On convex functions on the duals of Δ_2-Orlicz spaces

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ABSTRACT

In the dual L_{Φ^*} of a Δ_2-Orlicz space L_Φ, we show that a proper (resp. finite) convex function is lower semicontinuous (resp. continuous) for the Mackey topology $\tau(L_{\Phi^*}, L_\Phi)$ if and only if on each order interval $[-\zeta, \zeta] = \{\xi : -\zeta \leq \xi \leq \zeta\}$ ($\zeta \in L_{\Phi^*}$), it is lower semicontinuous (resp. continuous) for the topology of convergence in probability. For this purpose, we provide the following Komlós type result: every norm bounded sequence $(\xi_n)_n$ in L_{Φ^*} admits a sequence of forward convex combinations $\bar{\xi}_n \in \mathrm{conv}(\xi_n, \xi_{n+1}, ...)$ such that $\sup_n |\bar{\xi}_n| \in L_{\Phi^*}$ and $\bar{\xi}_n$ converges a.s.

Key Words: Orlicz spaces, Mackey topology, Komlós's theorem, convex functions, order closed sets, risk measures

1 Introduction

Notation. We use the usual probabilistic notation. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $L_0 := L_0(\Omega, \mathcal{F}, \mathbb{P})$ stands for the space of (classes modulo equality $\mathbb{P}$-a.s. of) finite measurable functions equipped with the complete metrisable vector topology τ_{L_0} of convergence in $\mathbb{P}$ (in probability). As usual, we identify a measurable function with the class it generates. We write $\mathbb{E}[\xi] := \int_\Omega \xi d\mathbb{P}$ whenever it makes sense, and $L_p := L_p(\Omega, \mathcal{F}, \mathbb{P})$, $p \in [1, \infty]$, denote the standard Lebesgue spaces.

Problems in financial mathematics often involve convex functions on the dual E' of a Banach space E. Dealing with such f, the lower semicontinuity (lsc) and continuity for the Mackey topology $\tau(E', E)$ are basic; the former ($\Leftrightarrow \sigma(E', E)$-lsc) is necessary and sufficient (by the Hahn-Banach theorem) for the dual representation

$$f(x') = \sup_{x \in E}(\langle x, x' \rangle - f^*(x)),\ x' \in E'; \quad \text{where } f^*(x) = \sup_{x' \in E'}(\langle x, x' \rangle - f(x'))$$

Generally speaking, $\tau(E', E)$ is not easy to deal with, but its restrictions to bounded sets often have a nice description. The best known case is $L_\infty = L_1'$: on bounded sets, $\tau(L_\infty, L_1)$ coincides with the topology of L_0, a fortiori metrisable (this result is due to Grothendieck; see [11], pp.222-223). Hence by the Krein-Šmulian theorem, we have

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Proposition 1.1. *For proper convex functions f on L_∞, the following are equivalent:*

(1) f is $\sigma(L_\infty, L_1)$-lsc, equivalently $\tau(L_\infty, L_1)$-lsc;
(2) f is sequentially $\tau(L_\infty, L_1)$-lsc;
(3) f is lsc on bounded sets for the topology of convergence in probability.

The following result for the $\tau(L_\infty, L_1)$-continuity is also known for convex risk measures (e.g. [12, 6]), and it remains true for finite convex functions; but we could not find a relevant reference, so we include a short proof in the Appendix.

Proposition 1.2. *For any convex function $f : L_\infty \to \mathbb{R}$, the following are equivalent:*

(1) f is $\tau(L_\infty, L_1)$-continuous;
*(2) f is **sequentially** $\tau(L_\infty, L_1)$-continuous;*
(3) f is continuous for the topology of convergence in probability on bounded sets.

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a (finite coercive) ***Young function***, i.e. an even convex function with $\Phi(0) = 0$ and $\lim_{x\to+\infty} \frac{\Phi(x)}{x} = +\infty$. Then $\mathbb{B}_\Phi := \{\xi \in L_0 : \mathbb{E}[\Phi(\xi)] \leq 1\}$ is a closed convex solid subset of L_0 bounded in L_1 containing a non-zero constant, thus it generates a Banach lattice with the closed unit ball $\mathbb{B}_\Phi$, called the ***Orlicz space***:

$$L_\Phi := \bigcup_{\lambda>0} \lambda\mathbb{B}_\Phi = \{\xi \in L_0 : \exists\lambda > 0 \text{ with } \mathbb{E}[\Phi(\lambda\xi)] < \infty\},$$

given the norm $\|\xi\|_\Phi := \inf\{\lambda > 0 : \xi \in \lambda\mathbb{B}_\Phi\}$ and ***a.s. pointwise order***. In general, $L_\infty \subset L_\Phi \subset L_1$ with continuous injections. The conjugate $\Phi^*(y) := \sup_x(xy - \Phi(x))$ is again a (finite coercive) Young function, so the Orlicz space L_{Φ^*} is similarly defined.

A Young function Φ is said to satisfy the ***Δ_2-condition***, denoted by $\Phi \in \Delta_2$, if $\limsup_{x\to\infty} \Phi(2x)/\Phi(x) < \infty$, or equivalently

$$p_\Phi := \inf_{x\geq 0} p_\Phi(x) := \inf_{x\geq 0}\left(\sup_{y>x} \frac{y\Phi'(y)}{\Phi(y)}\right) < \infty, \tag{1.1}$$

where Φ' is the left-derivative of Φ (see [19], Th. II.2.3). If $\Phi \in \Delta_2$, the dual L'_Φ of L_Φ is identified via $\langle\xi, \eta\rangle = \mathbb{E}[\xi\eta]$ with L_{Φ^*} given an equivalent norm $\|\xi\|_{(\Phi^*)} := \sup_{\eta\in\mathbb{B}_\Phi} \mathbb{E}[\eta\xi]$; more precisely $\|\xi\|_{\Phi^*} \leq \|\xi\|_{(\Phi^*)} \leq 2\|\xi\|_{\Phi^*}$, and $\mathbb{E}[\eta\xi] \leq \|\eta\|_\Phi\|\xi\|_{(\Phi^*)}$. In particular, L_Φ is reflexive if both $\Phi, \Phi^* \in \Delta_2$; the condition is also necessary if $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless. In the sequel, we suppose $\Phi \in \Delta_2$ unless otherwise mentioned.

Our basic interest is to understand the $\tau(L_{\Phi^*}, L_\Phi)$-lower semicontinuity and continuity of convex functions through the sequential convergence in probability on bounded sets. At this point, we note that there are two possible interpretations of "bounded sets"; norm bounded sets, and ***order bounded*** sets, that is, those $A \subset L_{\Phi^*}$ contained in an ***order interval*** $[-\zeta, \zeta] := \{\xi : -\zeta \leq \xi \leq \zeta\}$, $0 \leq \zeta \in L_{\Phi^*}$, i.e. ***dominated in*** L_{Φ^*}. Since $[-\zeta, \zeta] \subset \|\zeta\|_{\Phi^*}\mathbb{B}_{\Phi^*}$, the order bounded sets are norm bounded, and in L_∞, the two notions of boundedness are identical.

