ATTAINABLE CLAIMS WITH P'TH MOMENTS

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Abstract. The paper deals with the following question arising naturally in Mathematical Finance: "What is the good notion of the space of attainable claims in $L^p(\mathbb{P})$?"

Fix $1 \leq p \leq \infty$ and a semi-martingale $S = (S_t)_{t \in \mathbb{R}_+}$ which is locally in $L^p(\mathbb{P})$. It is natural to define the space K_p^s of simple p-attainable claims as random variables $(H \cdot S)_\infty$ where H are simple predictable integrands satisfying suitable integrability conditions insuring that $(H \cdot S)_\infty \in L^p(\mathbb{P})$. Then one may define the closure K_p of K_p^s in $L^p(\mathbb{P})$ and ask whether the elements of K_p may be written as $(H \cdot S)_\infty$ for appropriate (not necessarily simple) predictable integrands and study the duality relation between the space K_p and the equivalent martingale measures $\mathbb{Q}$ for S with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{Q})$, where $\frac{1}{p} + \frac{1}{q} = 1$.

In the case of continuous processes S, the situation turns out to be nice and K_p consists precisely of the random variables $(H \cdot S)_\infty$ such that H is predictable and $H \cdot S$ is a uniformly integrable martingale with respect to each equivalent martingale measure $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{P})$.

In the case of processes with jumps the situation is more intriguing: it turns out that K_p should be replaced by the space D_p which is the intersection of the L^p closures of $K_p^s - L_+^p(\mathbb{P})$ and $K_p^s + L_+^p(\mathbb{P})$. This "sandwich-like" definition of D_p may be interpreted naturally in economic terms and this notion allows to prove a general theorem analogous to the theorem for the continuous case.

We also give examples which are intended to convince the reader that the concepts defined in the paper are indeed the natural notions for attainable claims with p'th moments.

Sommaire

Le présent papier traite la question que l'on rencontre de manière naturelle dans la finance mathématique: "Quels sont les biens contingents atteignables dans $L^p(\mathbb{P})$?"

Prenons $1 \leq p \leq \infty$ et une semi-martingale $S = (S_t)_{t \in \mathbb{R}_+}$,localement dans $L^p(\mathbb{P})$. Il est naturel de definir l'espace K_p^s des biens contingents p–atteignables simples, comme l'espace des variables aléatoires $(H \cdot S)_\infty$ où H est un processus simple prévisible, vérifiant une condition d'intégrabilité, assurant que $(H \cdot S)_\infty \in L^p(\mathbb{P})$. On peut définir la fermeture K_p de K_p^s dans $L^p(\mathbb{P})$ et se demander si les eléments de K_p peuvent être écrits comme $(H \cdot S)_\infty$ pour un processus prévisible, pas nécessairement simple, et on peut étudier la dualité entre K_p et l'ensemble des mesures $\mathbb{Q}$ qui sont des mesures de martingale pour S, avec $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{Q})$, où $\frac{1}{p} + \frac{1}{q} = 1$.

Dans le cas d'un processus continu S, la situation est agréable et K_p est précisément l'espace des variables aléatoires $(H \cdot S)_\infty$ telles que H est prévisible et $H \cdot S$ est une martingale équi–intégrable par rapport à toute mesure equivalente de martingale $\mathbb{Q}$ avec $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{P})$.

Dans le cas d'un processus avec sauts la situation est plus intriguante: l'espace K_p doit être remplacée par l'espace D_p, qui est l'intersection des fermetures dans L^p de $K_p^s - L_+^p(\mathbb{P})$ et $K_p^s + L_+^p(\mathbb{P})$. Cette situation "sandwich" de D_p peut être interpretée de façon naturelle en termes économiques et la notion permet de démontrer un théorème analogue au cas d'un processus continu.

Nous donnons aussi des exemples qui doivent convaincre le lecteur que ces notions sont en effet les bonnes notions de biens contingents atteignables avec moment d'ordre p.

1. Introduction

We consider an $\mathbb{R}^d$-valued càdlàg semi-martingale $(S_t)_{t\in\mathbb{R}_+}$. As usual in Mathematical Finance, S models the price process of d stocks and economic agents are allowed to trade on these stocks. An easy way to formalize this concept is the notion of *simple integrand* (we refer to [P 90] for the theory of stochastic integration), which is a linear combination of processes of the form

$$H = f \cdot \chi_{]\!]T_1,T_2]\!]}$$

where $T_1 \le T_2$ are finite stopping times and the $\mathbb{R}^d$-valued random variable f is $\mathcal{F}_{T_1}$-measurable. We then may form the stochastic integral

$$(H \cdot S)_t = f \cdot (S_{T_2 \wedge t} - S_{T_1 \wedge t})$$

and the random variable

$$(H \cdot S)_\infty = f \cdot (S_{T_2} - S_{T_1}).$$

The interpretation is that H defines the *trading strategy* of buying at time T_1 the amount of $f(\omega) = (f^1(\omega), \cdots, f^d(\omega))$ units of the stocks $S = (S^1, \cdots, S^d)$ and selling it at time T_2. At time $t \in [0,\infty]$ the random variable $(H \cdot S)_t(\omega)$ describes the cumulated gain (or loss) up to time t and $(H \cdot S)_\infty(\omega)$ the final result, if an agent follows the trading strategy H. We call the linear space spanned by the above random variables $(H \cdot S)_\infty$ the space of *claims attainable* by simple integrands or by simple trading strategies.

The concept of simple trading strategies does not require any sophisticated concepts from the theory of stochastic integration and allows the obvious economic interpretation given above. But, of course, the natural question arises, what happens if "one passes to the limit". Recall that even in the most basic example of Mathematical Finance, the replicating portfolio for a european call-option in the Black-Scholes economy, one has to go beyond the concept of simple integrands and allow more general predictable processes H as trading strategies.

There is a well-established theory of stochastic integration for a semi-martingale S and there is a precise concept of S*-integrable* predictable process H, so that the semi-martingale

$$(H \cdot S)_t = \int_0^t H_u dS_u$$

is well defined (see e.g., [P 90]). But this notion is too wide to be useful in Mathematical Finance. For example, if S is standard Brownian motion with its natural filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, R. Dudley [D 77] has shown that *any* $\mathcal{F}_\infty$-measurable random variable f may be represented as $f = (H \cdot S)_\infty$ for some S-integrable predictable process H.

Hence some additional *admissibility condition* is needed to define a concept useful in Mathematical Finance.

One possible way, which goes back to Harrison-Pliska [HP 81], is to require in addition to the S-integrability of H that the process $(H \cdot S)$ is uniformly bounded from below (compare [DS 94]).

This notion of admissibility has an obvious interpretation as a budget constraint of the economic agents, avoids paradoxes coming from doubling strategies (see [HP 81]) and allows a satisfactory duality relation between admissible attainable claims and $\mathbb{P}$-absolutely continuous local martingale measures for S (see [DS 94]).

