Subspaces of L_p Isometric to Subspaces of ℓ_p

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Abstract. We present three results on isometric embeddings of a (closed, linear) subspace X of $L_p = L_p[0,1]$ into ℓ_p. First we show that if $p \notin 2\mathbb{N}$, then X is isometrically isomorphic to a subspace of ℓ_p if and only if some, equivalently every, subspace of L_p which contains the constant functions and which is isometrically isomorphic to X, consists of functions having discrete distribution. In contrast, if $p \in 2\mathbb{N}$ and X is finite-dimensional, then X is isometrically isomorphic to a subspace of ℓ_p^N, where the positive integer N depends on the dimension of X, on p, and on the chosen scalar field. The third result, stated in local terms, shows in particular that if p is not an even integer, then no finite-dimensional Banach space can be isometrically universal for the 2-dimensional subspaces of L_p.

0. Introduction

Statement of Results

The starting point of the present paper was a question of Albrecht Pietsch: "Which finite-dimensional subspaces of L_p are isometrically isomorphic to subspaces of ℓ_p?" The purpose of this note is to give a comprehensive answer to Pietsch's question. Curiously, the answer depends on the arithmetic nature of p and splits into two cases. In the first case p is an arbitrary positive number but not an even integer, in the second case p is a positive even integer (in symbols $p \in 2\mathbb{N}$). In the first case, our criterion extends to arbitrary, not necessarily finite-dimensional, closed linear subspaces of L_p. In any case, the answer is independent of whether the basic scalar field, which we denote by $\mathbb{K}$, is the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$.

Accordingly, the answer is stated in two parts, Theorem A and Theorem B, which are proved in Sections 1 and 2, respectively. The first theorem reads as follows:

Theorem A. *Let $0 < p < \infty$, $p \notin 2\mathbb{N}$. Then a closed linear subspace X of L_p is isometrically isomorphic to a subspace of ℓ_p if and only if every, equivalently some, unital subspace which is isometrically isomorphic to X, consists of functions with discrete distribution.*

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Recall that a subspace of L_p is **unital** if it contains the constant functions. As we shall see (Lemma 1.2), every non-zero closed linear subspace of L_p is in fact isometrically isomorphic to a unital one.

¿From Theorem A we can easily derive that if $0 < p < \infty$ and $p \notin 2\mathbb{N}$, then 2-dimensional Hilbert space is not isometrically isomorphic to a subspace of ℓ_p (cf. Corollary 1.8). A weaker result, with ℓ_p replaced by ℓ_p^n ($n \in \mathbb{N}$), was discovered by Yu. Lyubich [Ly].

Theorem B. *Let $p \in 2\mathbb{N}$. Then every finite-dimensional subspace of L_p is isometrically isomorphic to a subspace of ℓ_p.*

Moreover, for each $n \in \mathbb{N}$ there exists an $N \in \mathbb{N}$, depending on n, p and $\mathbb{K}$, such that every n-dimensional subspace of L_p is isometrically isomorphic to a subspace of ℓ_p^N.

Denote by $N(n,p,\mathbb{K})$ the least N with the property described in the 'moreover part' of Theorem B, and by $H(n,p,\mathbb{K})$ the smallest of all integers H such that n-dimensional Hilbert space is isometrically isomorphic to a subspace of ℓ_p^H. In recent years, the numbers $H(n,p,\mathbb{K})$ have been object of intensive study (see the survey [Kö] and the references there). The second part of Section 2 is devoted to a discussion of the relationship between the numbers $H(n,p,\mathbb{K})$ and $N(n,p,\mathbb{K})$. In particular, we show that $H(2,4,\mathbb{K}) = N(2,4,\mathbb{K})$ for both, $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. As a byproduct we obtain the surprising fact that every 2-dimensional subspace of real L_4 has a symmetric basis of symmetric norm one.

To describe the topic of Section 3 recall that a Banach space Y is **isometrically universal** for a given family $\mathcal{F}$ of Banach spaces provided every member of $\mathcal{F}$ is isometrically isomorphic to a subspace of Y. In this language, the 'moreover part' of Theorem B can be restated as follows: if $p \in 2\mathbb{N}$ then given any $n \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that ℓ_p^N is isometrically universal for the family of all n-dimensional subspaces of L_p.

This is in marked contrast to the strongly 'negative' character of

Theorem C. *Let $1 \le p < \infty$, $p \notin 2\mathbb{N}$, and let X be a Banach space which is isomorphic either to L_p or to ℓ_p. Then, for each $k = 2, 3, \ldots$, there is no finite-dimensional Banach space which is isometrically universal for all k-dimensional subspaces of X.*

Theorem C will be obtained as a simple consequence of Theorem 3.2 which is the main result of Section 3. It is stated in terms of 'Local Theory of Banach Spaces' and connects, for a given Banach space X, the problem of non-existence of finite-dimensional Banach spaces which are isometrically universal for all 2-dimensional subspaces of X to 'type and cotype' of X.

Tools and Notation

Let $0 < p < \infty$, and let μ be a non-negative measure defined on a σ-field Σ of subsets of a set Ω. By $L_p(\mu) = L_p(\Omega, \Sigma, \mu)$ we denote the usual quasi-Banach space of μ-equivalence classes of scalar valued functions on Ω which are p-absolutely integrable against μ. The quasi-norm on

$L_p(\mu)$ (which is a norm when $p \geq 1$) will be denoted by $\|\cdot\|_p$, or by $\|\cdot\|_{L_p(\mu)}$ if necessary; so

$$\|f\|_p \;=\; \Big(\int_\Omega |f|^p d\mu \Big)^{1/p} \qquad \text{for} \quad f \in L_p(\mu) \;.$$

If ν is a probability (=normalized) measure on a σ-field of subsets of a set Ω and $f : \Omega \to \mathbb{K}$ is a ν-measurable function (=random variable), then the **distribution** of f is the probability measure on $\mathcal{B}(\mathbb{K})$ denoted by μ_f an defined by

$$\mu_f(A) \;=\; \nu(f^{-1}(A)) \qquad \text{for} \qquad A \in \mathcal{B}(\mathbb{K}) \;.$$

Here we denote, for a topological space X, by $\mathcal{B}(X)$ the σ-field of all Borel subsets of X. The distribution is said to be **discrete** if it is of the form

$$\mu_f \;=\; \sum_{j\in\mathbb{N}} a_j \delta_{x_j} \;,$$

where the x_j's and a_j's are scalars, with $a_j \geq 0$ for all j and $\sum_j a_j = 1$; by δ_x we denote the measure of total mass 1 concentrated at x.

We are going to consider the following standard L_p spaces:

- $L_p := L_p([0,1], \mathcal{B}, \lambda)$; here $\mathcal{B} = \mathcal{B}([0,1])$, and λ is the usual Lebesgue probability measure on $\mathcal{B}$.
- ℓ_p is the space of all infinite p-absolutely summable sequences; so Ω is $\mathbb{N}$, Σ is the collection of all subsets of $\mathbb{N}$, and μ is the usual counting measure.
- Given $N \in \mathbb{N}$, the spaces ℓ_p^N and L_p^N are obtained by taking, for any N-point set M, μ to be the counting measure, resp. the normalized counting measure, on the subsets of M.

The main tool to prove Theorem A is the following

Equimeasurability Lemma 0.1. *Suppose that $0 < p < \infty$, $p \notin 2\mathbb{N}$, and that μ and ν are probability measures. If $f \in L_p(\mu)$ and $g \in L_p(\nu)$ are such that*

$$\|1 + z\cdot f\|_p \;=\; \|1 + z\cdot g\|_p \qquad \text{for all} \quad z \in \mathbb{K} \;,$$

then $\mu_f = \mu_g$.

The Equimeasurability Lemma will also enable us to prove Lemma 3.3, and this will eventually lead to a proof of Theorem C.

