $\mathcal{H}^1$ martingale and that we also may require the process ϕ to be bounded (compare [DS96]). As easily seen, local martingales are sigma-martingales. In the local martingale case the predictable process ϕ can be chosen to be decreasing and this characterises the local martingales among the sigma-martingales. The concept of sigma martingale is therefore more general than the concept of local martingale. The set $\mathcal{M}^e(S)$ denotes the set of all equivalent probability measures $\mathbb{Q}$ on $\mathcal{F}$ such that S is a $\mathbb{Q}$-sigma-martingale. It is an easy exercise to show that the set $\mathcal{M}^e(S)$ is a convex set. We suppose that this set is nonempty and we will refer to elements of $\mathcal{M}^e(S)$ as equivalent sigma martingale measures. We refer to [DS96] for more details and for a discussion of the concept of sigma martingales. We also remark that if S is a semi martingale and if ϕ is strictly positive, bounded and predictable, then the sets of stochastic integrals with respect to S and with respect to $\phi \cdot S$ are the same. This follows easily from the formula $H \cdot S = \frac{H}{\phi} \cdot (\phi \cdot S)$.

5.1 Optional Decomposition Theorem. *Let $S = (S_t)_{t \in \mathbb{R}_+}$ be an $\mathbb{R}^d$-valued semi-martingale, such that the set $\mathcal{M}^e(S) \neq \emptyset$, and $V = (V_t)_{t \in \mathbb{R}_+}$ a real valued semi-martingale, $V_0 = 0$ such that, for each $Q \in \mathcal{M}^e(S)$, the process V is a Q-local-super-martingale. Then there is an S-integrable $\mathbb{R}^d$-valued predictable process H such that $(H \cdot S) - V$ is increasing.*

5.2 Remark. The Optional Decomposition Theorem is proved in [KQ95] in the setting of $\mathbb{R}^d$-valued continuous processes. The important—and highly non-trivial—extension to not necessarily continuous processes was achieved by D. Kramkov in his beautiful paper [K96]. His proof relies on some of the arguments from [DS94] and therefore he was forced to make the following hypotheses: The process S is assumed to be a locally bounded $\mathbb{R}^d$-valued semi-martingale and V is assumed to be uniformly bounded from below. Later H. Föllmer and Y. Kabanov [FK95] gave a proof of the Optional Decomposition Theorem based on Lagrange-multiplier techniques which allowed them to drop the local boundedness assumption on S. Very recently Föllmer and Kramkov, [FKr96] gave another proof of this result.

In the present paper our techniques—combined with the arguments of D. Kramkov—allow us to abandon the one-sided boundedness assumption on the process V and to pass to the—not necessarily locally bounded—setting for the process S.

For the economic interpretation and relevance of the Optional Decomposition Theorem we refer to [KQ95] and [K96].

We start the proof with some simple lemmas. The first one—which we state without proof—resumes the wellknown fact that a local martingale is locally in $\mathcal{H}^1$.

5.3 Lemma. *For a $\mathbb{P}$-local-super-martingale V we may find a sequence $(T_j)_{j\geq 1}$ of stopping times increasing to infinity and $\mathbb{P}$-integrable functions $(w_j)_{j\geq 1}$ such that the stopped supermartingales V^{T_j} satisfy*

$$|V^{T_j}| \leq w_j \qquad \text{a.s., for } j \in \mathbb{N}.$$

q.e.d.

The next lemma is due to D. Kramkov ([K96], lemma 5.1) and similar to lemma 4.13 above.

5.4 Lemma. *In the setting of the Optional Decomposition Theorem 5.1 there is a semi-martingale W with $W - V$ increasing, such that W is a $\mathbb{Q}$-local-super-martingale, for each $\mathbb{Q} \in \mathcal{M}^e(S)$ and which is maximal in the following sense: for each semi-martingale $\tilde{W}$ with $\tilde{W} - W$ increasing and such that $\tilde{W}$ is a $\mathbb{Q}$-local-super-martingale, for each $\mathbb{Q} \in \mathcal{M}^e(S)$, we have $W = \tilde{W}$.*

q.e.d.

Proof of the Optional Decomposition Theorem 5.1. For the given semi-martingale V we find a maximal semi-martingale W as in the preceding lemma 5.4. We shall find an S-integrable predictable process H such that we obtain a representation of the process W as the stochastic integral over H, i.e.,

$$W = H \cdot S$$

which will in particular prove the theorem.

Fix $\mathbb{Q}_0 \in \mathcal{M}^e(S)$ and apply lemma 5.3 to the $\mathbb{Q}_0$-local-super-martingale W to find $(T_j)_{j\geq 1}$ and $w_j \in L^1(\Omega, \mathcal{F}, \mathbb{Q}_0)$. Note that it suffices—similarly as in [K96]—to prove theorem 5.1 locally on the stochastic intervals $]\!]T_{j-1}, T_j]\!]$. Hence we may and do assume that $|W| \leq w$ for some $\mathbb{Q}_0$-integrable weight-function $w \geq 1$. Since S is a sigma-martingale for the measure Q_0, we can by the discussion preceding the theorem 5.1, and without loss of generality, assume that S is an $\mathcal{H}^1(Q_0)$ martingale. So we suppose that the weight function w also satisfies $|S| \leq w$, where $|\,.\,|$ denotes any norm on $\mathbb{R}^d$.

Fix the real numbers $0 \le u < v$ and consider the process ${}^uW^v$ "starting at u and stopped at time v", i.e.,

$$ {}^uW^v_t = W_{t \wedge v} - W_{t \wedge u}, $$

which is a $\mathbb{Q}$-local-super-martingale, for each $\mathbb{Q} \in \mathcal{M}^e(S)$, and such that $|{}^uW^v| \le 2w$.