In the present paper we adopt a different point of view by using approximation with respect to the norm of $L^p(\mathbb{P})$. This approach goes back — at least — to Harrison-Kreps ([HK 79], [K 81]) and arises naturally— at least for $p = 2$ — in the context of optimization (compare [St 90], [Schw 93], [MS 94b]). Fix $p \in [1, \infty]$; we want to investigate the question which S-integrable stochastic integrals $(H \cdot S)$ make sense in the context of the space $L^p(\mathbb{P})$ and what the resulting subspace K_p of $L^p(\mathbb{P})$ spanned by the random variables $(H \cdot S)_\infty$ is.

An aspect of central importance will be the duality relation between the space K_p of attainable claims and the martingale measures $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{P})$. This line or research is stimulated by ([K 81], [KLSX 91], [KQ 92], [Ja 92], [AS 94], [DS 94]).

If S is a martingale under $\mathbb{P}$ it is clear what the natural notions are: If $1 < p \leq \infty$ then we are led to consider those S-integrable predictable processes H such that $(H \cdot S)_t$ remains bounded in $L^p(\mathbb{P})$ and, if $p = 1$, such that $(H \cdot S)_t$ is uniformly integrable. A celebrated result of M. Yor ([Y 78], for the vector-valued case see [J 79]), states that the resulting space K_1 formed by the random variables $(H \cdot S)_\infty$, where H runs through the predictable processes such that $(H \cdot S)$ is uniformly integrable, is *closed* in $L^1(\mathbb{P})$. It follows that $K_p = K_1 \cap L^p(\mathbb{P})$ is closed in $L^p(\mathbb{P})$; in the case $p = 2$ this classical (and very easy) result was observed by Kunita-Watanabe ([KW 67]).

But the problem at hand is more delicate if S is only assumed to be a *semi-martingale* under $\mathbb{P}$ and — as usual in Mathematical Finance — a martingale only under some measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. In this case one may proceed as follows: one calls a predictable S-integrable process admissible if $(H \cdot S)$ remains bounded in the space $S^p(\mathbb{P})$ of semi-martingales (with a uniform integrability condition added in the case $p = 1$) and define K_p as the space formed by the respective random variables $(H \cdot S)_\infty$. Then the crucial question arises whether or not the space K_p is closed in $L^p(\mathbb{P})$ (compare [Schw 93], [MS 94a], [DMSSS 94]). This will not be the case, in general, and can be shown only under rather strong requirements on the process S. Loosely speaking, one only gets nice results if S is not too far from being a (local) martingale under $\mathbb{P}$.

In the present paper we adopt a more general notion of admissible stochastic integrals which is designed to insure under very weak assumptions on S (the existence of an equivalent local martingale measure $\mathbb{Q}$ for S with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{P})$) that K_p (as well as the space D_p to be defined below) are closed in $L^p(\mathbb{P})$. To define these notions we have to be more formal and fix precisely the setting.

Throughout the paper $p \in [1, \infty]$ will be arbitrary (but fixed) and q will denote the conjugate exponent, $\frac{1}{p} + \frac{1}{q} = 1$. On $L^p(\mathbb{P})$ we shall consider the norm topology, if $1 \leq p < \infty$, and the weak-star topology $\sigma(L^\infty, L^1)$, if $p = \infty$. S will be an $\mathbb{R}^d$-valued càdlàg semi-martingale based on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ which is locally in $L^p(\mathbb{P})$ in the following sense: There exists a sequence $(U_n)_{n=1}^\infty$ of localizing stopping times increasing to infinity such that, for each $n \in \mathbb{N}$, the family $\{S_T : T$ stopping time, $T \leq U_n$ is bounded in $L^p(\mathbb{P})$ (and uniformly integrable in the case $p = 1$). In the case $p = 1$ this notion is known in the literature under the name "S is locally of class D".

A predictable $\mathbb{R}^d$-valued process H which is a linear combination of processes of the form

$$H = f\chi_{]\!]T_1, T_2]\!]}$$

where $T_1 \leq T_2$ are finite stopping times dominated by some U_n and where f is in $L^\infty(\Omega, \mathcal{F}_{T_1}, \mathbb{P})$ will be called a *simple p-admissible integrand* for S. We denote by K_p^s the subspace of $L^p(\mathbb{P})$

$$K_p^s = \{(H \cdot S)_\infty : H\text{-simple and } p\text{-admissible}\}.$$

Similarly as in [DS 94] we let

$$\mathcal{M}_q^e(\mathbb{P}) = \{\mathbb{Q} \sim \mathbb{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{P}), S \text{ is a } \mathbb{Q}\text{-local martingale}\}$$

and

$$\mathcal{M}_q(\mathbb{P}) = \{\mathbb{Q} \ll \mathbb{P} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L^q(\mathbb{P}), S \text{ is a } \mathbb{Q}\text{-local martingale}\}$$

denote the set of $\mathbb{P}$-equivalent (resp. $\mathbb{P}$-absolutely continuous) probability measures on $\mathcal{F}$ having q'th moments and such that S is a local martingale under $\mathbb{Q}$. Throughout the paper we shall make the assumption $\mathcal{M}_q^e(\mathbb{P}) \neq \emptyset$, which is natural in our context (compare [HK 79], [St 90], [DS 94]).

One easily deduces from Hölder's inequality that our definition of simple p-admissible integrands was designed in such a way that

$$\mathbb{E}_\mathbb{Q}(f) = 0 \qquad \text{for } f \in K_p^s, \mathbb{Q} \in \mathcal{M}_q(\mathbb{P}). \tag{1}$$

In the easy case when Ω is finite it is a simple matter of linear algebra to check that (1) defines a complete duality relation between K^s and $\mathcal{M}(\mathbb{P})$ (the subscripts p and q being superfluous in this case): Under our standing assumption $\mathcal{M}^e(\mathbb{P}) \neq \emptyset$ the random variable f (resp. the probability measure $\mathbb{Q}$) is in K^s (resp. in $\mathcal{M}(\mathbb{P})$) iff $\mathbb{E}_\mathbb{Q}(f) = 0$ for all $\mathbb{Q} \in \mathcal{M}(\mathbb{P})$ (resp. for all $f \in K^s$) (compare [HP 81], [DS 94]). But, of course, we cannot expect that this duality relation carries over in a naive way to the case when Ω is not finite any more, the most obvious obstacle being that K_p^s will not necessarily be closed in $L^p(\mathbb{P})$. This leads us to the central concepts of this paper.

1.1 Notation. *In the above setting let*

$$K_p = \overline{K_p^s}$$

$$\textit{and} \qquad D_p = \overline{K_p^s - L_+^p} \cap \overline{K_p^s + L_+^p},$$

where the bar denotes the closure with respect to the norm topology of $L^p(\mathbb{P})$, for $1 \le p < \infty$, and with respect to the σ^*-topology, for $p = \infty$.