The lemma is a special case of a more general result discovered independently by A.I. Plotkin [Pl1], [Pl2], [Pl3] and by W. Rudin [Ru]; cf. also [Har], Theorem 1.1. For further information we refer to [Ko] and the references given there.

Our proof of Theorem B is based upon the classical theorem by C. Carathéodory on convex combinations in $\mathbb{R}^n$ (see e.g. [DGK]). The method goes back to J.H.B. Kemperman's [Ke] research in empirical statistics; in fact, his result can be adopted to get the qualitative version of Theorem B.

Our strategy to prove Theorem C has its origin in a paper of C. Bessaga [Be]. Solving a problem of S. Mazur [SB], he proved that there is no finite-dimensional Banach space which is isometrically universal for all 2-dimensional Banach spaces. Hence, given any integer $k \geq 2$, no finite-dimensional Banach space can be isometrically universal for all k-dimensional Banach spaces. Bessaga's method was considerably extended by V.L. Klee, Jr. [Kl], and in fact our proof of Theorem 3.2 is based upon Klee's criterion on estimates of the topological dimension of families of finite-dimensional Banach spaces amitting finite-dimensional universal spaces (cf. [Kl], §3). We will outline Klee's argument in the Appendix to Section 3.

In Sections 2 and 3, we use the concept of Banach-Mazur compactum. Recall that given $n \in \mathbb{N}$ and n-dimensional Banach space E and F, the (multiplicative) Banach-Mazur distance is defined by

$$d(E,F) := \inf \{c : \exists\, u : E \to F \text{ such that } \|x\| \leq \|u(x)\| \leq c \cdot \|x\| \quad \forall\, x \in E\} .$$

The Banach-Mazur 'metric'

$$\delta(E,F) := \log d(E,F)$$

induces a genuine metric (also denoted by $\delta(\,\cdot\,,\,\cdot\,)$) on

$$BM_n := \text{ the set of isometry classes of } n\text{-dimensional Banach spaces.}$$

With respect to this metric BM_n is compact; it is called the n-th **Banach-Mazur compactum**.

In the sequel, we shall identify an element in BM_n with any of its representing n-dimensional Banach spaces. Accordingly, given a Banach space X, we define

$$BM_n(X) := \{E \subset X : E \in BM_n\} .$$

The following is well-known and easy to prove:

Lemma 0.2. *If X is finite-dimensional then $BM_n(X)$ is compact; more precisely, $BM_n(X)$ is compact when it is identified with a subset of the space (BM_n, δ).*

Throughout this paper, terms like 'isomorphism', 'isomorphic embedding', 'isometric isomorphism', ... have their usual meaning; see [DS]. Frequently, we shall use 'subspace' instead of 'closed linear subspace', 'operator' instead of 'bounded linear map', etc., and occasionally even 'isometric' instead of 'isometrically isomorphic'.

We employ standard set theoretic notation. The number of elements in a set Ω is denoted by $|\Omega|$ (where $|\Omega| = \infty$ if Ω is infinite). We write 2^{Ω} for the family of all subsets of Ω, and we denote the indicator function of a subset C of Ω by 1_C. Instead of 1_{Ω}, we simply write 1.

The support of a function $f : \Omega \to \mathbb{K}$ is the set $\{\omega \in \Omega \colon f(\omega) \neq 0\}$. We adopt the convention to understand equalities and inclusions involving measurable sets modulo null sets. Accordingly, if f is an equivalence class of measurable functions $\Omega \to \mathbb{K}$, we denote by supp f the support of any representative of f. Let X be a space of such (equivalence classes of) functions. We say that $f \in X$ has **full support** in X if $\mu(\text{supp } g \setminus \text{supp } f) = 0$ for any $g \in X$. Clearly, if μ is a probability measure, then $f \in L_p(\mu)$ has full support (in $L_p(\mu)$) iff $\mu(\text{supp } f) = 1$.

It should be clear how to extend the preceding results to spaces $L_p(\mu)$ where μ is an arbitrary infinite, σ - finite, separable measure.

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1. Proof of Theorem A

We begin with two known lemmas. The first one was essentially established by S. Banach (cf. Chapter IX in [Ba]). In a general setting, it is due to J. Lamperti [La]. We present here a simple proof of the original result based on a formula taken from [Ro], p. 48.

Lemma 1.1. *Let* $0 < p < \infty$ *and suppose that* $f \in L_p$ *satisfies* (i) $\|f\|_p = 1$ *and* (ii) $\lambda(\text{supp } f) = 1$. *Then there exists an isometric isomorphism of* L_p *onto itself, say* v, *such that* $vf = 1$ *and* $|vg| \leq 1$ *for all* $g \in L_p$ *with* $|g| \leq |f|$.

Proof. Let us put $\Phi(t) := \int_0^t |f(s)|^p \, ds$ for $t \in [0,1]$. It follows from (i) and (ii) that Φ is an absolutely continuous, strictly increasing homeomorphism of $[0,1]$ onto itself. Moreover, $\frac{d\Phi}{dt} = |f|^p$ λ - a.e. By (i), $\mu := |f|^p d\lambda$ is a probability measure on $\mathcal{B}$.

Clearly, if $0 \leq a < b \leq 1$ then

$$\mu([a,b]) = \int_a^b |f(t)|^p dt = \Phi(b) - \Phi(a) = \lambda(\Phi([a,b])) \ .$$

The measure μ is obviously regular. Since Φ is absolutely continuous, this is also true for $\lambda \circ \Phi$. Regular measures on $\mathcal{B}$ are uniquely determined by their values on closed subintervals, hence

$$\mu(A) = \lambda(\Phi(A)) \quad \text{for every} \quad B \in \mathcal{B} \ .$$

Let $\Psi : [0,1] \to [0,1]$ be the inverse of Φ. Since this is a homeomorphism, the map $u : L_p(\mu) \to L_p$ defined by $u(h) = h \circ \Psi$ is an isometric isomorphism onto.

On the other hand, it follows from (ii) that $w(g) := g/f$ also defines an isometric isomorphism w of L_p onto itself. Thus $v = u \circ w$ maps L_p isometrically onto itself. Obviously,

$$vg = \frac{g \circ \Psi}{f \circ \Psi} \qquad \text{for} \quad g \in L_p \ ,$$

so that v has the desired properties. ■

Lemma 1.2. *Let $0 < p < \infty$. Then every non-zero subspace X of L_p is isometrically isomorphic to a unital subspace of L_p.*

Proof. Together with L_p, X is separable. Hence, by [Har], Lemma 3.2, there exists a $g \in X$ with full support in X. Note that $\lambda(A) > 0$ since $X \neq \{0\}$. Consider the probability space $(A, \mathcal{B}_A, \lambda_A)$ where $\mathcal{B}_A := \{B \in \mathcal{B} : B \subset A\}$ and λ_A is the conditional probability defined by $\lambda_A(B) = \lambda(B) \cdot \lambda(A)^{-1}$ for $B \in \mathcal{B}_A$. Note that $f \mapsto \lambda(A)^{-1/p} f$ defines an isometric isomorphism of X onto a subspace of $L_p(\lambda_A)$ and that λ_A is a purely nonatomic separable probability measure.

It follows from [Hal], Chap. VIII, Theorem 3, that the measure algebras $(A, \mathcal{B}_A, \lambda_A)$ and $([0,1], \mathcal{B}, \lambda)$ are isometrically isomorphic. The isometric isomorphism of the algebras induces an isometric isomorphism, say u, of $L_p(\lambda_A)$ onto L_p such that $\lambda_A(\text{supp } g) = \lambda(\text{supp } u(g))$. In particular, $\lambda(\text{supp } u(g)) = 1$, and Lemma 1.1 provides us with an isometric isomorphism v of L_p onto itself such that $v(u(g)) = 1$. Hence X is isometrically isomorphic to the unital subspace $vu(X)$ of L_p. ■

Proof of Theorem A. Necessity:

Let $p \notin 2\mathbb{N}$. Let X be a unital subspace of L_p which is isometrically isomorphic to a subspace of ℓ_p, via an isometric isomorphism w.