The core of this papar is a few variants of ***Komlós's theorem*** in the dual L_{Φ^*} of a Δ_2-Orlicz space L_Φ. The classical Komlós theorem [13] states that any bounded sequence $(\xi_n)_n$ in L_1 has a subsequence $(n_k)_k$ as well as $\xi \in L_1$ such that for any further subsequence $(n_{k(i)})_i$, the ***Cesàro means*** $\frac{1}{n}\sum_{i\leq N} \xi_{n_{k(i)}}$ converges a.s. to ξ. The basic form of our variants (Theorem 3.6) asserts that under the stronger assumption of boundedness in L_{Φ^*} and convergence in $\mathbb{P}$, a subsequence can be chosen so that the

Cesàro means are ***order bounded*** in L_{Φ^*} as well. Its practically useful consequence (Corollary 3.10) is that any ***norm bounded*** sequence in L_{Φ^*}, not necessarity convergent in $\mathbb{P}$, has an ***order bounded*** (and a.s. convergent) sequence of ***forward convex combinations*** $\zeta_n \in \mathrm{conv}(\xi_k; k \geq n)$, $n \geq 1$. Moreover, if $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, this version of Komlós theorem characterises the Δ_2-Orlicz spaces (Theorem 3.12).

In view of the Krein-Šmulian theorem, this form of Komlós theorem yields that a ***convex*** set $C \subset L_{\Phi^*}$ is $\sigma(L_{\Phi^*}, L_\Phi)$-closed if (and only if[1]) it is ***order closed***:

$$\forall \zeta \in L_{\Phi^*},\ C \cap [-\zeta, \zeta] \text{ is closed in } L_0. \tag{1.2}$$

In terms of functions, this reads as: a proper convex function f on L_{Φ^*} is $\sigma(L_{\Phi^*}, L_\Phi)$-lsc if (and only if) f is lsc for the topology of L_0 on order intervals in L_{Φ^*}, or explicitly $f(\xi) \leq \liminf_n f(\xi_n)$ whenever $\xi_n \to \xi$ in $\mathbb{P}$ and $\sup_n |\xi_n| \in L_{\Phi^*}$ (Theorem 4.4). A similar characterisation of the $\tau(L_{\Phi^*}, L_\Phi)$-continuity is also given (Theorem 4.5).

The question of the weak* closedness of order closed convex sets in L_{Φ^*} is raised by [5] in the context of representation of convex risk measures. They claimed in [5, Lemma 6] that this is the case because $\sigma(L_{\Phi^*}, L_\Phi)$ has the following property:

(C) if $\xi_\alpha \to \xi$ in $\sigma(L_{\Phi^*}, L_\Phi)$, there exist a sequence of indices $(\alpha_n)_n$ and $\zeta_n \in \mathrm{conv}(\xi_{\alpha_k}; k \geq n)$, $n \geq 1$, such that $\zeta_n \to \xi$ a.s. and $\sup_n |\zeta_n| \in L_{\Phi^*}$.

Unfortunately, this is not correct; (C) holds (if and) ***only if*** L_Φ is reflexive ([10]). For $(\zeta_n)_n$ in (C) converges in $\sigma(L_{\Phi^*}, L_\Phi)$, thus (C) would imply that for any convex set $C \subset L_{\Phi^*}$, its weak* closure coincides with the ***sequential weak* closure*** $C_{(1)} := \{\xi : \xi = w^*\text{-}\lim_n \xi_n \text{ with } (\xi_n)_n \subset C\}$, while any non-reflexive Banach space has a convex set C in the dual such that $C_{(1)}$ is not weak* closed ([18, Th. 2]; see [17] for the history of problem of sequential weak* closures which goes back to Banach [4]). On the other hand, Corollary 3.10 shows that the property (C) holds for ***bounded nets*** (recall that convergent nets need not be bounded).

2 Mackey Topology on Orlicz Spaces

The following criterion for $\sigma(L_\Phi, L_{\Phi^*})$-compact sets is known (e.g. [19], Th. IV.5.1), but we include a short proof in the Appendix. Here the Δ_2-condition is not necessary.

Lemma 2.1. *(Regardless of $\Phi \in \Delta_2$,) a set $A \subset L_\Phi$ is relatively $\sigma(L_\Phi, L_{\Phi^*})$-compact if and only if for each $\xi \in L_{\Phi^*}$, $A\xi := \{\eta\xi : \eta \in A\}$ is uniformly integrable.*

Lemma 2.2. *$\tau(L_{\Phi^*}, L_\Phi)$ is finer than the restriction of τ_{L_0} to L_{Φ^*}, and*

$$\forall \zeta \in L_{\Phi^*},\ \tau(L_{\Phi^*}, L_\Phi)|_{[-\zeta,\zeta]} = \tau_{L_0}|_{[-\zeta,\zeta]}. \tag{2.1}$$

In particular, $\tau(L_{\Phi^*}, L_\Phi)$ is metrisable on order bounded sets. If $\Phi \in \Delta_2$, we have

$$\sigma(L_{\Phi^*}, L_\Phi)|_{\mathbb{B}_{\Phi^*}} \subset \tau_{L_0}|_{\mathbb{B}_{\Phi^*}} \subset \tau(L_{\Phi^*}, L_\Phi)|_{\mathbb{B}_{\Phi^*}}. \tag{2.2}$$

[1] Regardless of $\Phi \in \Delta_2$ and convexity, $\sigma(L_{\Phi^*}, L_\Phi)$-closed $\Rightarrow$ order closed $\Rightarrow$ norm closed since L_Φ is identified with the ***order continuous dual of*** L_{Φ^*} and norm convergent sequences have order convergent subsequences; see e.g. [20, Ch. 14] for details and unexplained terminologies.

Proof. The (image in L_Φ) of $\mathbb{B}_{L_\infty}$ is $\sigma(L_\Phi, L_{\Phi^*})$-compact, thus defines a Mackey continuous seminorm $\xi \mapsto \sup_{\eta\in\mathbb{B}_{L_\infty}} |\mathbb{E}[\xi\eta]| = \mathbb{E}[|\xi|] \geq \mathbb{E}[|\xi| \wedge 1]$, so $\tau(L_{\Phi^*}, L_\Phi)$ is finer than the restriction of τ_{L_0}. On the other hand, for any $\sigma(L_\Phi, L_{\Phi^*})$-compact set $A \subset L_\Phi$ and $\zeta \in L_{\Phi^*}$, one has $\lim_N \sup_{\eta\in A} \mathbb{P}(|\eta| \vee |\zeta| > N) = 0$, and $p_A(\xi) := \sup_{\eta\in A} |\mathbb{E}[\eta\xi]| \leq \sup_{\eta\in A} \mathbb{E}[|\eta\zeta|\mathbb{1}_{\{|\eta|\vee|\zeta|>N\}}] + N^2\mathbb{E}[|\xi| \wedge 1]$, $\forall N \in \mathbb{N}$, on $[-\zeta, \zeta]$. A standard diagonalisation procedure then shows that p_A is τ_{L_0}-continuous on $[-\zeta, \zeta]$, and we see that $\tau(L_{\Phi^*}, L_\Phi)|_{[-\zeta,\zeta]} \subset \tau_{L_0}|_{[-\zeta,\zeta]}$. Finally, if $\Phi \in \Delta_2$, so $L_{\Phi^*} = L'_\Phi$, $\mathbb{B}_{\Phi^*}$ is $\sigma(L_{\Phi^*}, L_\Phi)$-compact, thus $\eta\mathbb{B}_{\Phi^*}$, $\eta \in L_\Phi$, are uniformly integrable. Thus $\xi_n \in \mathbb{B}_{\Phi^*}$ and $\xi_n \to \xi$ in $\mathbb{P}$ imply $\mathbb{E}[\eta\xi_n] \to \mathbb{E}[\eta\xi]$ ($\forall\eta \in L_\Phi$), i.e. $\xi_n \to \xi$ in $\sigma(L_{\Phi^*}, L_\Phi)$, which proves (2.2). □

In the last part, the assumption $\Phi \in \Delta_2$ is used only to ensure that bounded sequences are relatively $\sigma(L_{\Phi^*}, L_\Phi)$-compact. Thus the same argument shows that:

Corollary 2.3. *(Regardless of $\Phi \in \Delta_2$,) if a sequence $(\xi_n)_n$ in L_{Φ^*} is null in $\mathbb{P}$ and $\sigma(L_{\Phi^*}, L_\Phi)$-convergent, then $\xi_n \to 0$ in $\sigma(L_{\Phi^*}, L_\Phi)$.*

Remark 2.4. On $\mathbb{B}_{\Phi^*}$, $\tau(L_{\Phi^*}, L_\Phi)$ is not generally the same as the topology of L_0. For example, if $A_n \in \mathcal{F}$ are disjoint with $\mathbb{P}(A_n) > 0$, $\xi_n = \mathbb{P}(A_n)^{-1/2}\mathbb{1}_{A_n}$ form a sequence in $\mathbb{B}_{L_2}$, null in $\mathbb{P}$, but $\|\xi_n\|_2 \equiv 1$, while $\tau(L_2, L_2)$ is the norm topology. ♦

Proposition 2.5. *If $\Phi \in \Delta_2$, the following are equivalent for all convex $C \subset L_{\Phi^*}$:*

(1) C is $\sigma(L_{\Phi^*}, L_\Phi)$-closed;
*(2) C is **sequentially** $\sigma(L_{\Phi^*}, L_\Phi)$-closed;*
(3) for each $\lambda > 0$, $C \cap \lambda\mathbb{B}_{\Phi^*}$ is closed in L_0, or equivalently, $\xi_n \in C$ ($\forall n$), $\xi_n \to \xi$ in $\mathbb{P}$ and $\sup_n \|\xi_n\|_{\Phi^*} < \infty$ imply $\xi \in C$.

Proof. By the Krein-Šmulian theorem, (1) ⇔ $C \cap \lambda\mathbb{B}_{\Phi^*}$, $\lambda > 0$, are $\sigma(L_{\Phi^*}, L_\Phi)$-closed, and the three kinds of closedness are the same for $C \cap \lambda\mathbb{B}_{\Phi^*}$ by (2.2). □

3 Komlós-Type Results

In the sequel, we suppose $\Phi \in \Delta_2$ so that $L_{\Phi^*} = L'_\Phi$ unless otherwise mentioned.

Recall that $\Phi \in \Delta_2$ if and only if $p_\Phi = \inf_{x\geq 0} p_\Phi(x) < \infty$ where $p_\Phi(x) = \sup_{y>x} \frac{y\Phi'(y)}{\Phi(y)}$ (see (1.1)). Let $q_\Phi := \lim_{x\to\infty} \frac{p_\Phi(x)}{p_\Phi(x)-1} = \frac{p_\Phi}{p_\Phi-1} > 1$ with the convention $1/0 = \infty$.

Lemma 3.1. *For any $1 \leq q < q_\Phi$, L_{Φ^*} has **an upper q-estimate**, that is, there exists a constant $C_{q,\Phi^*} > 0$ such that for any $n \in \mathbb{N}$ and disjointly supported $\xi_1, ..., \xi_n \in L_{\Phi^*}$ (i.e. $\xi_k = \xi_k\mathbb{1}_{A_k}$ with $A_k \in \mathcal{F}$ pairwise disjoint),*

$$\left\|\sum_{k\leq n} \xi_k\right\|_{(\Phi^*)} \leq C_{q,\Phi^*}\left(\sum_{k\leq n} \|\xi_k\|^q_{(\Phi^*)}\right)^{1/q}. \tag{3.1}$$

Proof. The case $q = 1$ is trivial (we can take $C_{q,\Phi^*} = 1$), and note that $1 < q < \frac{p_\Phi}{p_\Phi-1}$ if and only if $q = \frac{p}{p-1}$ for some $p \in (p_\Phi(x_0), \infty)$ and $x_0 > 0$. Fix such q, p and x_0. Then $\Psi(x) := \frac{\Phi(x_0)}{x_0}x\mathbb{1}_{[0,x_0]}(x) + \Phi(x)\mathbb{1}_{(x_0,\infty)}(x)$ is a Δ_2-Young function with $\Psi(x) = \Phi(x)$ for $x \geq x_0$, thus $L_\Psi = L_\Phi$ with equivalent norms (since $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, see [19], Th. V.1.3); hence there exists a $C > 0$ such that

$$C^{-1}\|\cdot\|_{(\Psi^*)} \leq \|\cdot\|_{(\Phi^*)} \leq C\|\cdot\|_{(\Psi^*)}. \tag{3.2}$$

Moreover, $\Psi(x) > 0$ for $x > 0$ and $p_\Psi(0) = 1 \vee p_\Phi(x_0) = p_\Phi(x_0) < p < \infty$; in particular, for any $\lambda \geq 1$ and $x > 0$, $\log \frac{\Psi(\lambda x)}{\Psi(x)} = \int_1^\lambda \frac{tx\Psi'(tx)}{\Psi(tx)} \frac{dt}{t} \leq p \log \lambda$, hence

$$\Psi(\lambda x) \leq \lambda^p \Psi(x) \text{ for } x > 0,\ \lambda \geq 1. \tag{3.3}$$

Therefore $1 = \mathbb{E}[\Psi(\eta/\|\eta\|_\Psi)] \leq \|\eta\|_\Psi^{-p}\mathbb{E}[\Psi(\eta)]$ for $0 < \|\eta\|_\Psi \leq 1$, where the first equality is another consequence of $\Phi \in \Delta_2$. Hence we have

$$\|\eta\|_\Psi \leq \mathbb{E}[\Psi(\eta)]^{1/p} \text{ for all } \eta \in \mathbb{B}_\Psi. \tag{3.4}$$

Now if $\xi_k = \xi_k 1\!\!1_{A_k} \in L_{\Phi^*} = L_{\Psi^*}$ with $A_k \in \mathcal{F}$ disjoint, then for any $\eta \in \mathbb{B}_\Psi$,

$$\begin{aligned}\mathbb{E}\left[\left(\sum_{k\leq n}\xi_k\right)\eta\right] &\leq \sum_{k\leq n}\|\xi_k\|_{(\Psi^*)}\|\eta 1\!\!1_{A_k}\|_\Psi \overset{(3.4)}{\leq} \sum_{k\leq n}\|\xi_k\|_{(\Psi^*)}\mathbb{E}[\Psi(\eta)1\!\!1_{A_k}]^{1/p} \\ &\leq \left(\sum_{k\leq n}\|\xi_k\|_{(\Psi^*)}^q\right)^{1/q}\left(\sum_{k\leq n}\mathbb{E}[\Psi(\eta)1\!\!1_{A_k}]\right)^{1/p} \leq \left(\sum_{k\leq n}\|\xi_k\|_{(\Psi^*)}^q\right)^{1/q},\end{aligned}$$

since $\sum_{k\leq n}\mathbb{E}[\Psi(\eta)1\!\!1_{A_k}] \leq \mathbb{E}[\Psi(\eta)] \leq 1$. Taking the supremum over $\eta \in \mathbb{B}_\Psi$,

$$\frac{1}{C}\left\|\sum_{k\leq n}\xi_k\right\|_{(\Phi^*)} \overset{(3.2)}{\leq} \left\|\sum_{k\leq n}\xi_k\right\|_{(\Psi^*)} \leq \left(\sum_{k\leq n}\|\xi_k\|_{(\Psi^*)}^q\right)^{\frac{1}{q}} \overset{(3.2)}{\leq} C\left(\sum_{k\leq n}\|\xi_k\|_{(\Phi^*)}^q\right)^{\frac{1}{q}}. \qquad \square$$