The interpretation of the elements in K_p is obvious: a random variable $f \in L^p$ is in K_p if it can be approximated by random variables $(H \cdot S)_\infty$ where H is simple and p-admissible. The space D_p (which clearly contains K_p) has a more intriguing interpretation: a random variable $f \in L^p$ is in D_p if it may be approximately "sandwiched" between elements of K_p^s, i.e. if there are simple p-admissible strategies H^+ and H^- such that $((H^+ \cdot S)_\infty - f)_-$ as well as $((H^- \cdot S)_\infty - f)_+$ both are small with respect to the topology on $L^p(\mathbb{P})$. An economic agent will wish to approximate f by either $(H^+ \cdot S)_\infty$ or $(H^- \cdot S)_\infty$ depending on whether she wants to buy or sell the contingent claim modelled by the random variable f. Although the definition of D_p might seem weird at first glance, a moment's reflection reveals that it is quite natural from an economic point of view.

Theorem 1.2 stated below, which is one of the main results of this paper, shows that the notion of D_p is such that there is a satisfactory duality between D_p and $\mathcal{M}_q(\mathbb{P})$ and that the elements of D_p may be written as $(H \cdot S)_\infty$ for a precisely defined class of integrands H.

1.2 Theorem. *Let $1 \le p \le \infty$, q its conjugate exponent, S a semi-martingale locally in $L^p(\mathbb{P})$ such that $\mathcal{M}_q^e(\mathbb{P}) \neq \emptyset$, and $f \in L^p(\mathbb{P})$. The following assertions are equivalent:*

- (i) *$f \in D_p$.*
- (ii) *There is an S-integrable predictable process H such that, for each $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$, the process $(H \cdot S)$ is a uniformly integrable $\mathbb{Q}$-martingale converging to f in the norm of $L^1(\mathbb{Q})$.*
- (ii') *There is an S-integrable predictable process H such that, for each $\mathbb{Q} \in \mathcal{M}_q(\mathbb{P})$, the process $(H \cdot S)$ is a uniformly integrable $\mathbb{Q}$-martingale converging to f in the norm of $L^1(\mathbb{Q})$.*
- (iii) *$\mathbb{E}_\mathbb{Q}(f) = 0$ for each $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$.*
- (iii') *$\mathbb{E}_\mathbb{Q}(f) = 0$ for each $\mathbb{Q} \in \mathcal{M}_q(\mathbb{P})$.*

Of course, the question arises, whether the concept of "sandwich-able" contingent claims is vacuous in the sense that we always have $K_p = D_p$. An example (for the case $p = 2$) given in section 3 below, shows that this is not always the case, i.e., there are situations where D_p strictly contains K_p. From theeconomic point of view this leads to an interesting (and paradoxical) interpretation: in example 3.1 there is a contingent claim $f_0 \in L^2(\mathbb{P})$ such that, for every simple 2-admissible integrand H we have $\|(H \cdot S)_\infty - f_0\|_{L^2(\mathbb{P})} \ge 1$ but, for $\varepsilon > 0$, there are simple

2-admissible integrands H^+ and H^- such that $\|((H^+ \cdot S) - f_0)_-\|_{L^2(\mathbb{P})} < \varepsilon$ and $\|((H^- \cdot S) - f_0)_+\|_{L^2(\mathbb{P})} < \varepsilon$. Hence f_0 may or may not be approximated in the L^2-sense by simple 2-admissible integrands depending on whether agents are allowed to "throw away money" or not. Note that this phenomenon occurs although the process S does not permit any arbitrage opportunities (remember: we assumed $\mathcal{M}_q^e(\mathbb{P}) \neq \emptyset$).

On the other hand, in section 2 we shall show that the above phenomenon can only occur if S has jumps, i.e., for continuous processes S we always have that $K_p = D_p$.

The paper is organized in the following way: In section 2 we proof theorem 1.2 and the anounced result on continuous processes. In section 3 we construct two counter-examples.

2. Results and Proofs

We now pass to the proof of the Theorem 1.2 stated in the introduction above.

Proof of Theorem 1.2. (i) $\Rightarrow$ (iii'): For $\mathbb{Q} \in \mathcal{M}_q(\mathbb{P})$ we have that $\mathbb{Q}$ takes values ≤ 0 on $\overline{K_p^s - L_+^p(\mathbb{P})}$ and values ≥ 0 on $\overline{K_p^s + L_+^p(\mathbb{P})}$, hence $\mathbb{Q}$ vanishes on D_p.

(iii) $\Leftrightarrow$ (iii') As (iii') $\Rightarrow$ (iii) is trivial we have to show that (iii) $\Rightarrow$ (iii'). Suppose to the contrary that there is $f \in L^p(\mathbb{P})$ such that $\mathbb{E}_{\mathbb{Q}}(f) = 0$ for all $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$ but such that there is $\mathbb{Q}_0 \in \mathcal{M}_q(\mathbb{P})$ with $\mathbb{E}_{\mathbb{Q}_0}(f) \neq 0$. Fix $\mathbb{Q}_1 \in \mathcal{M}_q^e(\mathbb{P})$ (remember: $\mathcal{M}_q^e(\mathbb{P}) \neq \emptyset$) and note that $\frac{1}{2}(\mathbb{Q}_0 + \mathbb{Q}_1) \in \mathcal{M}_q^e(\mathbb{P})$ and $\mathbb{E}_{\frac{1}{2}(\mathbb{Q}_0+\mathbb{Q}_1)}(f) \neq 0$, a contradiction.

(iii') $\Rightarrow$ (i): If $f \notin \overline{K_p^s - L_+^p(\mathbb{P})}$ then the Hahn-Banach theorem provides us with an element g of $L_+^q(\mathbb{P})$ vanishing on K_p^s – so that, after normalisation, it is the density of a non-negative probability measure R in $\mathcal{M}_q(\mathbb{P})$ – and such that $E_{\mathbb{P}}(fg) = E_R(f) > 0$.

The case $f \notin \overline{K_p^s + L_+^p(\mathbb{P})}$ is similar.

(i) $\Rightarrow$ (ii'): Fix $f \in D_p$ and $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$; first note that the identity mapping considered as an operator from $L^p(\mathbb{P})$ to $L^1(\mathbb{Q})$ is well defined and continuous. Hence we have that

$$f \in \overline{K_p^s + L_+^1(\mathbb{Q})}^{L^1(\mathbb{Q})}$$

which means that there is a sequence $(f_n)_{n=1}^{\infty} = ((H_n \cdot S)_\infty)_{n=1}^{\infty}$ in K_p^s such that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}(f - f_n)_+ = 0.$$

But from the martingale property and (iii) – which we have already proved to being equivalent to (i) – we get for each $n \in \mathbb{N}$

$$\mathbb{E}_{\mathbb{Q}}(f_n) = \mathbb{E}_{\mathbb{Q}}(f) = 0,$$

which implies that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{Q}}(|f - f_n|) = 0,$$

i.e., f is in the $L^1(\mathbb{Q})$-closure of K_p^s.