By [Har], Lemma 3.4, $\gamma := w(1)$ has full support in $w(X)$. Put $S := \text{supp } \gamma$. Clearly, $w(X)$ can be regarded as a subspace of ℓ_p^S, where $\ell_p^S = \{\xi \in \ell_p : \xi(s) = 0 \text{ for } s \notin S\}$. ($S$ is not necessarily a finite set!) Define the discrete measure ν on 2^S by

$$\nu(B) := \sum_{s \in B} |\gamma(s)|^p \qquad \text{for} \quad B \subset S \ .$$

Note that ν is a probability measure because $\nu(S) = \sum_{s \in S} |\gamma(s)|^p = \|w(1)\|_{\ell_p}^p = 1$. Next define $u : \ell_p^S \to L_p(\nu)$ by

$$u(\xi) := \Big(\frac{\xi(s)}{\gamma(s)} \cdot \text{sign}\, \gamma(s) \Big)_{s \in S} \qquad (\xi \in \ell_p^S) \ .$$

This is an isometric isomorphism with $u(\gamma) = 1$. Hence uw is an isometric isomorphism of X onto the unital subspace $uw(X)$ of $L_p(\nu)$ such that $uw(1) = 1$.

Now pick $f \in X$. Clearly, if $f = z \cdot 1$ for some $z \in \mathbb{K}$, then f has discrete distribution. If $f \neq z \cdot 1$ for all $z \in \mathbb{K}$, then uw carries the 2-dimensional subspace of X spanned by 1 and f onto the 2-dimensional subspace of $L_p(\nu)$ spanned by 1 and $uw(f)$. Recall that $p \notin 2\mathbb{N}$ and invoke the Equimeasurability Lemma 0.1 to conclude that $uw(f)$ has discrete distribution since f does.

Proof of Theorem A. Sufficiency:

Let X be a (unital) subspace of L_p such that each $f \in X$ has discrete distribution. Under the additional hypothesis that X is finite-dimensional the proof is easy.

In fact, let then $\{f_1, \ldots, f_k\}$ be a basis for X. Since each f_n $(1 \le n \le k)$ has discrete distribution, we may write

$$f_n = \sum_{j \in J(n)} z_{j,n} 1_{C(j,n)}$$

where either $J(n) = \mathbb{N}$, or $J(n)$ is a finite interval of $\mathbb{N}$, the $z_{j,n}$'s belong to $\mathbb{K}$ $(j \in J(n))$ and $(C(j,n))_{j \in J(n)}$ is a partition of $[0,1]$ into Borel sets. Changing, if necessary, each $C(j,n)$ by an appropriate λ-null set, we may also assume that for every sequence $(j(n))_{n=1}^k \in \prod_{n=1}^k J(n)$ we have

$$\bigcap_{n=1}^{k} C(j(n), n) = \emptyset \qquad \text{whenever} \qquad \lambda\Big(\bigcap_{n=1}^{k} C(j(n), n)\Big) = 0 \ .$$

Let $\mathcal{A}$ be the Boolean algebra generated by the sets $C(j(n), n)$, $j(n) \in J(n)$, $n = 1, \ldots, k$. It is easy to verify that $\mathcal{A}$ is purely atomic; the atoms being all the sets $\bigcap_{n=1}^k C(j(n), n)$ of positive measure. Thus, if Y is the closed subspace of L_p generated by the characteristic functions of these atoms, then Y is isometrically isomorphic either to ℓ_p, or to ℓ_p^m for some $m \in \mathbb{N}$. Clearly $X \subset Y$ because $f_n \in Y$ for $n = 1, \ldots, k$.

In the general case the proof of the sufficiency part of Theorem A is much deeper. It is an immediate consequence of Theorem 1.3 below which was conjectured by the authors and essentially proved by S. Kwapień. His original elegant argument was more probabilistic in nature; it made extensive use of Prokhorov's Theorem and the concept of weak convergence of distribution measures (cf. [Pa]). We present here an altenative argument.

Theorem 1.3. *Let $0 < p < \infty$ and let X be a subspace of L_p such that each function in X has discrete distribution. Then X is isometrically isomorphic to a subspace of ℓ_p.*

The proof requires a bit of preparation.

We work on a fixed probability space (Ω, Σ, μ). Recall that (Ω, Σ, μ) can be decomposed into at most countably many sets $A_n, A_\infty \in \Sigma$ such that each A_n is an atom whereas A_∞ contains no atoms. If $\mu(A_\infty) = 1$, then (Ω, Σ, μ) is called **atomless** or **diffuse**. If $\mu(A_\infty) = 0$, then (Ω, Σ, μ) is called **purely atomic**. Suppose that $\mu(A_\infty) > 0$. If $g : \Omega \to \mathbb{K}$ is essentially injective in the sense that $\big\{(\omega, \omega') \in \Omega \times \Omega : \omega \neq \omega',\ g(\omega) = g(\omega')\big\}$ is a null set for the product measure $\mu \otimes \mu$, then g cannot have atomic distribution. Indeed, it suffices to remark that A_∞ is then essentially uncountable.

(The problem whether A_∞ has at least the power of the continuum is fundamental in set theory and leads to discussions on measurable cardinals, fortunately we do not need such results here.)

Given a set M of measurable functions $\Omega \to \mathbb{K}$, we denote the σ-algebra generated by the members of M by $\sigma(M)$, or by a related simpler symbol. We need a lemma on pairs of measurable functions $g, h : \Omega \to \mathbb{K}$. It is clear that g has atomic distribution if and only if $\sigma(g)$ is generated by a partition of Ω into at most countably many atoms. If the functions $g, h : \Omega \to \mathbb{K}$ have atomic distribution and $\alpha \in \mathbb{K}$, then $g + \alpha h$ also has atomic distribution, and the partition generating $\sigma(g + \alpha h)$ is coarser than $\sigma(g, h)$.

Lemma 1.4. *Suppose that g and h have atomic distribution. Then there is, for each $\varepsilon > 0$, an $\alpha \in \mathbb{K}$ such that $|\alpha| \le \varepsilon$ and*

$$\sigma(g + \alpha h) \;=\; \sigma(g, h) \;.$$

Proof. Let (A_s) and (B_t) be the (at most countable) partitions of Ω generating $\sigma(g)$ and $\sigma(h)$, respectively. Then $g = \sum_s \xi_s 1_{A_s}$ and $h = \sum_t \eta_t 1_{B_t}$ where $\xi_s \ne \xi_{s'}$ if $s \ne s'$ and $\eta_t \ne \eta_{t'}$ if $t \ne t'$. Consider the set Z of all $(\xi_s - \xi_{s'})/(\eta_t - \eta_{t'})$ with $t \ne t'$. Since Z is at most countable, we may select $\alpha \in \mathbb{K}$ such that $|\alpha| \le \varepsilon$ and $\alpha \notin Z$. We claim that $\sigma(g + \alpha h) = \sigma(g, h)$.

For this it is enough to show that if $\omega \in A_s \cap B_t$ and $\omega' \in A_{s'} \cap B_{t'}$ are such that $(g + \alpha h)(\omega) = (g + \alpha h)(\omega')$, then $s = s'$ and $t = t'$. Note that $t = t'$ yields $\xi_s = \xi_{s'}$ and so $s = s'$ by the very definition. On the other hand, $t \ne t'$ cannot happen! In fact, together with $\xi_s + \alpha\eta_t = \xi_{s'} + \alpha\eta_{t'}$ this would lead to $\alpha = (\xi_s - \xi_{s'})/(\eta_{t'} - \eta_t) \in Z$: a contradiction. ■

We denote by $L_0(\mu)$ the space of all (equivalence classes of) measurable functions $\Omega \to \mathbb{K}$. Recall that with respect to convergence in measure, this is a complete metrizable topological vector space.