Corollary 3.2. *If $(\xi_n)_n$ is a norm bounded disjointly supported sequence in L_{Φ^*}, then*

$$\sup_n\left|\frac{\xi_1+\cdots+\xi_n}{n}\right| \in L_{\Phi^*} \quad \textit{and} \quad \left\|\frac{\xi_1+\cdots+\xi_n}{n}\right\|_{\Phi^*} \to 0.$$

Proof. Let $\xi_n = \xi_n 1\!\!1_{A_n}$ with $A_n \in \mathcal{F}$ disjoint, $a := \sup_n\|\xi_n\|_{(\Phi^*)} < \infty$, $1 < q < q_\Phi$, and $C = C_{q,\Phi^*}$ as in Lemma 3.1. Put $\bar{\xi}_n := \frac{\xi_1+\cdots+\xi_n}{n}$. Then $\|\bar{\xi}_n\|_{(\Phi^*)} \leq aC(nn^{-q})^{1/q} = aCn^{\frac{1}{q}-1} \to 0$. Next, observe that $\|\sup_n|\bar{\xi}_n|\|_{(\Phi^*)} = \sup_N\|\sup_{n\leq N}|\bar{\xi}_n|\|_{(\Phi^*)}$ and

$$\sup_{n\leq N}|\bar{\xi}_n| = \sum_{k\leq N}\left(\sup_{n\leq N}|\bar{\xi}_n|\right)1\!\!1_{A_k} = \sum_{k\leq N}\left(\sup_{k\leq n\leq N}\frac{1}{n}|\xi_k|\right)1\!\!1_{A_k} = \sum_{k\leq N}\frac{1}{k}|\xi_k|,$$

while $\left\|\sum_{k\leq N}\frac{1}{k}\xi_n\right\|_{(\Phi^*)} \leq aC\left(\sum_{k\leq N}\frac{1}{k^q}\right)^{1/q} \leq aC\left(\sum_{k=1}^\infty\frac{1}{k^q}\right)^{1/q} < \infty$, so $\sup_n|\bar{\xi}_n| \in L_{\Phi^*}$. $\square$

Noting that $\bar{\xi}_n = \frac{\xi_{n+1}+\cdots+\xi_{2n}}{n} = 2\frac{\xi_1+\cdots+\xi_{2n}}{2n} - \frac{\xi_1+\cdots+\xi_n}{n} \in \mathrm{conv}(\xi_n, \xi_{n+1}, \ldots)$, we get:

Corollary 3.3. *Any norm bounded disjoint sequence $(\xi_n)_n$ in L_{Φ^*} has an order bounded and norm null sequence of forward convex combinations $\bar{\xi}_n \in \mathrm{conv}(\xi_k; k \geq n)$.*

Since any subsequence of norm bounded disjoint sequence is again bounded and disjoint, the same conclusion holds for any subsequence; thus

Corollary 3.4. *Any norm bounded disjoint sequence in L_{Φ^*} is $\sigma(L_{\Phi^*}, L'_{\Phi^*})$-null.*

Remark 3.5. The last two corollaries could be derived also from the fact that the dual of a Banach lattice E has order continuous norm iff every norm bounded disjoint sequence in E is weakly null ([20, Th. 116.1] or [14, Th. 2.4.14]). In L_{Φ^*}, $(\xi_n)_n$ is disjoint in the lattice sense iff it is disjointly supported, while $L'_{\Phi^*} = L_\Phi \oplus L_\infty^\bigcirc$, where $L_\infty^\bigcirc$ is the polar of $L_\infty \subset L_{\Phi^*}$ in L'_{Φ^*}. The projections of L'_{Φ^*} onto L_Φ and onto $L_\infty^\bigcirc$ are order continuous (e.g. [3]). But $L_\infty^\bigcirc$ is an AL space, hence has order continuous norm (regardless of Δ_2; e.g. [20, Th. 133.6]), thus $\Phi \in \Delta_2$ implies that ($\|\cdot\|_{(\Phi)} = \|\cdot\|_{L'_{\Phi^*}}|_{L_\Phi}$, hence) $\|\cdot\|_{L'_{\Phi^*}}$ is order continuous, so bounded disjoint sequences are weakly null. ♦

Now we can state the basic version of our Komlós type result.

Theorem 3.6. *If $(\xi_n)_n$ is a norm bounded sequence in L_{Φ^*}, converging in $\mathbb{P}$ to some $\xi \in L_{\Phi^*}$, then there exists a subsequence $(\xi_{n_k})_k$ such that for any further subsequence $(\xi_{n_{k(i)}})_i$, the Cesàro means $\frac{1}{N}\sum_{k\le N}\xi_{n_{k(i)}}$ converge in order to ξ, i.e.*

$$\sup_N \left|\frac{1}{N}\sum_{i\le N}\xi_{n_{k(i)}}\right| \in L_{\Phi^*} \text{ and } \frac{1}{N}\sum_{i\le N}\xi_{n_{k(i)}} \xrightarrow{N} \xi \text{ a.s.} \tag{3.5}$$

Here the original bounded sequence $(\xi_n)_n$ is supposed to converge in $\mathbb{P}$, which is needed to ensure that the Cesàro means themselves of any subsequence converge in order. Without this a priori assumption, we still have a slightly weaker conclusion.

Theorem 3.7. *Any norm bounded sequence $(\xi_n)_n$ in L_{Φ^*} admits a subsequence $(\xi_{n_k})_k$ as well as $\xi \in L_{\Phi^*}$ such that for any subsequence $(\xi_{n_{k(i)}})_i$, the sequence of Cesàro means $\frac{1}{N}\sum_{k\le N}\xi_{n_{k(i)}}$ has a subsequence order convergent to ξ, i.e. there is a sequence $(N_l)_l$ with $\sup_l\left|\frac{1}{N_l}\sum_{i\le N_l}\xi_{n_{k(i)}}\right| \in L_{\Phi^*}$ and $\frac{1}{N_l}\sum_{i\le N_l}\xi_{n_{k(i)}} \to \xi$ a.s.*

Lemma 3.8 (cf. [16]). *If $\xi_n \to 0$ in $\mathbb{P}$ and if $(\Phi^*(\xi_n))_n$ is uniformly integrable, there exists a subsequence $(\xi_{n_k})_k$ such that $\sup_k|\xi_{n_k}| \in L_{\Phi^*}$; in particular $\xi_n \to 0$ in $\tau(L_{\Phi^*}, L_\Phi)$.*

Proof. The assumption implies $\mathbb{E}[\Phi^*(\xi_n)] \to 0$, so there is a subsequence $(\xi_{n_k})_k$ such that $\sum_k \mathbb{E}[\Phi^*(\xi_{n_k})] < \infty$. Noting that $\Phi^*(|\eta| \vee |\eta'|) = \Phi^*(\eta)\mathbb{1}_{\{|\eta|>|\eta'|\}} + \Phi^*(\eta')\mathbb{1}_{\{|\eta|\le|\eta'|\}} \le \Phi^*(\eta) + \Phi^*(\eta')$, a simple induction and the monotone convergence theorem show that

$$\mathbb{E}\left[\Phi^*\left(\sup_k|\xi_{n_k}|\right)\right] \le \lim_m \mathbb{E}\left[\Phi^*\left(\sup_{k\le m}|\xi_{n_k}|\right)\right] \le \lim_m \sum_{k\le m}\mathbb{E}[\Phi^*(\xi_{n_k})] \le \sum_{k=1}^\infty \mathbb{E}[\Phi^*(\xi_{n_k})] < \infty.$$