The rest of the proof follows an argument of C. Stricker ([St 90], rem. III.2): We may identify f – as well as each f_n – with a uniformly integrable martingale $(f_t)_{t\in\mathbb{R}_+}$ by letting $f_t = \mathbb{E}_{\mathbb{Q}}(f \mid \mathcal{F}_t)$. We now are in the position to apply the theorem of M. Yor ([Y 78], cor. 2.5.2, for the vector-valued version see [J 79]) – to exhibit a predictable integrand H with the desired properties with respect to $\mathbb{Q}$.

We still have to show that H also has the desired properties with respect to each $R \in \mathcal{M}_q(\mathbb{P})$. We have to show that $(H \cdot S)_t = \mathbb{E}_R[f|\mathcal{F}_t]$ for each $t \in \mathbb{R}_+$, which will readily show that the R-almost surely defined stochastic integral $H \cdot S$ is indeed a R-uniformly integrable martingale. As $R \in \mathcal{M}_q(\mathbb{P})$ we have that each $(H_n \cdot S)$ is an R-uniformly integrable martingale so that $(H_n \cdot S)_t = \mathbb{E}[(H_n \cdot S)_\infty|\mathcal{F}_t]$. By the same argument as above $((H_n \cdot S)_\infty)_{n=1}^\infty$ converges to f in $L^1(R)$ and therefore $((H_n \cdot S)_t)_{n=1}^\infty$ is a Cauchy sequence in $L^1(R)$ that converges to $(H \cdot S)_t$ in $L^1(\mathbb{Q})$ and therefore also to $(H \cdot S)_t$ in $L^1(R)$. This shows that $(H \cdot S)$ is indeed a R-uniformly integrable martingale converging to f in $L^1(R)$ thus finishing the proof of the implication (i)$\Rightarrow$ (ii').

(ii') $\Rightarrow$ (ii) is trivial and

(ii) $\Rightarrow$ (iii) is obvious.

q.e.d.

2.2 Remark. (a) In the case $1 < p \le \infty$ we can also formulate the equivalent characterisation

(ii") There is an S-integrable predictable process H_t such that, for each $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$ (or, equivalently, for each $\mathbb{Q} \in \mathcal{M}_q(\mathbb{P})$), the process $(H \cdot S)_t$ is a $\mathbb{Q}$-martingale converging to f in the norm of $H^1(\mathbb{Q})$.

It suffices to remark that the identity mapping from $L^p(\mathbb{P})$ to $L^1(\mathbb{Q})$ is in fact a continuous operator into $H^1(\mathbb{Q})$.

Indeed, if $f \in L^p(\mathbb{P}), (f_t)_{t\in\mathbb{R}_+} = (\mathbb{E}[f|\mathcal{F}_t])_{t\in\mathbb{R}_+}$ is the associated martingale and $f^* = \sup_t |f_t|$ is the maximal function, there is a constant C_p such that $\|f^*\|_p \le C_p\|f\|_p$ and therefore $\|f\|_{\mathcal{H}^1(\mathbb{Q})} \le c\|f^*\|_{L^1(\mathbb{Q})} \le cC_p\|f\|_p$.

(b) Note that in the theorem above we did not assert that the process $(H \cdot S)_t$ converges to f in $L^p(\mathbb{P})$. There is simply no reason for this assertion to hold if $1 \le p < \infty$: it is easy to construct examples of (continuous) processes, satisfying the assumptions of the theorem and such that $(H \cdot S)_t$ does not converge to f in $L^p(\mathbb{P})$.

(c) Questions leading to more demanding counter-examples are the following: Can one replace in (i) above D_p by K_p, which seems at first glance a much more natural object than D_p? Or: Can one replace the requirement "for each $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$" in (ii) by the requirement "for some $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$"?

We shall see in section 3 below that the question to both answers is no, in general. However, we shall presently show that the answer to the first question is yes for the case of continuous processes.

2.2 Theorem. *Suppose that $S = (S_t)_{t \in \mathbb{R}_+}$ is a continuous process, let $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and suppose that $\mathcal{M}_q^e(\mathbb{P}) \neq \emptyset$.*

Then $K_p = D_p$.

Proof. Suppose first that $1 \le p < \infty$. Let $f \in D_p$; by theorem 1.2 there is an S-integrable predictable process H such that $((H \cdot S)_t)_{t \in \mathbb{R}_+}$ is a uniformly integrable martingale with respect to each $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$ and such that almost surely

$$f = \lim_{t \to \infty} (H \cdot S)_t.$$

The definition of D_p implies the existence of sequences $(H^{+,n})_{n=1}^\infty$ and $(H^{-,n})_{n=1}^\infty$ of simple p-admissible integrands such that, for

$$f^{+,n} = (H^{+,n} \cdot S)_\infty \quad \text{and} \quad f^{-,n} = (H^{-,n} \cdot S)_\infty$$

we have that $((f - f^{+,n})_+)_{n=1}^\infty$ and $((f - f^{-,n})_-)_{n=1}^\infty$ tend to zero in $L^p(\mathbb{P})$. By the argument used in the proof of theorem 1.2 we deduce that, in fact, $(f - f^{+,n})_{n=1}^\infty$ and $(f - f^{-,n})_{n=1}^\infty$ tend to zero in $L^1(\mathbb{Q})$, for each $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$. It follows from the martingale property that, more generally, for each $\mathbb{Q} \in \mathcal{M}_q^e(\mathbb{P})$ and each (not necessarily finite) stopping time T, we have

$$(1) \quad \lim_{n \to \infty} \|((H - H^{+,n}) \cdot S)_T\|_{L^1(\mathbb{Q})} = 0 \quad \text{and} \quad \lim_{n \to \infty} \|((H - H^{-,n}) \cdot S)_T\|_{L^1(\mathbb{Q})} = 0$$

We have to show that, for $\epsilon > 0$, there is a simple p-admissible integrand H^ε such that

$$\|f - (H^\varepsilon \cdot S)_\infty\|_{L^p(\mathbb{P})} < \varepsilon.$$

For $C \in \mathbb{R}_+$ let

$$T_C = \inf\{t : |(H \cdot S)_t| \ge C\},$$

so that

$$\lim_{C \to \infty} \mathbb{P}\{T_C < \infty\} = 0.$$

Let $\delta > 0$ to be specified below and find $C = C(\delta) > 1$ such that

$$\mathbb{P}\{T_C < \infty\} < \delta.$$

Some warning seems in order here: There is no reason that $(H \cdot S)_{T_C}$ converges to $(H \cdot S)_\infty$, as $C \to \infty$ with respect to the norm of $L^p(\mathbb{P})$ (compare, e.g., example 3.1 below). We have to be more careful and to use the continuity of S in a nontrivial way.