We are going to consider subsets M of $L_0(\mu)$ with ¿the following property:

(*) *For every sequence (f_n) in M there is a sequence (ε_n) of positive real numbers such that whenever (a_n) is a sequence of scalars such that $|a_n| \le \varepsilon_n \ \forall n \in \mathbb{N}$, then $\sum_n a_n f_n$ converges in measure and its limit belongs to M.*

This condition is satisfied if M is a closed subspace of $L_p(\mu)$ for $0 \le p < \infty$, or a weak$*$ closed subspace of $L_\infty(\mu)$. But M could also just be a sufficiently big generator of a suitable space of functions on Ω. Thus Theorem 1.3 is a special case of the following

Theorem 1.5. *If $M \subset L_0(\mu)$ satisfies $(*)$ and every $g \in M$ has atomic distribution, then there is an at most countable partition of Ω into sets $A_k \in \Sigma$ such that for each $f \in M$ there are scalars $a_k(f)$ with*

$$f \;=\; \sum_k a_k(f) 1_{A_k}$$

(convergence in $L_0(\mu)$).

Proof. We start with the case where M is a separable subset of $L_0(\mu)$. Let $(f_n)_{n\in\mathbb{N}}$ be a dense sequence in M. We may assume that Σ coincides with $\sigma(M) = \sigma(f_n : n \in \mathbb{N})$. In fact, otherwise a standard reduction procedure could be applied to arrange for such a situation. We may also assume that (f_n) separates the points of Ω in the sense that if $\omega, \omega' \in \Omega$ are such that $f_n(\omega) = f_n(\omega')$ for all n, then $\omega = \omega'$. Again, should this not a priori be the case, we could pass from Ω to an appropriate set of equivalence classes of points in Ω.

All what is then needed to show is that $\sigma(\{f_n : n\})$ is purely atomic. The idea is to find scalars a_n such that $g = \sum_n a_n f_n$ exists in M and is essentially injective in the sense explained above. Since g has atomic distribution, it is clear that there cannot be any diffuse part in (Ω, Σ, μ).

The construction is as follows. Fix (ε_n) as in $(*)$. The trick is to select a_n's with $|a_n| \leq \varepsilon_n$ so that (a_n) tends to zero so quickly that $g = \sum_n a_n f_n$ must be injective. We proceed by induction.

For $n = 1$ we simply take $a_1 = \varepsilon_1$ (which we can take to be 1 if we like). Suppose that, for some n, we have already found $a_1, \ldots, a_n$ such that $\sigma(\sum_{k=1}^n a_k f_k) = \sigma(f_1, \ldots, f_n)$. Let $(D_j^{(n)})_j$ be the (at most countable) family of atoms of $\sigma(f_1, \ldots, f_n)$. Choose $N_n \in \mathbb{N}$ so big that $\mu\big(\bigcup_{j>N_n} D_j^{(n)}\big) \leq 3^{-n}$. On $\bigcup_{j\leq N_n} D_j^{(n)}$, the function $a_1 f_1 + \ldots + a_n f_n$ assumes N_n distinct values $\xi_1, \ldots, \xi_{N_n}$. Choose $0 < \delta_n \leq \delta_{n-1}$ such that $\delta_n \leq |\xi_j - \xi_{j'}|$ for all $j, j' \leq N_n$ with $j \neq j'$ and fix m_{n+1} so that $\mu\{|f_{n+1}| > m_{n+1}\} \leq 3^{-n}$. Then we can find a_{n+1} such that

(i) $|a_{n+1}| \leq \varepsilon_{n+1}$,

(ii) $|a_{n+1}| \leq m_{n+1}^{-1} 3^{-n} \delta_n$,

(iii) $\sigma(a_1 f_1 + \ldots + a_{n+1} f_{n+1}) = \sigma(a_1 f_1 + \ldots + a_n f_n, f_{n+1})$.

In the last step we have applied Lemma 1.4. Note that the induction hypothesis allows us to write $\sigma(a_1 f_1 + \ldots + a_n f_n, f_{n+1}) = \sigma(f_1, \ldots, f_n, f_{n+1})$.

Let now $g = \sum_n a_n f_n$. Because of $\Sigma = \sigma(f_n : n \in \mathbb{N})$, $\{(\omega, \omega') : g(\omega) = g(\omega')\}$ is an element of $\Sigma \otimes \Sigma$. For each n, define now $E_n \in \Sigma \otimes \Sigma$ by

$$E_n := \Big(\big(\bigcup_{j\leq N_n} D_j^{(n)} \big) \times \big(\bigcup_{j \leq N_n} D_j^{(n)} \big) \Big)^c \cup \bigcup_{k>n} \{(\omega, \omega') : |f_k(\omega)| > m_k \text{ or } |f_k(\omega')| > m_k\} \ .$$

Here A^c stands for the complement of a set A. A straightforward calculation reveals that

$$(\mu \otimes \mu)(E_n) \ \leq \ 3^{-n} + 3^{-n} + \sum_{k>n} (3^{-k} + 3^{-k}) \ = \ 3^{-n+1} \ .$$

The Borel-Cantelli Lemma (cf. [Kh], p.7) now implies that for $\mu \otimes \mu$ almost all choices of $(\omega, \omega') \in \Omega \times \Omega$, there is an $n_0 = n_0(\omega, \omega')$ such that $(\omega, \omega') \in E_n^c$ for all $n \geq n_0$.

Suppose now that in addition ω, ω' are such that $g(\omega) = g(\omega')$ and let $n \geq n_0$. Clearly, ω, ω' both belong to $\bigcup_{j\leq N_n} D_j^{(n)}$ and $\max\{|f_k(\omega)|, |f_k(\omega')|\} \leq m_k$ whenever $k > n$. Our assumptions on $(a_k)_k$ imply that

$$\Big| \sum_{k>n} a_k f_k(\omega) \Big| \ \leq \ \sum_{k>n} |a_k| |f_k(\omega)| \ \leq \ \sum_{k>n} 3^{-k} \delta_k \ \leq \ \delta_n \sum_{k>n} 3^{-k} \ = \ 3^{-n} \delta_n / 2 \ .$$

Similar for ω'. Thus taking into account that $g(\omega) = g(\omega')$ we get

$$\Big| \sum_{k\leq n} a_k f_k(\omega) - \sum_{k\leq n} a_k f_k(\omega') \Big| \ = \ \Big| \sum_{k>n} a_k f_k(\omega') - \sum_{k>n} a_k f_k(\omega) \Big| \ \leq \ 3^{-n} \delta_n \ ,$$

hence ω, ω' are in the same atom $D_j^{(n)}$ for some $j \le N_n$. Thus $\sum_{k\le n} a_k f_k(\omega) = \sum_{k\le n} a_k f_k(\omega')$ and so, by our choice of the sequence $(a_s)_s$, we have $f_k(\omega) = f_k(\omega')$ for $k \le n$. But $n \ge n_0$ was arbitrary, hence $f_k(\omega) = f_k(\omega')$ for all k.

This leads to $\omega = \omega'$ and completes the proof for the case when M is separable, viz. $\sigma(X)$ is generated by a sequence of functions. The general case can be reduced to this one as follows.

Let $S := \{\Sigma_i : i \in I\}$ be the collection of all σ-algebras which are generated by sequences in M. S is a σ-lattice in the sense that for each countable family $(\Sigma_{i_n})_n$ in S there is a $j \in I$ such that $\Sigma_{i_n} \subset \Sigma_j$ for all n.