Hence $\sup_k|\xi_{n_k}| \in L_{\Phi^*}$. In particular, $\xi_{n_k} \to 0$ in $\tau(L_{\Phi^*}, L_\Phi)$ by (2.1). Since the assumptions on $(\xi_n)_n$ are inherited to any subsequence, we deduce that every subsequence has a $\tau(L_{\Phi^*}, L_\Phi)$-null subsequence; hence $(\xi_n)_n$ itself is $\tau(L_{\Phi^*}, L_\Phi)$-null. □

Proof of Theorems 3.6 and 3.7. Let $(\xi_n)_n$ be a norm bounded sequence in L_{Φ^*}, a fortiori bounded in L_1. Komlós's theorem yields a subsequence, still denoted by $(\xi_n)_n$, and a $\xi \in L_1$ such that the Cesàro means of any further subsequence converges a.s. to ξ; then $\xi \in L_{\Phi^*}$ by Fatou's lemma. We can normalise $(\xi_n)_n$ so that $\xi = 0$ and $\|\xi_n\|_{\Phi^*} \le 1$ ($\Leftrightarrow \mathbb{E}[\Phi^*(\xi_n)] \le 1$). Then the Kadec–Pełczyński theorem (e.g. [1, Lemma 5.2.8]) applied to the bounded sequence $(\Phi^*(\xi_n))_n$ yields a subsequence $(\zeta_n)_n$ of (ξ_n) as well as a disjoint sequence $(A_n)_n$ in $\mathcal{F}$ such that $(\Phi^*(\zeta_n\mathbb{1}_{A_n^c}))_n$ is uniformly integrable. Let $\zeta_n^r := \zeta_n\mathbb{1}_{A_n^c}$ and $\zeta_n^s := \zeta_n\mathbb{1}_{A_n}$ so that $\zeta_n = \zeta_n^r + \zeta_n^s$.

Now if the original sequence $(\xi_n)_n$ converges in $\mathbb{P}$ (to 0 by the reduction above), then $(\zeta_n)_n \subset (\xi_n)_n$ as well as $(\zeta^r)_n$ are null in $\mathbb{P}$. Since $(\Phi^*(\zeta^r_n))_n$ is uniformly integrable, Lemma 3.8 yields a subsequence $(n_k)_k$ of positive integers such that $\eta' := \sup_k |\zeta^r_{n_k}| \in L_{\Phi^*}$. On the other hand, $(\zeta^s_n)_n$ (and any of its subsequence) is a norm bounded disjoint sequence, hence Corollary 3.2 shows that for any subsequence $(k(i))_i$,

$$\sup_N \left| \frac{1}{N} \sum_{i \le N} \zeta_{n_{k(i)}} \right| \le \sup_N \left| \frac{1}{N} \sum_{i \le N} \zeta^r_{n_{k(i)}} \right| + \sup_N \left| \frac{1}{N} \sum_{i \le N} \zeta^s_{n_{k(i)}} \right| \le \eta' + \sup_N \left| \frac{1}{N} \sum_{i \le N} \zeta_{n_{k(i)}} \right| \in L_{\Phi^*}.$$

Since $\frac{1}{N} \sum_{i \le N} \zeta_{n_{k(i)}} \to 0$ a.s. by construction, we have Theorem 3.6.

Next, if $(\zeta_n)_n$ is not null in $\mathbb{P}$, we can no longer hope for a "universal bound" for the regular part $(\zeta^r_n)_n$. However, once a subsequence $(n_k)_k$ is chosen we get

$$\bar{\zeta}_N := \frac{1}{N} \sum_{k \le N} \zeta_{n_k} = \frac{1}{N} \sum_{k \le N} \zeta^r_{n_k} + \frac{1}{N} \sum_{k \le N} \zeta^s_{n_k} =: \bar{\zeta}^r_N + \bar{\zeta}^s_N \to 0 \text{ in } \mathbb{P},$$

by the construction of $(\zeta_n)_n$. Again by Corollary 3.2, $(\bar{\zeta}^s_N)_N$ is order bounded and norm null. In particular, $\bar{\zeta}^r_N = \bar{\zeta}_N - \bar{\zeta}^s_N \to 0$ in $\mathbb{P}$, and $(\Phi^*(\bar{\zeta}^r_N))_N$ is uniformly integrable since Φ^* is convex. Thus by Lemma 3.8, we find a subsequence $(N(i))_i$ such that $(\bar{\zeta}^r_{N(i)})_i$, hence $(\bar{\zeta}_{N(i)})_i = (\bar{\zeta}^r_{N(i)} + \bar{\zeta}^s_{N(i)})_i$ too, are order bounded. □

Since $(\bar{\zeta}^r_N)_N$ in the last paragraph is null in $\mathbb{P}$ and $(\Phi^*(\bar{\zeta}^r_N))_N$ is uniformly integrable, it is null in $\tau(L_{\Phi^*}, L_\Phi)$ by the last part of Lemma 3.8. Thus we have also:

Corollary 3.9. *Any norm bounded sequence $(\xi_n)_n$ in L_{Φ^*} has a subsequence $(\xi_{n_k})_k$ and $\xi \in L_{\Phi^*}$ such that for any further subsequence $(n_{k(i)})_i$, $\frac{1}{N} \sum_{i \le N} \xi_{n_{k(i)}} \to \xi$ in $\tau(L_{\Phi^*}, L_\Phi)$.*

At the moment, it is not clear if one can drop the assumption of convergence in $\mathbb{P}$ in Theorem 3.6, or equivalently if the Cesàro means in Theorem 3.7 are order bounded without passing to a further subsequence. This question is left for a future work. In applications, however, this point does not much matter; since any norm bounded sequence in L_{Φ^*} (a fortiori bounded in L_1) has an a.s. convergent sequence of forward convex combinations by the usual Komlós theorem, and convex combinations of convex combinations are convex combinations (cf. Cesàro means of Cesàro means are not Cesàro means), we get the following utility grade version of Theorem 3.6.

Corollary 3.10. *Any norm bounded sequence $(\xi_n)_n$ in L_{Φ^*} admits a sequence of forward convex combinations $\bar{\xi}_n \in \mathrm{conv}(\xi_k; k \ge n)$ as well as a $\xi \in L_{\Phi^*}$ such that $\bar{\xi}_n \to \xi$ in order, i.e. $\sup_n |\bar{\xi}_n| \in L_{\Phi^*}$ and $\bar{\xi}_n \to \xi$ a.s.*

Regarding the property (*C*) of [5], we can confirm that it is true for ***bounded nets***, while the boundedness cannot be dropped as noted in the introduction.

Corollary 3.11 ((*C*) for bounded nets; cf. [5], Lemma 6). *If $(\xi_\alpha)_\alpha$ is a **bounded** net in L_{Φ^*} converging in $\sigma(L_{\Phi^*}, L_\Phi)$ to some $\xi \in L_{\Phi^*}$, there exist a sequence of indices $(\alpha_n)_n$ as well as $\zeta_n \in \mathrm{conv}(\xi_{\alpha_k}; k \ge n)$ such that $\sup_n |\zeta_n| \in L_{\Phi^*}$ and $\zeta_n \to \xi$ a.s.*

Proof. By the continuity of $(L_{\Phi^*}, \sigma(L_{\Phi^*}, L_\Phi)) \to (L_1, \sigma(L_1, L_\infty))$, $\xi_\alpha \to \xi$ in $\sigma(L_1, L_\infty)$ as well, and by $L_\infty = L_1'$, one finds a sequence of indices $(\alpha_n)_n$ and $\eta_n \in \mathrm{conv}(\xi_{\alpha_k}; k \ge n)$ with $\|\xi - \eta_n\|_1 \le 1/2^n$, so $\eta_n \to \xi$ a.s. Since $((\xi_\alpha)_\alpha$, hence$)$ $(\eta_n)_n$ is bounded in L_{Φ^*}, Corollary 3.10 yields $\zeta_n \in \mathrm{conv}(\eta_k; k \ge n) \subset \mathrm{conv}(\xi_{\alpha_k}; k \ge n)$ with $\sup_n |\zeta_n| \in L_{\Phi^*}$. □

Finally, when $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, these Komlós type results characterise the Δ_2-Orlicz spaces; in this case, $\Phi \in \Delta_2$ if (and only if) $\lim_n \|\xi \mathbb{1}_{\{|\xi|>n\}}\|_{(\Phi)} = 0$ for every $\xi \in L_\Phi$ (i.e. $\|\cdot\|_{(\Phi)}$ is order continuous on L_Φ; see [20, Th. 133.4]).