Let

$$h = (H \cdot S)_{T_C}$$

and define, for $n \in \mathbb{N}$

$$h^{+,n} = (H^{+,n} \cdot S)_{T_C} \quad \text{and} \quad h^{-,n} = (H^{-,n} \cdot S)_{T_C}.$$

We deduce from (1) that $(h^{+,n})_{n=1}^\infty$ and $(h^{-,n})_{n=1}^\infty$, converge to h in $L^1(\mathbb{Q})$ and therefore in measure. For $\eta > 0$, again to be specified below, we therefore have that, for $n = n(\eta)$ sufficiently big,

$$\mathbb{P}\{|h - h^{+,n}| > 1\} < \eta \quad \text{and} \quad \mathbb{P}\{|h - h^{-,n}| > 1\} < \eta.$$

Fix n such that the above inequalities hold true and define the $\mathcal{F}_{T_C}$-measurable sets A^+ and A^- by

$$\begin{aligned} A^+ &= \{T_C < \infty\} \cap \{h = -C\} \cap \{|h - h^{+,n}| \le 1\} \quad \text{and} \\ A^- &= \{T_C < \infty\} \cap \{h = +C\} \cap \{|h - h^{-,n}| \le 1\}, \end{aligned}$$

so that A^+ and A^- are disjoint subsets of $\{T_C < \infty\}$ covering this set up to a set $B = \{T_C < \infty\} \setminus (A^+ \cup A^-)$ of $\mathbb{P}$-measure at most $\mathbb{P}(B) < 2\eta$.

Define the predictable integrand $\tilde{H}$

$$\begin{aligned} \tilde{H} = H\chi_{]\!]0,T_C]\!]} &+ H^{+,n}\chi_{]\!]T_C,\infty[\![}\,\chi_{A^+} \\ &+ H^{-,n}\chi_{]\!]T_C,\infty[\![}\,\chi_{A^-}, \end{aligned}$$

and define the stopping time T as the first moment after T_C when $(\tilde{H} \cdot S)_t = 0$. We want to show that

$$\|f - \tilde{f}\|_{L^p(\mathbb{P})} < \epsilon \tag{2}$$

if $\delta = \delta(\epsilon) > 0$ and $\eta = \eta(C(\delta), \epsilon) > 0$ are sufficiently small, where

$$\tilde{f} = (\tilde{H} \cdot S)_T.$$

As

$$\|f - \tilde{f}\|_{L^p(\mathbb{P})} \le \|(f - \tilde{f})\chi_{A^+}\|_{L^p(\mathbb{P})} + \|(f - \tilde{f})\chi_{A^-}\|_{L^p(\mathbb{P})} + \|(f - \tilde{f})\chi_B\|_{L^p(\mathbb{P})} \tag{3}$$

it will suffice to show that each of the three terms on the right hand side is less than $\varepsilon/3$. As regards the last one note that

$$\|f\chi_B\|_{L^p(\mathbb{P})} < \varepsilon/6 \qquad \text{and} \qquad \|\tilde{f}\chi_B\|_{L^p(\mathbb{P})} \leq C(2\eta)^{\frac{1}{p}} < \varepsilon/6$$

if $\eta = \eta(C(\delta), \varepsilon)$ is small enough.

As regards the first two terms in (3) we only estimate the first one (the second being analogous): we split the set A^+ into $A^+ \cap \{T < \infty\}$ and $A^+ \cap \{T = \infty\}$. For the former set we may estimate

$$\|(f - \tilde{f})\chi_{A^+ \cap \{T<\infty\}}\|_{L^p(\mathbb{P})} = \|f\chi_{A^+\cap\{T<\infty\}}\|_{L^p(\mathbb{P})} \leq \|f\chi_{A^+}\|_{L^p(\mathbb{P})},$$

which is smaller than $\epsilon/6$ if $\delta = \delta(\varepsilon) > 0$ is sufficiently small as $f \in L^p(\mathbb{P})$. For the second set we may estimate

$$\begin{aligned}
&\|(f - \tilde{f})\chi_{A^+\cap\{T=\infty\}}\|_{L^p(\mathbb{P})} \\
&\leq \|(1 + |f - f^{+,n}|)\chi_{A^+\cap\{T=\infty\}}\|_{L^p(\mathbb{P})} \\
&\leq \|(1 + (f - f^{+,n})_+)\chi_{A^+\cap\{T=\infty\}}\|_{L^p(\mathbb{P})} + \|(1 + (f - f^{+,n})_-)\chi_{A^+\cap\{T=\infty\}}\|_{L^p(\mathbb{P})} \\
&\leq \|(1 + (f - f^{+,n})_+)\chi_{A^+\cap\{T=\infty\}}\|_{L^p(\mathbb{P})} + \|(2 + f_-)\chi_{A^+\cap\{T=\infty\}}\|_{L^p(\mathbb{P})}.
\end{aligned}$$

In the last line we have used the fact $f^{+,n}$ is less than or equal to 1 on $\{T = \infty\}$. If we choose $\delta = \delta(\varepsilon) > 0$ small enough and $n = n(\varepsilon, \eta)$ big enough the above expression is smaller than $\varepsilon/6$.

Summing up we have shown (2): given $\varepsilon > 0$ choose $\delta = \delta(\varepsilon) > 0$, then $C = C(\delta) > 0$, $\eta = \eta(C, \varepsilon) > 0$ and finally $n = n(\varepsilon, \eta) \in \mathbb{N}_0$. However, we are not yet finished, as $\tilde{H}$ is a simple p-admissible integrand only after the stopping time T_C. But it is standard to approximate $(H \cdot S)_{T_C}$ by the stochastic integral of a simple p-admissible integrand $\hat{H}$ supported by $[\![0, T_C]\!]$ such that $|(\hat{H} \cdot S)_t|$ is bounded by C and

$$\|(\hat{H} \cdot S)_{T_C} - (H \cdot S)_{T_C}\|_{L^p(\mathbb{P})} < \epsilon.$$

For the convenience of the reader we isolate this argument in the subsequent lemma 1.3.

Modifying $\tilde{H}$ on $[\![0, T_C]\!]$ in the indicated way we obtain the desired simple integrand H^ϵ for which

$$\|f - (H^\epsilon \cdot S)_\infty\|_{L^p(\mathbb{P})} < 2\epsilon,$$

thus finishing the proof.

The case $p = \infty$ is easy: simply note that, for $Q \in \mathcal{M}_1^e(P)$, $(L^\infty(\mathbb{P}), \sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P})))$, may be identified with $(L^\infty(\mathbb{Q}), \sigma(L^\infty(\mathbb{Q}), L^1(\mathbb{Q})))$.

q.e.d.