Claim. *There is an S-integrable $2w$-admissible predictable process ${}^uH^v$, which we may choose to be supported by the interval $]u, v]$, such that*

$$ ({}^uH^v \cdot S)_\infty = ({}^uH^v \cdot S)_v \ge \ f = {}^uW^v_v = \ {}^uW^v_\infty. $$

Assuming this claim for a moment, we proceed similarly as D. Kramkov ([K96], proof of th. 2.1.): fix $n \in \mathbb{N}$ and denote by $\mathcal{T}(n)$ the set of time indices

$$ \mathcal{T}(n) = \left\{ \frac{j}{2^n} : 0 \le j \le n\,2^n \right\} $$

and denote by H^n the predictable process

$$ H^n = \sum_{j \ge 1}^{n2^n} {}^{(j-1)2^{-n}}H^{j2^{-n}}, $$

where we obtain ${}^{(j-1)2^{-n}}H^{j2^{-n}}$ as a $2w$-admissible integrand as above with $u = (j-1)2^{-n}$ and $v = j2^{-n}$. Clearly H^n is a $2w$-admissible integrand such that the process indexed by $\mathcal{T}(n)$

$$ ((H^n \cdot S)_{j2^{-n}} - W_{j2^{-n}})_{j=0,\dots,n2^n} $$

is increasing.

By applying theorem D to the $\mathbb{Q}_0$-local martingale S—and by passing to convex combinations, if necessary—the process

$$ \tilde{W}_t = \lim_{s \underset{>}{\rightarrow} t, s \in \mathbb{Q}_+} \lim_{n \to \infty} (H^n \cdot S)_s $$

is welldefined and we may find a predictable S-integrable process H such that $H \cdot S - \tilde{W}$ is increasing; as $W - \tilde{W}$ is increasing too, we obtain in particular that $H \cdot S - W$ is increasing.

As $H \cdot S \ge W$ we deduce from [AS92] that, for each $\mathbb{Q} \in \mathcal{M}^e(S)$, $H \cdot S$ is a $\mathbb{Q}$-local martingale and a $\mathbb{Q}$-super-martingale. By the maximality condition of W we must have $H \cdot S = W$ thus finishing the proof of the Optional Decomposition Theorem 5.1.

We still have to prove the claim. This essentially follows from corollary 4.12.

Let us define L^∞_w to be the space of all measurable functions g such that $\frac{g}{w}$ is essentially bounded. This space is the dual of the space $L^1_{w^{-1}}(\mathbb{Q}_0)$ of functions g such that $\mathbb{E}_{\mathbb{Q}_0}[w\,|g|] < \infty$. By the Banach-Dieudonné theorem or the Krein-Smulian theorem (see [DS94] for a similar application), it follows from corollary 4.12 that the set

$$B = \{\, h \mid |\varepsilon h| \le w \text{ and } \varepsilon h \in C_{1,2w} \text{ for some } \varepsilon > 0 \,\},$$

is a weak* closed convex cone in L^∞_w (the set $C_{1,2w}$ was defined in 4.10 above). Now as easily seen, if the claim were not true, then the said function f is not in B. Since $B - L^\infty_{w\,+} \subset B$ we have by Yan's separation theorem ([Y80]), that there is a strictly positive function $h \in L^1_{w^{-1}}$ such that $\mathbb{E}_{\mathbb{Q}_0}[hf] > 0$ and such that $\mathbb{E}_{\mathbb{Q}_0}[hg] \le 0$ for all $g \in B$. If we normalise h so that $\mathbb{E}_{\mathbb{Q}_0}[h] = 1$ we obtain an equivalent probability measure $\mathbb{Q}$, $d\mathbb{Q} = h\,d\mathbb{Q}_0$ such that $\mathbb{E}_{\mathbb{Q}}[f] > 0$. But since S is dominated by the weight function w, we have that the measure $\mathbb{Q}$ is an equivalent martingale measure for the process S. The process W is therefore a local-super-martingale under $\mathbb{Q}$. But the density h is such that $\mathbb{E}_{\mathbb{Q}}[w] < \infty$ and therefore the process ${}^uW^v$, being dominated by $2w$, is a genuine super-martingale under $\mathbb{Q}$. However this is a contradiction to the inequality $\mathbb{E}_{\mathbb{Q}}[f] > 0$. This ends the proof of the claim and the proof of the optional decomposition theorem.

q.e.d.

5.5 Remark. Let us stress out that we have proved above that in theorem 5.1 for each process W with $W - V$ increasing, W a $\mathbb{Q}$-local-super-martingale for each $\mathbb{Q} \in \mathcal{M}^e(S)$ and W being maximal with respect to this property in the sense of lemma 5.4, we obtain the semi-martingale representation $W = H \cdot S$.

5.6 Remark. Referring to the notation of the proof of the optional decomposition theorem and the claim made in it , the fact that the cone B is weak* closed in L^∞_w yields a duality equality as well as the characterisation of maximal elements in the set of w-admissible outcomes. These results are parallel to the results obtained in the case of locally bounded price processes. We refer to a our forthcoming paper [DS96] for more details.

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ETH-Zentrum, CH-8092 Zürich, Schweiz
E-mail address: `delbaen@math.ethz.ch`

Universität Wien, Brünnerstr. 72, A-1210 Wien Austria
E-mail address: `Walter.Schachermayer@univie.ac.at`