Theorem 3.12. *Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, and let Φ be a (finite coercive) Young function (**not a priori assumed** Δ_2). Then the following are equivalent:*

(1) $\Phi \in \Delta_2$;
(2) every norm bounded sequence in L_{Φ^*} has a subsequence with $\tau(L_{\Phi^*}, L_\Phi)$-convergent Cesàro means;
(3) every norm bounded sequence in L_{Φ^*} has a $\sigma(L_{\Phi^*}, L_\Phi)$-convergent sequence of forward convex combinations;
(4) every norm bound sequence in L_{Φ^*} has an order bounded sequence of forward convex combinations.

Proof. (1) $\Rightarrow$ (4) is Corollary 3.10, (1) $\Rightarrow$ (2) is Corollary 3.9, and (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3) are clear. It remains to prove (3) $\Rightarrow$ (1). Since $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, $\Phi \notin \Delta_2$ yields some $0 \le \zeta_0 \in \mathbb{B}_\Phi$ with $\lim_n \sup_{\eta \in \mathbb{B}_{\Phi^*}} \mathbb{E}[\zeta_0 \eta \mathbb{1}_{\{\zeta_0 > n\}}] = \lim_n \|\zeta_0 \mathbb{1}_{\{\zeta_0 > n\}}\|_{(\Phi)} > 0$, so $\zeta_0 \mathbb{B}_{\Phi^*}$ is not uniformly integrable, hence there are $0 \le \eta_n \in \mathbb{B}_{\Phi^*}$, ***disjoint*** sets $A_n \in \mathcal{F}$, $n \ge 1$, and $\varepsilon > 0$ such that $\mathbb{E}[\zeta_0 \eta_n \mathbb{1}_{A_n}] \ge \varepsilon$ $(\forall n)$. Then the bounded sequence $(\eta_n \mathbb{1}_{A_n})_n$ has no $\sigma(L_{\Phi^*}, L_\Phi)$-convergent forward convex combinations: if $\xi_n \in \mathrm{conv}(\eta_k \mathbb{1}_{A_k}; k \ge n)$, $n \ge 1$, then $\xi_n \to 0$ in $\mathbb{P}$ since A_n are disjoint, so the only possible $\sigma(L_{\Phi^*}, L_\Phi)$-limit is 0 by Corollary 2.3, which is impossible since $\mathbb{E}[\zeta_0 \xi_n] \ge \inf_k \mathbb{E}[\zeta_0 \eta_k \mathbb{1}_{A_k}] \ge \varepsilon$. □

4 Closedness of Convex Sets

Now we deduce from Corollary 3.10 that

Theorem 4.1. *A **convex** subset $C \subset L_{\Phi^*}$ is $\sigma(L_{\Phi^*}, L_\Phi)$-closed if and only if for every $\zeta \in L_{\Phi^*}$, the intersection $C \cap [-\zeta, \zeta]$ is closed in L_0 (i.e. order closed).*

Proof. The necessity is clear since $[-\zeta, \zeta]$ is closed in L_0 and $\tau(L_{\Phi^*}, L_\Phi)|_{[-\zeta,\zeta]} = \tau_{L_0}|_{[-\zeta,\zeta]}$. For the sufficiency, it suffices that $C \cap \lambda \mathbb{B}_{\Phi^*}$, $\lambda > 0$, are closed in L_0 (Proposition 2.5). Pick a sequence $(\xi_n)_n$ in $C \cap \lambda \mathbb{B}_{\Phi^*}$ with $\xi_n \to \xi$ in $\mathbb{P}$. Corollary 3.10 yields a sequence $\bar{\xi}_n \in \mathrm{conv}(\xi_k; k \ge n) \subset C$ (by convexity) with $\zeta := \sup_n |\bar{\xi}_n| \in L_{\Phi^*}$, and $\bar{\xi}_n \to \xi$ a.s. But $\lambda \mathbb{B}_{\Phi^*}$ and $C \cap [-\zeta, \zeta]$ are τ_{L_0}-closed, hence $\xi \in C \cap [-\zeta, \zeta] \cap \lambda \mathbb{B}_{\Phi^*}$. □

To the best of our knowledge, this criterion for the weak*-closedness is only known for ***solid sets*** (i.e. $A \subset L_{\Phi^*}$ with $\zeta \in A$ and $|\xi| \le |\zeta| \Rightarrow \xi \in A$); see [2, Th. 4.20]. But convex functions with solid lower level sets are symmetric, so exclude all non-trivial monotone convex functions, especially convex risk measures. Also, since $\sigma(L_{\Phi^*}, L_\Phi)|_{[-\zeta,\zeta]} \subset \tau_{L_0}|_{[-\zeta,\zeta]} = \tau(L_{\Phi^*}, L_\Phi)|_{[-\zeta,\zeta]}$, $\zeta \in L_{\Phi^*}$ (by (2.1) and (2.2)), the condition is also equivalent to: $C \cap [-\zeta, \zeta]$, $\zeta \in L_{\Phi^*}$, are $\sigma(L_{\Phi^*}, L_\Phi)$-closed.

Remark 4.2. After our results were presented in Vienna Congress on Mathematical Finance, 12–14 September 2016 (https://fam.tuwien.ac.at/events/vcmf2016/), and after a discussion with Niushan Gao, he and his collaborators [9] came up with their own proof of Theorem 4.1. They used a different technique which in our opinion will not yield a Komlós type theorem. The problem to get a Komlós type theorem was suggested by Hans Föllmer during the aforementioned Vienna conference. ♦

While the Mackey and weak* closed convex sets in the dual of a Banach space are the same, ***sequentially*** Mackey closed convex sets need not be (sequentially) weak* closed. For instance, $A = \{(\alpha_n)_n \in \ell_1 : \alpha_1 = \sum_{n\ge 2} \alpha_n\}$ is norm closed but not sequentially weak* closed in $\ell_1 = c_0'$ (see [4]), while since $\tau(\ell_1, c_0)$-convergent ***sequences*** are norm convergent, A is sequentially $\tau(\ell_1, c_0)$-closed. In our situation, however, since $\tau(L_{\Phi^*}, L_\Phi)|_{[-\zeta,\zeta]} = \tau_{L_0}|_{[-\zeta,\zeta]}$, $\zeta \in L_{\Phi^*}$, are metrisable, Theorem 4.1 implies that

Corollary 4.3. ***Sequentially*** *$\tau(L_{\Phi^*}, L_\Phi)$-closed convex sets in L_{Φ^*} are $\sigma(L_{\Phi^*}, L_\Phi)$-closed.*

Now the dual representation of proper convex functions on L_{Φ^*}, or equivalently the $\sigma(L_{\Phi^*}, L_\Phi)$-lsc ($\Leftrightarrow \tau(L_{\Phi^*}, L_\Phi)$-lsc), is characterised as follows.