The preceding reasoning applied to the set $M_i := \{f \in M : f$ is Σ_i-measurable$\}$ shows that each Σ_i is purely atomic. For each $i \in I$, we set $\varphi(i) := \sum_D \mu(D)^2$ where summation is over all atoms D of Σ_i. It is clear that $\varphi(i) \ge \varphi(j)$ whenever $\Sigma_i \subset \Sigma_j$, with equality only when $\Sigma_i = \Sigma_j$ (up to sets of measure zero). Since S is a σ-lattice and since $\varphi(i) > 0$ for each i, we get that φ attains its minimum: $\varphi(j) = \min\{\varphi(i) : i \in I\}$ for some $j \in I$. It follows that Σ_j is the biggest of all our σ-algebras, that is, that $\Sigma_j = \sigma(M)$.

The proof is complete. ∎

Corollary 1.6. *Suppose that X is a Banach space of measurable functions on Ω, that $X \hookrightarrow L_0(\mu)$ is continuous, and that $M \subset X$ is a closed absolutely convex subset whose linear span is dense in X for the $L_0(\mu)$ topology. If all functions in M have purely atomic distribution, then the same is true for all functions in X; there is even a partition of Ω into at most countably many atoms A_n such that each $f \in X$ has a representation $f = \sum_k a_k(f) 1_{A_k}$, the $a_k(f)$'s being suitably chosen scalars.*

The next corollary is an immediate consequence of Theorem A.

Corollary 1.7. *If $0 < p < \infty$ is not an even integer, then a subspace X of L_p is isometrically isomorphic to a subspace of ℓ_p if and only if every 2-dimensional subspace of X is isometrically isomorphic to a subspace of ℓ_p.*

The last result of this section follows from Corollary 1.7 and ¿from the fact that if $p \ne 2$, then no infinite-dimensional subspace of ℓ_p can be isomorphic to a Hilbert space.

Corollary 1.8. *If $0 < p < \infty$ is not an even integer, then the 2-dimensional Hilbert space ℓ_2^2 cannot be isometrically isomorphic to a subspace of ℓ_p.*

It is interesting to compare Theorem A with Theorem 5.4 of N.J. Kalton and D. Werner [KW] (see also [GK] where the case $p = 1$ is studied). They proved that if $1 < p < \infty$ and if X is a subspace of L_p, then the unit ball of X is compact in the norm topology of L_1 if and only if X is 'nearly isometric' to a subspace of ℓ_p in the sense that, for each $\varepsilon > 0$, there exists an isomorphism $u_\varepsilon : X \to \ell_p$ such that $\|x\| \le \|u_\varepsilon(x)\| \le (1+\varepsilon)\|x\|$ for all $x \in X$. Clearly every finite-dimensional subspace of L_p has this property. An interesting infinite-dimensional example is the **Müntz space** $M_p(\{t^{n_k}\})$; it is defined to be the subspace of L_p generated by the monomials

t^{n_k}, $k \in \mathbb{N}$, where (n_k) is such that $\sum_{k=1}^{\infty} n_k^{-1} < \infty$. Combining Theorem 5.4 of [KW] with results of [Sch] we infer that $M_p(\{t^{n_k}\})$ is nearly isometric to a subspace of ℓ_p for $1 < p < \infty$, whereas Theorem A yields that, at least for $p \notin 2\mathbb{N}$, $M_p(\{t^{n_k}\})$ is not isometrically isomorphic to a subspace of ℓ_p.

2. Proof of Theorem B

In this section, p will be an even integer: $p = 2s$ with $s \in \mathbb{N}$.

Proof of Theorem B. It suffices to show that there exist an integer N, depending on n, p and $\mathbb{K}$, and a subset $\mathcal{A}_n$ of $BM_n(L_p)$, dense with respect to the Banach-Mazur metric, such that $\mathcal{A}_n \subset BM_n(\ell_p^N)$. Continuity of the embedding and compactness of $BM_n(\ell_p^N)$ will then yield $BM_n(L_p) \subset BM_n(\ell_p^N)$.

We choose $\mathcal{A}_n$ as follows. Fix $r \in \mathbb{N}$ and use the intervals $I_{k,r} := \left[\frac{k-1}{r}, \frac{k}{r}\right]$, $k = 1, \ldots, r$, to define the finite rank operator $u_r : L_p \to L_p$ by

$$u_r(f) := \sum_{k=1}^{r} r \cdot \Big(\int_0^1 f(t) \cdot 1_{I_{k,r}}(t)\, dt \Big) \cdot 1_{I_{k,r}} \qquad (f \in L_p) \ .$$

Let $\mathcal{A}_{n,r} \subset BM_n(L_p)$ be the family generated by all n-dimensional subspaces of $u_r(L_p)$, and put

$$\mathcal{A}_n := \bigcup_{r \in \mathbb{N}} \mathcal{A}_{n,r} \ .$$

We claim that $\mathcal{A}_n$ is dense in $BM_n(L_p)$. Indeed, it is well-known and easy to check that the operators u_r are norm-one projections such that $\lim_r \|u_r(1_{[a,b]}) - 1_{[a,b]}\|_p = 0$ for every interval $[a,b] \subset [0,1]$. It follows that the sequence (u_r) converges strongly to the identity operator on L_p. By the Banach-Steinhaus principle, it tends to the identity uniformly on compact sets. In particular, if F is an n-dimensional subspace of L_p then, given ε with $0 < \varepsilon < 1$, we get

$$(1-\varepsilon)\|f\|_p \ \le\ \|u_r(f)\|_p \ \le\ \|f\|_p$$

for r sufficiently large. Hence $d(F, u_r(F)) \le (1-\varepsilon)^{-1}$. This proves the claim.

Clearly, $u_r(1) = 1$, so that if F is unital then so is $u_r(F)$. In order to complete the proof of Theorem B, it is therefore enough to show that if E is a unital n-dimensional subspace of some $u_r(L_p)$, then E is isometrically isomorphic to a subspace of ℓ_p^N for some N independent of E and r.

We present the argument in the case $\mathbb{K} = \mathbb{C}$.

Given $n \in \mathbb{N}$, we choose $N = N_{\mathbb{C}}$ to be the number of elements of the subset $S = S(n,p,\mathbb{C})$ of $\mathbb{Z}_+^{n-1} \times \mathbb{Z}_+^{n-1} \setminus \{(0,0)\}$ defined by

$$S := \Big\{ (\alpha,\beta) \in \mathbb{Z}_+^{n-1} \times \mathbb{Z}_+^{n-1} : \sum_{j=1}^{n-1} \alpha_j \le s \ \text{and} \ \sum_{j=1}^{n-1} \beta_j \le s \Big\} \ .$$

Here $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $(0,0)$ denotes the sequence in $\mathbb{Z}_+^{n-1} \times \mathbb{Z}_+^{n-1}$ with zero coordinates only.

In the sequel, we shall apply standard multiindex notation. So, if $\alpha = (\alpha_j)_{j=1}^{n-1} \in \mathbb{Z}_+^{n-1}$ and $z = (z_j)_{j=1}^{n-1} \in \mathbb{C}^{n-1}$, then $z^\alpha := \prod_{j=0}^{n-1} z_j^{\alpha_j}$. Here 0^0 is understood to be 1. We also set $\overline{z} := (\overline{z_j})_{j=1}^{n-1}$, $\operatorname{Re} z := (\operatorname{Re} z_j)_{j=1}^{n-1}$ and $\operatorname{Im} z := (\operatorname{Im} z_j)_{j=1}^{n-1}$, and we use corresponding notation for $(n-1)$-tuples of scalar valued functions f_n in the pointwise sense. For example, if we are given $F = (f_j)_{j=1}^{n-1}$ with $\mathbb{C}$-valued functions f_j, then we write $(\operatorname{Re} F)^\alpha = \prod_{j=0}^{n-1} (\operatorname{Re} f_j)^{\alpha_j}$ for $\alpha \in \mathbb{Z}_+^{n-1}$.