2.3 Lemma. *Let $(S_t)_{t\in\mathbb{R}_+}$ be a continuous local martingale with respect to a probability measure $\mathbb{Q}$ and H a predictable integrand such that $H \cdot S$ is bounded by 1 in absolute value.*

Then there exists a sequence $(H^n)_{n=1}^{\infty}$ of ∞-admissible simple integrand S such that $H^n \cdot S$ is bounded by 1 in absolute value, for each $n \in \mathbb{N}$, and such that $(H^n \cdot S)_\infty$ converges almost surely to $(H \cdot S)_\infty$.

Proof. By the very construction of the stochastic integral there is a sequence $(\hat{H}^n)_{n=1}^{\infty}$ of simple integrands such that $(\hat{H}^n \cdot S)_\infty$ converges to $(H \cdot S)_\infty$ in the norm of $L^2(\mathbb{Q})$. As S is locally bounded one easily verifies that we may assume (by stopping) that the integrands $\hat{H}^n$ are ∞-admissible (see the definition in the introduction). By Doob's inequality the maximal functions $((H - \hat{H}^n) \cdot S)^*_\infty$ tends to zero in $L^2(\mathbb{Q})$ and therefore in measure. By passing to a subsequence we may suppose that $((H - \hat{H}_n) \cdot S)^*_\infty$ is less than n^{-1} on a set of $\mathbb{Q}$-measure bigger than $1 - 2^{-n}$. Define the stopping times $T_n = \inf\{t : |(\hat{H}_n \cdot S)_t| \geq 1 + n^{-1}\}$ and the integrands

$$H^n = \frac{1}{1 + n^{-1}} \hat{H}_n \chi_{]\!]0,T_n]\!]}.$$

The sequence $(H^n)_{n=1}^{\infty}$ satisfies the requirement of the lemma.

q.e.d.

3. Two Counterexamples

This section is devoted to the construction of two counter-examples.

3.1 Example. *We construct a uniformly bounded discrete adapted stochastic process $S = (S_t)_{t=0}^{\infty}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F})_{t=0}^{\infty}, \mathbb{P})$ with the following properties.*

1. *There exists an equivalent martingale measure $\mathbb{Q}$ for S with density function $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$.*
2. *K_2 is strictly contained in D_2.*

Construction of Example 3.1. We work on $\Omega = \mathbb{N}$. Denote, for $t \in \mathbb{N}_0$,

$$\begin{aligned}
A_1^t &= \{3t + 1\}, \\
A_2^t &= \{3t + 2\}, \\
A_3^t &= \{3t + 3\}, \\
B^t &= \{3t + 1, 3t + 2, \dots\} = \bigcup_{s \geq t} (A_1^s \cup A_2^s \cup A_3^s), \\
C^t &= \{3t + 2, 3t + 3, \dots\} = B^t \setminus A_1^t.
\end{aligned}$$

In order to keep track of the right order of magnitude of the sequences constructed below we shall use the following notation: For sequences $(a_t)_{t=0}^\infty$ and $(b_t)_{t=0}^\infty$ of positive numbers we write $a_t \approx b_t$ if there are constants $c, C > 0$ such that $ca_t < b_t < Ca_t$ for all t sufficiently big.

$\mathcal{F}$ will denote the sigma-algebra of all subsets of Ω and we shall define measures $\mathbb{P}$ and $\mathbb{Q}$ on $\mathcal{F}$: let (formally) $\mathbb{P}(A_1^{-1}) = -1$ and define recursively, for $t \geq 0$,

$$\begin{aligned}
\mathbb{P}(A_1^t) = 2^{-1}(1-2^{-(2t-1)})\mathbb{P}(A_1^{t-1}) \quad &= 2^{-(t+1)}\prod_{s=1}^{t}(1-2^{-(2s-1)}) \approx 2^{-t},\\
\mathbb{P}(A_2^t) = \mathbb{P}(A_3^t) = 2^{-(2t+2)}\mathbb{P}(A_1^t) \quad &= 2^{-(3t+3)}\prod_{s=1}^{t}(1-2^{-(2s-1)}) \approx 2^{-3t},
\end{aligned}$$

and (formally) $\mathbb{Q}(A_1^{-1}) = 4$ and, for $t \geq 0$,

$$\begin{aligned}
\mathbb{Q}(A_1^t) &= 2^{-3}\mathbb{Q}(A_1^{t-1}) && = 2^{-(3t+1)} \approx 2^{-3t},\\
\mathbb{Q}(A_2^t) &= 2^{-(t+2)}\mathbb{Q}(A_1^t) && = 2^{-(4t+3)} \approx 2^{-4t},\\
\mathbb{Q}(A_3^t) &= (1-2^{-2}-2^{-(t+2)})\mathbb{Q}(A_1^t) && = (1-2^{-2}-2^{-(t+2)})2^{-(3t+1)} \approx 2^{-3t}.
\end{aligned}$$

Let us try to explain the idea behind this definition: we start with letting $\mathbb{P}(A_1^0) = \mathbb{Q}(A_1^0) = 2^{-1}$ so that $\mathbb{P}(C^0) = \mathbb{Q}(C^0) = 2^{-1}$. For each $t \in \mathbb{N}_0$ the set C^t is broken into

$$C^t = A_2^t \cup A_3^t \cup A_1^{t+1} \cup C^{t+1}.$$

The mass of C^t is divided amongst these 4 sets such that

$$\mathbb{P}(A_1^{t+1}) = \mathbb{P}(C^{t+1}) \qquad \text{and} \qquad \mathbb{Q}(A_1^{t+1}) = \mathbb{Q}(C^{t+1}). \tag{1}$$

In the case of $\mathbb{P}$ the mass of C^t is distributed among the 4 sets above with the weights $\{2^{-(2t+2)}, 2^{-(2t+2)}, 2^{-1}(1-2^{-(2t+1)}), 2^{-1}(1-2^{-(2t+1)}\})$ and in the case of $\mathbb{Q}$ with the weights $\{2^{-(t+2)}, (1-2^{-2}-2^{-(t+2)}), 2^{-3}, 2^{-3}\}$.

Clearly the measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is uniformly bounded.