Theorem 4.4. *For a proper convex function f on L_{Φ^*}, the following are equivalent:*

(1) f is $\sigma(L_{\Phi^*}, L_\Phi)$-lsc, or equivalently $f(\xi) = \sup_{\eta\in L_\Phi}(\mathbb{E}[\eta\xi] - f^*(\eta))$, $\xi \in L_{\Phi^*}$;
(2) f is ***sequentially*** *$\tau(L_{\Phi^*}, L_\Phi)$-lsc;*
(3) f is τ_{L_0}-lsc on every order interval $[-\zeta, \zeta]$ ($\zeta \in L_{\Phi^*}$), or equivalently order lsc: $f(\xi) \le \liminf_n f(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L_{Φ^*}.

For the $\tau(L_{\Phi^*}, L_\Phi)$-continuity, we have

Theorem 4.5. *For any convex function $f : L_{\Phi^*} \to \mathbb{R}$, the following are equivalent:*

(1) f is $\tau(L_{\Phi^*}, L_\Phi)$-continuous on L_{Φ^*};
(2) f is ***sequentially*** *$\tau(L_{\Phi^*}, L_\Phi)$-continuous on L_{Φ^*};*
(3) f is ***sequentially*** *$\tau(L_{\Phi^*}, L_\Phi)$-continuous on* ***closed balls*** *$\lambda\mathbb{B}_{\Phi^*}$ ($\lambda > 0$);*
(4) f is ***sequentially*** *$\tau(L_{\Phi^*}, L_\Phi)$-continuous on* ***order intervals****;*
(5) f is τ_{L_0}-continuous on ***order intervals****, or equivalently order continuous, i.e. $f(\xi) = \lim_n f(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L_{Φ^*}.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial; (4) $\Leftrightarrow$ (5) since $\tau(L_{\Phi^*}, L_\Phi)$ coincides on order bounded sets with τ_{L_0}. Suppose (5). Then, by Theorem 4.4, $f = f^{**}$, so by Moreau's theorem [15], it suffices that each $\Lambda_c := \{\eta \in L_\Phi : f^*(\eta) \le c\}$, $c \in \mathbb{R}$, is $\sigma(L_\Phi, L_{\Phi^*})$-compact. By Young's inequality, for any $\lambda > 0$, $\xi \in L_{\Phi^*}$ and $\eta \in \Lambda_c$,

$$|\mathbb{E}[\eta\xi\mathbb{1}_A]| = \mathbb{E}[\eta\xi\mathbb{1}_A] \vee \mathbb{E}[\eta(-\xi)\mathbb{1}_A] \le \frac{1}{\lambda}(f(\lambda\xi\mathbb{1}_A) \vee f(-\lambda\xi\mathbb{1}_A) + c) \tag{4.1}$$

which implies that $\Lambda_c\xi$, $\xi \in L_{\Phi^*}$, are uniformly integrable, thus Λ_c is $\sigma(L_\Phi, L_{\Phi^*})$-compact. For if $\Lambda_c\xi$ were not uniformly integrable, there would be $\varepsilon > 0$, $A_n \in \mathcal{F}$ and $\eta_n \in \Lambda_c$ such that $\mathbb{P}(A_n) \le 2^{-n}$ and $|\mathbb{E}[\eta_n\xi\mathbb{1}_{A_n}]| \ge \varepsilon$; here note that $\mathbb{E}[|\zeta|\mathbb{1}_A] \ge 2\varepsilon$ implies either $|\mathbb{E}[\zeta\mathbb{1}_{A\cap\{\zeta>0\}}]| \ge \varepsilon$ or $|\mathbb{E}[\zeta\mathbb{1}_{A\cap\{\zeta<0\}}]| \ge \varepsilon$ and $\mathbb{P}(A \cap \{\zeta \gtrless 0\}) \le \mathbb{P}(A)$. But since $|\lambda\xi\mathbb{1}_{A_n}| \le \lambda|\xi|$ and $\lambda\xi\mathbb{1}_{A_n} \to 0$ in $\mathbb{P}$ for each $\lambda > 0$, (5) and (4.1) together with a diagonal argument show that $|\mathbb{E}[\eta_n\xi\mathbb{1}_{A_n}]| \to 0$, a contradiction. □

The property that f is (sequentially) τ_{L_0}-continuous on every closed ball implies (via (5)) the Mackey continuity of f. The converse implication holds for all finite convex functions if and only if $\tau(L_{\Phi^*}, L_\Phi)|_{\mathbb{B}_{\Phi^*}} = \tau_{L_0}|_{\mathbb{B}_{\Phi^*}}$. Indeed, seminorms generating the Mackey topology are finite valued Mackey continuous convex functions. As we saw in Remark 2.4, this is not the case if $\Phi(x) = x^2$; more generally, it fails whenever $\Phi^* \in \Delta_2$ (then L_Φ is reflexive). Precisely when $\tau(L_{\Phi^*}, L_\Phi)$ coincide with τ_{L_0} on $\mathbb{B}_{\Phi^*}$ is a subtle question which is left for further investigation.

Remark 4.6. In the proof of (5) ⇒ (1), we only used the facts that $f = f^{**}$ and $f|_{[-\zeta,\zeta]}$ is τ_{L_0}-continuous at 0, from which we derived that f is $\tau(L_{\Phi^*}, L_\Phi)$-continuous at 0. Thus if f is a priori supposed to be $\sigma(L_{\Phi^*}, L_\Phi)$-lsc on L_{Φ^*} (or any of its equivalents in Theorem 4.4), and $f(\xi_0) < \infty$ (we can suppose $\xi_0 = 0$ by translation), the following remain equivalent: (1′) f is $\tau(L_{\Phi^*}, L_\Phi)$-continuous at ξ_0, (2′) f sequentially $\tau(L_{\Phi^*}, L_\Phi)$-continuous at ξ_0, (3′) $f(\xi_0) = \lim_n f(\xi_n)$ whenever $\xi_n \to \xi_0$ in $\tau(L_{\Phi^*}, L_\Phi)$ and $\sup_n \|\xi_n\|_{\Phi^*} < \infty$, (4′) the same but with $|\xi_n| \le \zeta$ for some $\zeta \in L^+_{\Phi^*}$, (5′) the same but with $\xi_n \to \xi_0$ in $\mathbb{P}$ and $|\xi_n| \le \zeta$ for some $\zeta \in L^+_{\Phi^*}$. ♦

4.1 Application to Monetary Utility Functions

In utility theory, ***concave*** functions $u : L_{\Phi^*} \to \mathbb{R} \cup \{-\infty\}$ satisfying the following properties are called ***monetary utility functions*** (see e.g. [7, 8]):

$$u(0) = 0;\ \xi \in L_{\Phi^*},\ \xi \ge 0 \ \Rightarrow\ u(\xi) \ge 0; \tag{4.2}$$

$$a \in \mathbb{R},\ \xi \in L_{\Phi^*} \ \Rightarrow\ u(\xi + a) = u(\xi) + a. \tag{4.3}$$

Since $-u$ is a convex function, which is called a ***convex risk measure***, Theorems 4.4 and 4.5 with obvious change of sign characterise the basic regularities of u for the Mackey topology $\tau(L_{\Phi^*}, L_\Phi)$. (4.2) and (4.3) then give an even better description.