We use ‘ $<$ ’ to denote lexicographic order in $\mathbb{Z}_+^{n-1}$.

Since $E \subset u_r(L_p) \subset L_\infty$ and since E is unital, E admits a basis $\{f_0, f_1, \ldots, f_{n-1}\}$ with $f_0 = 1$ and $\|f_j\|_\infty = 1$ for $j = 1, \ldots, n-1$. Put $F := (f_j)_{j=1}^{n-1}$. Then, for any choice of $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$,

$$
\begin{aligned}
\| \sum_{j=0}^{n-1} a_j f_j \|_p^p &= \int_0^1 \big(\sum_{j=0}^{n-1} a_j f_j \big)^s \big(\sum_{j=0}^{n-1} \overline{a_j f_j} \big)^s \, dt \\
&= \sum_{\alpha \in \mathbb{Z}_+^{n-1}} c_{\alpha,\alpha} \int_0^1 F^\alpha \overline{F}^\alpha + \sum_{\substack{(\alpha,\beta) \in S \\ \alpha \neq \beta}} c_{\alpha,\beta} \int_0^1 F^\alpha \overline{F}^\beta \, dt \ ,
\end{aligned}
$$

where the $c_{\alpha,\beta}$ are polynomials in the variables $(a_j)_{j=0}^{n-1}$ and $(\overline{a_j})_{j=0}^{n-1}$ such that $c_{\alpha,\beta} = \overline{c_{\beta,\alpha}}$. Coupling the pairs (α,β) and (β,α) and using the identity

$$
c_{\alpha,\beta} F^\alpha \overline{F}^\beta + c_{\beta,\alpha} F^\beta \overline{F}^\alpha \ = \ 2 \operatorname{Re} c_{\alpha,\beta} \operatorname{Re} (F^\alpha \overline{F}^\beta) - 2 \operatorname{Im} c_{\beta,\alpha} \operatorname{Im} (F^\beta \overline{F}^\alpha)
$$

we get

$$
\Big\| \sum_{j=0}^{n-1} a_j f_j \Big\|_p^p \ = \ c_{0,0} + \sum_{(\alpha,\beta) \in S} d_{\alpha,\beta} m_{\alpha,\beta} \tag{2.1}
$$

where

$$
m_{\alpha,\beta} \ := \ \begin{cases} \int_0^1 F^\alpha \overline{F}^\alpha \, dt & \text{for} \quad \alpha = \beta \neq 0 \\ \int_0^1 \operatorname{Re} (F^\alpha \overline{F}^\beta) \, dt & \text{for} \quad \alpha < \beta \\ \int_0^1 \operatorname{Im} (F^\beta \overline{F}^\alpha) \, dt & \text{for} \quad \alpha > \beta \end{cases} \tag{2.2}
$$

and

$$
d_{\alpha,\beta} \ = \ \begin{cases} c_{\alpha,\alpha} & \text{for} \quad \alpha = \beta \neq 0 \\ 2 \ \operatorname{Re} c_{\alpha,\beta} & \text{for} \quad \alpha < \beta \\ -2 \ \operatorname{Im} c_{\beta,\alpha} & \text{for} \quad \alpha > \beta \ . \end{cases}
$$

By construction,

$$
m \ := \ (m_{\alpha,\beta})_{(\alpha,\beta) \in S}
$$

is an element of $\mathbb{R}^N$.

Next we consider the manifold $L \subset \mathbb{R}^N$ defined by

$$
L \ = \ \{ w(z) = \big(w_{\alpha,\beta}(z) \big)_{(\alpha,\beta) \in S} : z \in D^{n-1} \} \ ,
$$

where D is the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ in the complex plane and

$$
w_{\alpha,\beta}(z) \ = \ \begin{cases} z^\alpha \, \overline{z}^\alpha & \text{for} \quad \alpha = \beta \neq 0 \\ \operatorname{Re} (z^\alpha \, \overline{z}^\beta) & \text{for} \quad \alpha < \beta \\ \operatorname{Im} (z^\beta \, \overline{z}^\alpha) & \text{for} \quad \alpha > \beta \ . \end{cases}
$$

Note that the function $z \mapsto w(z)$ is a homeomorphism from D^{n-1} onto L and that, if $\delta^{(j)} = (\delta_{j,k})_{k=1}^{n-1}$ is the j'th standard unit vector in $\mathbb{Z}_+^{(n-1)}$, then

$$w_{0,\delta^{(j)}}(z) - i \cdot w_{\delta^{(j)},0}(z) \;=\; z_j$$

for all $z \in D^{n-1}$ and $j = 1, \ldots, n-1$.

We are going to work with the convex hull of L,

$$C \;:=\; \operatorname{conv} L \;.$$

We start by showing that

$$m \;\in\; C \;. \tag{2.3}$$

Indeed, since the f_j's belong to $u_r(L_p)$, they are constant on each of the intervals $I_{k,r}$, $k = 1, \ldots, r$. It follows that the functions $\mathrm{Re}\,(F^\alpha \overline{F}^\beta)$ and $\mathrm{Im}\,(F^\beta \overline{F}^\alpha)$ do have the same property. We put

$$z^{(k)} \;:=\; \left(f_j\Big(\frac{k-\frac{1}{2}}{r}\Big)\right)_{j=1}^{r} \quad \text{and} \quad w^{(k)} \;:=\; w(z^{(k)}) \qquad (k = 1, \ldots, r) \;.$$

¿From $\|f_j\|_\infty = 1$ for $j = 1, \ldots, n-1$ we infer that the $w^{(k)}$'s belong to L. Since $m = \frac{1}{r}\sum_{k=1}^{r} w^{(k)}$, the proof of (2.3) is complete.

Since $L \subset \mathbb{R}^N$ is compact and connected, it follows ¿from an improvement of Carathéodory's Theorem (see [Fe] and [DGK], 3.5, p. 117) that m is a convex combination of at most N elements of L. Hence there are vectors $\xi^{(1)}, \ldots, \xi^{(N)} \in L$ and numbers $\nu_1, \ldots, \nu_N \geq 0$ such that $\sum_{d=1}^{N} \nu_d = 1$ and

$$m \;=\; \sum_{d=1}^{N} \nu_d \, \xi^{(d)} \;. \tag{2.4}$$

Let $\mathcal{N} := \{1, \ldots, N\}$. Define a probability measure ν on $2^{\mathcal{N}}$ by

$$\nu(A) \;:=\; \sum_{d \in A} \nu_d \qquad (A \subset \mathcal{N}) \;.$$

Let $z^{(d)} = (z_j^{(d)})_{j=1}^{n-1}$ be the unique point of D^{n-1} such that $\xi^{(d)} = w(z^{(d)})$ for $d \in \mathcal{N}$. Define functions $v_j : \mathcal{N} \to \mathbb{C}$ by setting $v_0 := 1$ and

$$v_j(d) \;:=\; z_j^{(d)} \quad \text{for } d \in \mathcal{N} \quad \text{and} \quad j = 1, 2, \ldots, n-1 \;.$$

Let E_0 be the subspace of $L_p(\nu)$ spanned by the functions $v_0, \ldots, v_{n-1}$. We claim that E and E_0 are isometrically isomorphic. In fact, we will show that the map $f_j \mapsto v_j$ $(j = 0, \ldots, n-1)$ extends to an isometric isomorphism from E onto E_0.