Now we define a sequence $(f_t)_{t=0}^\infty$ of functions on Ω by

$$f_t = \begin{cases} 1, & \text{on } \ A_1^t,\\ -1, & \text{on } \ C^t,\\ 0, & \text{elsewhere.} \end{cases}$$

In view of (1) we have

$$\mathbb{E}_{\mathbb{P}}(f_t) = \mathbb{E}_{\mathbb{Q}}(f_t) = 0. \tag{2}$$

Let, for $t \in \mathbb{N}_0$,

$$a_t = 2^t \prod_{s=t+1}^{\infty} (1 - 2^{-(2s-1)}) \quad \approx \quad 2^t,$$

and define the function f on Ω by

$$f = \sum_{t=0}^{\infty} a_t f_t.$$

As for each $\omega \in \Omega$ the values $f_t(\omega)$ are eventually zero the above sum converges everywhere on Ω. It is elementary to calculate explicitly the values of f:

$$f = \begin{cases} c_t & \text{on} \quad A_1^t, \\ -b_t & \text{on} \quad A_2^t \cup A_3^t, \end{cases}$$

where

$$b_t = \sum_{s=0}^{t} a_s = \sum_{s=0}^{t} (2^s \prod_{r=s+1}^{\infty} (1 - 2^{-(2r-1)})) \approx 2^t \text{ and, more precisely, } \lim_{t \to \infty} b^t / 2^t = 2,$$

$$c_t = a_t - b_{t-1} = 2^t \prod_{s=t+1}^{\infty} (1 - 2^{-(2s-1)}) - \sum_{s=0}^{t-1} 2^s \prod_{r=s+1}^{\infty} (1 - 2^{-(2r-1)}) \quad \approx \quad 1.$$

Note that $f \in L^2(\mathbb{P})$ and that, for all $t \in \mathbb{N}_0$,

$$(f_t, f)_{\mathbb{P}} = 0, \tag{3}$$

where $(\cdot\,, \cdot)_{\mathbb{P}}$ denotes the inner product in $L^2(\mathbb{P})$. Indeed, letting F_n denote the n'th partial sum of f

$$F_n = \sum_{t=0}^{n} a_t f_t$$

and noting the biorthogonality of $(f_t)_{t=0}^{\infty}$ we have, for $n \geq t$,

$$\begin{aligned} (f_t, F_n)_{\mathbb{P}} &= a_t (f_t, f_t)_{\mathbb{P}} \\ &= a_t 2\mathbb{P}(A_1^t) \\ &= 2^t \prod_{s=t+1}^{\infty} (1 - 2^{-(2s-1)}) 2 (2^{-(t+1)} \prod_{s=1}^{t} (1 - 2^{-(2s-1)})) \\ &= \prod_{s=1}^{\infty} (1 - 2^{-(2s-1)}). \end{aligned}$$

On the other hand

$$
\begin{aligned}
\lim_{n\to\infty}(f_t, f-F_n)_{\mathbb{P}} &= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}((F_n - f)\chi_{C^n}) \\
&= \lim_{n\to\infty}\Big(\sum_{m=n+1}^{\infty} \mathbb{E}_{\mathbb{P}}(F_n - f)\chi_{A_1^m} + \sum_{m=n}^{\infty} \mathbb{E}_{\mathbb{P}}(F_n - f)\chi_{A_2^m \cup A_3^m}\Big) \\
&= \lim_{n\to\infty}\Big(\sum_{m=n+1}^{\infty} (-b_n - c_m)\,\mathbb{P}(A_1^m) + \sum_{m=n}^{\infty}(-b_n + b_m)\,\mathbb{P}(A_2^m \cup A_3^m)\Big) \\
&= -\lim_{n\to\infty} b_n \sum_{m=n+1}^{\infty} \mathbb{P}(A_1^m) \\
&= -\lim_{n\to\infty} 2^{n+1} \sum_{m=n+1}^{\infty} \Big(2^{-(m+1)} \prod_{s=1}^{m}(1 - 2^{-(2s-1)})\Big) \\
&= -\prod_{s=1}^{\infty}(1 - 2^{-(2s-1)})).
\end{aligned}
$$

Combining these two equalities we obtain (3).

For later use we observe that

$$
\mathbb{E}_{\mathbb{P}}(f\chi_{B^t}) = \sum_{s=t}^{\infty} c_t P[A_1^t] - \sum_{s=t}^{\infty} b_t P[A_2^t \cup A_3^t] \approx 2^{-t}. \tag{4}
$$

Also note that

$$
\begin{aligned}
\lim_{n\to\infty} \|(F_n - f)_+\|^2_{L^2(\mathbb{P})} &= \lim_{n\to\infty} \sum_{m=n+1}^{\infty} (b_m - b_n)^2 \mathbb{P}(A_2^m \cup A_3^m) \\
&= \lim_{n\to\infty} \sum_{m=n+1}^{\infty} 2^{2m}.2^{-3m} = 0.
\end{aligned} \tag{5}
$$

Now define, for $t \in \mathbb{N}_0$,

$$
g_t = \begin{cases} M_t \text{ on } A_2^t, \\ -m_t \text{ on } A_3^t, \\ 1 \text{ on } B^{t+1}, \end{cases}
$$

where the real numbers M_t and m_t will be chosen such that the relations

$$
(f, g_t)_{\mathbb{P}} = 0 \qquad \text{and} \qquad \mathbb{E}_{\mathbb{Q}}(g_t) = 0 \tag{6}
$$

hold true. Clearly these equations are satisfied iff M_t and m_t solve the two linear equations

$$
\begin{array}{llcl}
M_t\,.\,\mathbb{P}(A_2^t)(-b_t) & +m_t\,.\,\mathbb{P}(A_3^t)b_t & = & -\mathbb{E}_{\mathbb{P}}(f\chi_{B^{t+1}}) \\
M_t\,.\,\mathbb{Q}(A_2^t) & -m_t\,.\,\mathbb{Q}(A_3^t) & = & -\mathbb{Q}(B^{t+1})
\end{array}
$$

We can rearrange these equations to get

$$\begin{aligned} M_t - m_t &= \tfrac{\mathbb{E}_{\mathbb{P}}(f\chi_{B^{t+1}})}{\mathbb{P}(A_2^t)b_t} && \approx 2^t \\ M_t\,.\,(-\tfrac{\mathbb{Q}(A_2^t)}{\mathbb{Q}(A_3^t)}) + m_t &= \tfrac{\mathbb{Q}(B^{t+1})}{\mathbb{Q}(A_3^t)} && \approx 1 \end{aligned}$$

which yields, in view of $\frac{\mathbb{Q}(A_2^t)}{\mathbb{Q}(A_3^t)} \approx 2^{-t}$,

$$M_t \approx 2^t \qquad \text{and} \qquad m_t \approx 1.$$

Now we are ready to define the process S: let $S_0 \equiv 0$ and, for $t \geq 0$,

$$\begin{aligned} S_{2t+1} - S_{2t} &= 2^{-t} f_t, \\ S_{2t+2} - S_{2t+1} &= 2^{-2t} g_t. \end{aligned}$$