Theorem 4.7. *A monetary utility function $u : L_{\Phi^*} \to \mathbb{R} \cup \{-\infty\}$ is $\sigma(L_{\Phi^*}, L_\Phi)$-upper semicontinuous (or what is the same, $\tau(L_{\Phi^*}, L_\Phi)$-upper semicontinuous) if and only if it is continuous from above:*

$$\xi_n \downarrow \xi \ \Rightarrow\ u(\xi) = \lim_n u(\xi_n). \tag{4.4}$$

In this case, the dual representation of u can be written as

$$u(\xi) = \inf\{\mathbb{E}_Q[\xi] + c(Q) : c(Q) < \infty\}, \tag{4.5}$$

where Q runs through probabilities absolutely continuous w.r.t. $\mathbb{P}$ with $dQ/d\mathbb{P} \in L_\Phi$, $c(Q) = (-u)^*(-dQ/d\mathbb{P})$ and $\mathbb{E}_Q[\xi] = \mathbb{E}[\xi dQ/d\mathbb{P}]$.

Proof. The necessity is clear from Theorem 4.4 since $\xi_n \downarrow \xi$ implies $\xi_n \to \xi$ in order. For the sufficiency, we first show that (4.2)–(4.4) imply that u is monotone, i.e.

$$\xi, \eta \in L_{\Phi^*},\ \xi \le \eta \ \Rightarrow\ u(\xi) \le u(\eta) \tag{4.6}$$

We can suppose $u(\xi) = 0$ thanks to (4.3). For each $\varepsilon \in (0, 1)$, let $\alpha_\varepsilon = (1-\varepsilon)/\varepsilon$ so that $\zeta_\varepsilon := \eta + \varepsilon\xi^- + \alpha_\varepsilon(\eta + \varepsilon\xi^- - \xi) \ge 0$. Putting $\lambda_\varepsilon := \alpha_\varepsilon/(1+\alpha_\varepsilon) \in (0, 1)$, we have $\eta + \varepsilon\xi^- = \lambda_\varepsilon\xi + (1-\lambda_\varepsilon)\zeta_\varepsilon$, hence by the concavity, $u(\eta + \varepsilon\xi^-) \ge \lambda_\varepsilon u(\xi) + (1-\lambda_\varepsilon)u(\zeta_\varepsilon) \ge 0$. Then (4.4) shows that $u(\eta) = \lim_n u(\eta + n^{-1}\xi^-) \ge 0 = u(\xi)$. Now by Theorem 4.4 applied to the ***convex*** function $-u$, the $\sigma(L_{\Phi^*}, L_\Phi)$-upper semicontinuity of u is equivalent to the property that $u(\xi) \ge \limsup_n u(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L_{Φ^*}; given the monotonicity (4.6) of u, this is equivalent to (4.4). That the dual representation of $f = -u$ together with (4.2) and (4.3) yields (4.5) is standard. □

Note that if u is finite valued ($\mathbb{R}$-valued), (4.2) and (4.3) still imply (4.6) without assuming (4.4). For $\varepsilon \mapsto u(\eta + \varepsilon\xi^-)$ is continuous as a finite valued convex function on $\mathbb{R}$. One can easily see also that any monetary utility function that is $\tau(L_{\Phi^*}, L_\Phi)$-continuous at 0 is finite valued. For such u, Theorem 4.5 yields that

Theorem 4.8. *A monetary utility function $u : L_{\Phi^*} \to \mathbb{R}$ is $\tau(L_{\Phi^*}, L_\Phi)$-continuous if (and only if) it is continuous from below, i.e. $\xi_n \uparrow \xi \Rightarrow u(\xi) = \lim_n u(\xi_n)$.*

Proof. Given that u is finite, monotone and convave, the continuity from below implies the continuity from above. For if $\xi_n \downarrow \xi$, then $u(\xi) \geq \frac{1}{2}u(\xi_n) + \frac{1}{2}u(2\xi - \xi_n)$ by the concavity, so the continuity from below and the monotonicity imply $0 \leq u(\xi_n) - u(\xi) \leq u(\xi) - u(2\xi - \xi_n) \downarrow 0$ since $2\xi - \xi_n \uparrow \xi$. In particular, u is $\sigma(L_{\Phi^*}, L_\Phi)$-usc. On the other hand, again by the monotonicity, the continuity of u from below is equivalent to the property that $u(\xi) = \lim_n u(\xi_n)$ whenever $\xi_n \to \xi$ a.s. and $(\xi_n)_n$ is order bounded in L_{Φ^*}. The result now follows from Theorem 4.5. □

Appendix

Proof of Proposition 1.2. Only (3) ⇒ (1) deserves a proof. (3) implies, by Proposition 1.1, $f = f^{**}$, and $|\mathbb{E}[\eta \mathbb{1}_A]| \leq \frac{1}{n}(f(n\mathbb{1}_A) \vee f(-n\mathbb{1}_A) + c)$ for $A \in \mathcal{F}$ and $\eta \in L_1$ with $f^*(\eta) \leq c$ by Young's inequality; thus (3) implies that $\{\eta \in L_1 : f^*(\eta) \leq c\}$ is uniformly integrable, hence $\sigma(L_1, L_\infty)$-compact by the Dunford-Pettis theorem. Now Moreau's theorem [15] shows that f is $\tau(L_\infty, L_1)$-continuous. □

Proof of Lemma 2.1. For each $\xi \in L_{\Phi^*}$, $\eta \mapsto \eta\xi$ continuously maps $(L_\Phi, \sigma(L_\Phi, L_{\Phi^*}))$ into $(L_1, \sigma(L_1, L_\infty))$ since $\xi\zeta \in L_1$, $\forall \zeta \in L_\infty$. Thus if A is relatively $\sigma(L_\Phi, L_{\Phi^*})$-compact, its image $A\xi$ is relatively weakly compact in L_1, i.e. uniformly integrable. Conversely, if $A\xi$, $\xi \in L_{\Phi^*}$, are uniformly integrable, then $c_\xi := \sup_{\eta \in A} \mathbb{E}[|\eta\xi|] < \infty$ for each $\xi \in L_{\Phi^*}$, so A is pointwise bounded in the algebraic dual $L^\#_{\Phi^*}$ of L_{Φ^*}, and A is relatively $\sigma(L_1, L_\infty)$-compact in L_1. Thus if $(\eta_\alpha)_\alpha$ is a net in A with the pointwise limit $f(\xi) = \lim_\alpha \mathbb{E}[\eta_\alpha \xi]$ in $L^\#_{\Phi^*}$, there is a unique $\eta_0 \in L_1$ such that $f|_{L_\infty}(\xi) = \mathbb{E}[\eta_0\xi]$ for $\xi \in L_\infty$. Then for each $\xi \in L_{\Phi^*}$, $\mathbb{E}[|\eta_0\xi|] = \sup_n \mathbb{E}[\eta_0 \xi \mathbb{1}_{\{|\xi| \leq n\}} \mathrm{sgn}(\eta_0\xi)] = \sup_n f(\xi \mathbb{1}_{\{|\xi| \leq n\}} \mathrm{sgn}(\eta_0\xi)) \leq c_\xi$, hence $\eta_0 \in L_\Phi$, while $|f(\xi) - f(\xi \mathbb{1}_{\{|\xi| \leq n\}})| = |f(\xi \mathbb{1}_{\{|\xi| > n\}})| \leq \sup_{\eta \in A} \mathbb{E}[|\eta\xi| \mathbb{1}_{\{|\xi| > n\}}] \to 0$ since $A\xi$ is uniformly integrable; hence $f(\xi) = \mathbb{E}[\eta_0\xi]$. Therefore A is pointwise bounded and its $\sigma(L^\#_{\Phi^*}, L_{\Phi^*})$-closure in $L^\#_{\Phi^*}$ lies in L_Φ; hence A is relatively $\sigma(L_\Phi, L_{\Phi^*})$-compact. □

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