To prove this, put $V = (v_j)_{j=1}^{n-1}$ and notice first that

$$m_{\alpha,\beta} \;:=\; \begin{cases} \int_{\mathcal{N}} V^\alpha \, \overline{V}^\alpha \, d\nu & \text{for } \alpha = \beta \neq 0 \\ \int_{\mathcal{N}} \mathrm{Re}\,(V^\alpha \overline{V}^\beta) \, d\nu & \text{for } \alpha < \beta \\ \int_{\mathcal{N}} \mathrm{Im}\,(V^\beta \overline{V}^\alpha) \, d\nu & \text{for } \alpha > \beta \;. \end{cases} \tag{2.5}$$

In fact, if $\alpha < \beta$ then, in view of (2.4),

$$\int_{\mathcal{N}} \mathrm{Re}\,(V^\alpha \overline{V}^\beta)\,d\nu \;=\; \sum_{d=1}^{N} \nu_d \cdot \mathrm{Re}\Big(\prod_{j=0}^{n-1} v_j(d)^{\alpha_j}\overline{v}_j(d)^{\beta_j}\Big) \;=\; \sum_{d=1}^{N} \nu_d \cdot \mathrm{Re}\left((z^{(d)})^\alpha (\overline{z^{(d)}})^\beta\right)$$
$$= \sum_{d=1}^{N} \nu_d\, w_{\alpha,\beta}(z^{(d)}) \;=\; \sum_{d=1}^{N} \nu_d\, \xi^{(d)}_{l,\beta} \;=\; m_{\alpha,\beta}\ .$$

Similarly for $\alpha = \beta$ and $\alpha > \beta$.

Combining (2.1), (2.2) and (2.5) we get what we wanted: no matter how we choose complex numbers $a_0, \dots, a_{n-1}$, we have

$$\Big\| \sum_{j=0}^{n-1} a_j v_j \Big\|^p_{L_p(\nu)} \;=\; \int_{\mathcal{N}} \Big(\sum_{j=0}^{n-1} a_j v_j\Big)^s \Big(\sum_{j=0}^{n-1} \overline{a_j v_j}\Big)^s d\nu$$
$$= \sum_{\alpha\in\mathbb{Z}_+^{n-1}} c_{\alpha,\alpha} \int_{\mathcal{N}} V^\alpha \overline{V}^\alpha d\nu + \sum_{\substack{(\alpha,\beta)\in S\\ \alpha\neq\beta}} c_{\alpha,\beta} \int_{\mathcal{N}} V^\alpha \overline{V}^\beta d\nu$$
$$= c_{0,0} + \sum_{(\alpha,\beta)\in S} d_{\alpha,\beta} m_{\alpha,\beta} \;=\; \Big\| \sum_{j=0}^{n-1} a_j f_j \Big\|^p_p\ .$$

To complete the proof of Theorem B in the case $\mathbb{K} = \mathbb{C}$, just note that $L_p(\nu)$ embeds isometrically into ℓ_p^N via $g \mapsto \big(g(d)\nu_d^{1/p}\big)_{d=1}^N$. (This map is not necessarily onto because some of the ν_d's might be zero.)

The proof in the real case is similar but simpler. We define $N = N_{\mathbb{R}}$ to be the number of elements of the set

$$S(n,p,\mathbb{R}) \;=\; \{\alpha \in \mathbb{Z}_+^{n-1} : 1 \le \sum_{j=1}^{n-1} \alpha_j \le p\}\ . \qquad \blacksquare$$

Let now $N(n,p,\mathbb{K})$ denote the smallest of all integers N such that every n-dimensional subspace of L_p is isometrically isomorphic to a subspace of ℓ_p^N. It follows from our construction and a well-known combinatorial argument giving the number of different distributions of p identical balls in $n-1$ boxes that

$$\text{(2.6)} \qquad \begin{aligned} N(n,p,\mathbb{R}) \;&\le\; N_{\mathbb{R}} \;=\; \binom{n+p-1}{p} - 1\ ,\\ N(n,p,\mathbb{C}) \;&\le\; N_{\mathbb{C}} \;=\; \binom{n+s-1}{s}^2 - 1\ . \end{aligned}$$

Estimates for $N(n,p,\mathbb{K})$ from below can possibly be obtained using results ¿from [BLM] on almost isometric embeddings of finite-dimensional subspaces of L_p into ℓ_p^r. An alternative approach is as follows.

Let $H(n,p,\mathbb{K})$ denote the smallest of all integers H such that n-dimensional Hilbert space embeds isometrically into ℓ_p^H (all spaces are over $\mathbb{K}$). The numbers $H(n,p,\mathbb{K})$ have been objects of intensive study starting from [Mi] (cf. [Mi], in particular p. 284, Problem). For the state of affairs we refer to [Kö] and references given there, and to the very recent note [LS] where the case $n = 2$ is discussed.

Obviously,

$$H(n,p,\mathbb{K}) \le N(n,p,\mathbb{K}) \ .$$

The upper estimates for $H(n,p,\mathbb{K})$ given in [Kö], Proposition 2, are essentially the same as ours for $N(n,p,\mathbb{K})$ given in (2.6). This suggests the

CONJECTURE: $H(n,p,\mathbb{K}) = N(n,p,\mathbb{K})$.

We arrived at this conjecture discussing the problem with J. Lindenstrauss.

The next two propositions show that $H(2,4,\mathbb{K}) = N(2,4,\mathbb{K})$. Their proofs have been obtained in collaboration with S. Kwapień and S. Szarek.

Proposition 2.1. *Every 2-dimensional subspace of real L_4 is isometrically isomorphic to a subspace of real ℓ_4^3, hence $N(2,4,\mathbb{R}) = 3$.*

Recall that a sequence $(x_j)_{j=1}^n$ in an n-dimensional Banach space X is called an **Auerbach basis** provided there are linear functionals $x_1^*,\dots,x_n^*$ in X^* such that

$$\|x_j^*\| = \|x_j\| = x_j^*(x_j) = 1 \quad \text{and} \quad x_j^*(x_k) = 0 \quad \text{for} \quad j \ne k \quad (j,k = 1,\dots,n) \ .$$

It is well-known that every finite-dimensional Banach space admits an Auerbach basis (see e.g. [DJT], Lemma 6.26).

Lemma 2.2. *Let E be a 2-dimensional subspace of real $L_4(\mu)$, μ any measure. Then there exist $m_E \in [0,\frac{1}{3}]$ and an Auerbach basis (f_1,f_2) of E such that*

$$\int f_1^2 f_2^2 \, d\mu = m_E \ . \tag{2.7}$$

Moreover, if (g_1,g_2) is any Auerbach basis of E such that $m = \int g_1^2 g_2^2 \, d\mu \in [0,\frac{1}{3}]$, then $m = m_E$.

Any Auerbach basis (f_1,f_2) satisfying (2.7) has the following two properties:

$$(f_1,f_2) \ \text{ is a symmetric basis for } \ E \ . \tag{2.8}$$

(2.9) *The ellipse of maximal area inscribed into $B_E = \{g \in E : \|g\|_{L_4(\mu)} \le 1\}$ is given by*

$$\mathcal{E}_E = \{g \in E : \, g = \alpha_1 f_1 + \alpha_2 f_2 \text{ and } \alpha_1^2 + \alpha_2^2 \le 1\} \ .$$

In addition, if $0 \le m_E < \frac{1}{3}$, then the Auerbach basis satisfying (2.7) is unique. On the other hand, if $m_E = \frac{1}{3}$, then E is isometrically isomorphic to ℓ_2^2.