Clearly $(S_t)_{t=0}^{\infty}$ is a uniformly bounded process and it follows from (2) and (6) that S is a $\mathbb{Q}$-martingale with respect to its natural filtration $(\mathcal{F}_t)_{t=0}^{\infty}$. Note that each F_n is a simple integral on the process S, hence $F_n \in K_2^s$ and we obtain from (5) that

$$f \in \overline{K_2^s - L_+^2(\mathbb{P})}. \tag{7}$$

On the other hand we claim that for

$$G_n = F_n + b_n g_n$$

we have that

$$\lim_{n\to\infty} \|(G_n - f)_-\|_{L^2(\mathbb{P})} = 0 \tag{8}$$

which will readily imply that

$$f \in \overline{K_2^s + L_+^2(\mathbb{P})}. \tag{9}$$

and therefore, combining (7) and (9),

$$f \in D_2.$$

To prove (8) note that

$$\begin{aligned} \|(G_n - f)_-\|_{L^2(\mathbb{P})}^2 = \|(G_n - f) - \chi_{A_3^n}\|_{L^2(\mathbb{P})}^2 &\approx (2b_n)^2 \mathbb{P}(A_3^t) \\ &\approx 2^{2n}\, 2^{-3n}. \end{aligned}$$

Finally we shall show that f is orthogonal to K_2^s, whence in particular

$$f \in D_2 \setminus K_2. \tag{10}$$

Indeed, as for each $t \geq 0$ the support of $S_{t+1} - S_t$ is contained in an atom of $\mathcal{F}_t$, the space K_2^s of simple integrals consists of the linear span of $(f_t)_{t=0}^{\infty}$ and $(g_t)_{t=0}^{\infty}$ and therefore we obtain the assertion from (3) and (6). The construction of the example now is completed.

q.e.d.

3.2 Remark.

It is instructive to relate the above example to theorem 1.2. The representation

$$f = \sum_{t=0}^{\infty} a_t f_t = \sum_{t=0}^{\infty} a_t \, 2^t (S_{2t+1} - S_{2t})$$

is a representation of the random variable as a stochastic integral $(H \cdot S)_\infty$ on the process S. From theorem 1.2 in conjunction with (7) and (9) above, we deduce that the process $(H \cdot S)_t$ is a uniformly integrable martingale, with respect to each equivalent martingale measure R on Ω satisfying $\frac{dR}{d\mathbb{P}} \in L^2(\mathbb{P})$, converging to $f = (H \cdot S)_\infty$ with respect to the norm of $L^1(R)$; in particular for $\mathbb{Q}$ we have that f closes the $\mathbb{Q}$-martingale $(H \cdot S)_t$, a fact which also may easily be calculated directly.

But how is the situation with respect to the measure $\mathbb{P}$? It happens that in our example the process $(H \cdot S)_t$ is in fact an L^1-bounded martingale with respect to $\mathbb{P}$; but this martingale is not uniformly integrable with respect to $\mathbb{P}$ and f does not close it in $L^1(\mathbb{P})$.

We now pass to the second example answering negatively the second question asked in remark 2.2. It is closely related to the examplegiven in [Sch 93] as well as to the simplified version of this example given in [DS 94b]. In the present context it turns out to be more convenient to follow the track of the construction in [Sch93], if we are only interested in a discrete time example.

3.3 Example. *We construct a uniformly bounded discrete adapted process $(X_t)_{t=0}^{\infty}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^{\infty}, \hat{\mathbb{P}})$ and an equivalent measure $\mathbb{Q}$ on $\mathcal{F}$ with the following properties:*

1. *X is a martingale with respect to $\hat{\mathbb{P}}$ as well as with respect to $\mathbb{Q}$.*
2. *$\frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} \in L^2(\hat{\mathbb{P}})$*
3. *There is a predictable process H such that $((H \cdot X)_t)_{t=0}^{\infty}$ is a uniformly integrable martingale under $\mathbb{Q}$ converging (almost surely) to $(H \cdot X)_\infty \in L^2(\hat{\mathbb{P}})$, but such that $((H \cdot X)_t)_{t=0}^{\infty}$ is not a uniformly integrable martingale under $\hat{\mathbb{P}}$.*

Construction. Consider the example in section 2 from [S 93] from which we freely use the notation (the use of the notation X and $\hat{\mathbb{P}}$ above instead of the usual S and $\mathbb{P}$ was chosen in order to avoid confusion with the notation from [S 93]): define X by $X_0 \equiv 0$ and, for $t \geq 0$,

$$X_{t+1} - X_t = 4^{-t}(\hat{G}_{t+1} - \hat{G}_t).$$

We leave $\mathbb{Q}$ as in the original example, but we slightly modify $\mathbb{P}$ to define

$$\hat{\mathbb{P}} = [\bigotimes_{n=1}^{\infty} (\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1})] \otimes [\bigotimes_{n=1}^{\infty} ((1 - 4^{-n})\delta_1 + 4^{-n}\delta_{-1})].$$

Then we have

$$\begin{aligned}\hat{\mathbb{P}}(B_t) &= 2^{-t}\prod_{n=1}^{t}(1-4^{-n}) &&\approx 2^{-t}\\ \hat{\mathbb{P}}(C_t) &= 2^{-t-1}\prod_{n=1}^{t-1}(1-4^{-n})4^{-n} &&\approx 8^{-t}\\ \hat{\mathbb{P}}(D_t) &= 2^{-t}\prod_{n=1}^{t}(1-4^{-n}) &&\approx 2^{-t}\end{aligned}$$

Note that X is a bounded local martingale (and therefore a uniformly integrable martingale) with respect to $\mathbb{Q}$ as well as with respect to $\hat{\mathbb{P}}$. Let $H_t = 4^t$ so that

$$(H \cdot X)_t = \hat{G}_t,$$

which is a uniformly integrable martingale with respect to $\mathbb{Q}$ but not with respect to $\hat{\mathbb{P}}$.

Claim 1: $(H \cdot X)_\infty = \hat{G}_\infty \in L^2(\hat{\mathbb{P}})$.

Indeed, for $\omega \in C_t$ we have $\hat{G}_\infty(\omega) \approx 2^t$ and $\hat{\mathbb{P}}(C_t) \approx 8^{-t}$, while for $\omega \in D_t$ we have $\hat{G}_\infty(\omega) \approx 1$ and $\hat{\mathbb{P}}(D_t) \approx 2^{-t}$.

Claim 2: $\frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} \in L^2(\hat{\mathbb{P}})$. Indeed, for $\omega \in C_t$, we have $\frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} \approx 1$ and, for $\omega \in D_t$, we have $\frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} \approx 4^{-t}$, i.e., $\frac{d\mathbb{Q}}{d\hat{\mathbb{P}}}$ is even uniformly bounded.

The construction is complete.

q.e.d.

3.4 Remark. The example 3.3 is in discrete time; however, using the techniques from [Sch 93] or using the construction from [DS 94b] it may be translated into a continuous process in finite continuous time. On the other hand example 3.1 is resistent to such a translation as we have seen in theorem 2.2 above that, for continuous processes, we have $K_p = D_p$.

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