PROOF. Using the Hahn-Banach Extension Theorem, the duality between $L_4(\mu)$ and $L_{4/3}(\mu)$ and the fact that for any $0 \ne f \in L_4(\mu)$ there is a unique $g \in L_{4/3}(\mu)$ with $\|g\|_{4/3} = 1$ and $\int gf \, d\mu = \|f\|_4$, we infer that if (f_1,f_2) is an Auerbach basis for E, then the real-valued functions f_1^3 and f_2^3 represent the biorthogonal functionals to (f_1,f_2). Hence

$$\int f_1^3 f_2 \, d\mu = \int f_1 f_2^3 \, d\mu = 0 \ . \tag{2.10}$$

Next observe that if (f_1, f_2) is an Auerbach basis for E then, setting $m := \int f_1^2 f_2^2 d\mu$ and putting $\tilde{f}_1 := \dfrac{f_1 + f_2}{\sqrt[4]{2+6m}}$ and $\tilde{f}_2 := \dfrac{f_1 - f_2}{\sqrt[4]{2+6m}}$, we get another Auerbach basis for E with

$$\tilde{m} := \int \tilde{f}_1^2 \tilde{f}_2^2 \, d\mu = \frac{1-m}{1+3m} \ .$$

Clearly, $\min\{m, \tilde{m}\} \in [0, \frac{1}{3}]$ because $\int f_1^2 f_2^2 d\mu \in [0,1]$ by the Schwarz inequality. We have proved the existence of an Auerbach basis for E which satisfies (2.7) for some $m_E \in [0, \frac{1}{3}]$.

Now pick $\alpha_1, \alpha_2 \in \mathbb{R}$ arbitrarily. It follows from (2.10) that

$$\|\alpha_1 f_1 + \alpha_2 f_2\|_{L_4(\mu)}^4 = \alpha_1^4 + \alpha_2^4 + 6\alpha_1^2\alpha_2^2 m_E \ , \tag{2.11}$$

and we get (2.8) since the right hand side is a symmetric function of the variables α_1^2 and α_2^2.

To prove (2.9), put

$$\psi(f,g) = \alpha_1\beta_1 + \alpha_2\beta_2 \qquad \text{for} \quad f = \alpha_1 f_1 + \alpha_2 f_2 \quad \text{and} \quad g = \beta_1 f_1 + \beta_2 f_2$$

and note that ψ is a non-degenerate and symmetric bilinear form on $E \times E$ satisfying $\psi(g,g) \geq 0$ for all $g \in E$. The set $\mathcal{E}_E$ defined in (2.9) coincides with $\{g \in E \colon \psi(g,g) \leq 1\}$ and so it is an ellipse. Moreover, since $0 \leq m_E \leq \frac{1}{3}$, we can use (2.11) to obtain

$$\begin{aligned} > \|\alpha_1 f_1 + \alpha_2 f_2\|_{L_4(\mu)}^4 &= \psi(\alpha_1 f_1 + \alpha_2 f_2, \alpha_1 f_1 + \alpha_2 f_2)^2 - 6 \cdot \big(\frac{1}{3} - m_E\big)\alpha_1^2\alpha_2^2 \\ &\leq \psi(\alpha_1 f_1 + \alpha_2 f_2, \alpha_1 f_1 + \alpha_2 f_2)^2 \ . \end{aligned} \tag{2.12}$$

Note that for equality in (2.12) it is necessary and sufficient that either $m_E = \frac{1}{3}$, or that $m_E < \frac{1}{3}$ and $\alpha_1^2\alpha_2^2 = 0$. It also follows ¿from (2.12) that $\mathcal{E}_E \subset B_E$. Moreover, since (f_1, f_2) is an Auerbach basis, B_E is contained in the parallelogram

$$Q_E := \{g \in E : \ g = \alpha_1 f_1 + \alpha_2 f_2 \quad \text{with} \quad \max\{|\alpha_1|, |\alpha_2|\} \leq 1\} \ .$$

Then $\mathcal{E}_E$ is the ellipse of maximal area inscribed into Q_E since the affine transformation which takes f_1 and f_2 into the unit vectors $(1,0)$ and $(0,1)$ of $\mathbb{R}^2$, respectively, maps $\mathcal{E}_E$ onto the unit circle and Q_E onto the square $[-1,1] \times [-1,1]$. But $\mathcal{E}_E \subset B_E \subset Q_E$ so that $\mathcal{E}_E$ is also the ellipse of maximal area inscribed into B_E.

Let now (f_1', f_2') be another Auerbach basis for E satisfying (2.7), with f_j' replacing f_j $(j = 1,2)$ and some $m_E' \in [0, \frac{1}{3}]$ replacing m_E. Using the f_j''s, we define $\mathcal{E}_E'$ by (2.9) as before. Assume first that $0 \leq m_E < \frac{1}{3}$. Since the ellipse of maximal area inscribed into B_E is unique (see e.g. [T-J], Corollary 5.11), we conclude that $\mathcal{E}_E' = \mathcal{E}_E$. By what was stated right after (2.12), it is immediate that the boundaries $\partial\mathcal{E}_E$, $\partial\mathcal{E}_E'$ and ∂B_E satisfy

$$\partial\mathcal{E}_E \cap \partial B_E = \{\pm f_1, \pm f_2\} \quad \text{for} \quad m_E < 1/3 \quad \text{and} \quad \partial\mathcal{E}_E' \cap \partial B_E = \{\pm f_1', \pm f_2'\} \quad \text{for} \quad m_E' < 1/3 \ .$$

Hence the bases $\{f_1, f_2\}$ and $\{f_1', f_2'\}$ conincide up to a permutation, and this implies $m_E = m_E'$.

If $m_E = \frac{1}{3}$ or $m_E' = \frac{1}{3}$, then (2.12) is an identity for all $\alpha_1, \alpha_2 \in \mathbb{R}$. We get $B_E = \mathcal{E}_E = \mathcal{E}_E'$, and so E is isometrically isomorphic to ℓ_2^2. ■

Keeping the above notation, we may now state:

Corollary 2.3. *Suppose that E and E_1 are 2-dimensional subspaces of the real spaces $L_4(\mu)$ and $L_4(\mu_1)$, respectively. Then E and E_1 are isometrically isomorphic if and only if $m_E = m_{E_1}$.*

Proof. Apply the uniqueness of m_E and m_{E_1} and the formula (2.11). ■

Proof of Proposition 2.1. In view of Corollary 2.3, it suffices to establish

(2.13) *For every $m \in [0, \frac{1}{3}]$ there exists a 2-dimensional subspace E of real ℓ_4^3 such that $m_E = m$.*

For this we apply a continuity argument. Represent ℓ_4^3 as $\mathbb{R}^3$ with the corresponding canonical norm and let G_2^3 be the set of unit balls B_E of 2-dimensional subspaces $E \subset \ell_4^3$ considered as subsets of $\mathbb{R}^3$; so $B_E = E \cap B_{\ell_4^3}$. Let $\eta(\,\cdot\,,\,\cdot\,)$ be the Hausdorff distance between bounded subsets of $\mathbb{R}^3$ relative to the standard Euclidean metric of the latter. Consider the metric spaces (G_2^3, η) and $(\mathcal{E}_2^3, \eta)$. The second is defined similar as the first with B_E replaced by $\mathcal{E}_E$.

We claim that the function $B_E \mapsto m_E$ from (G_2^3, η) to $\mathbb{R}$ is continuous.

Indeed, if $\lim_n \eta(B_{E_n}, B_E) = 0$, then also $\lim_n \eta(\mathcal{E}_{E_n}, \mathcal{E}_E) = 0$ for the associated ellipses. Now, if $\limsup_n m_{E_n} < \frac{1}{3}$ then, by the uniqueness of the corresponding Auerbach bases satisfying (2.12), we also have $\lim_n \eta(\partial B_{E_n} \cap \partial \mathcal{E}_{E_n}, \partial B_E \cap \partial \mathcal{E}_E) = 0$, and $\lim_n m_{E_n} = m_E$ follows since, for each $F \in G_2^3$, m_F is expressed in terms of coordinates of the corresponding Auerbach basis of F. Suppose next that $\limsup_n m_{E_n} = \frac{1}{3}$. Passing to a subsequence if necessary we may assume that $\lim_n m_{E_n} = \frac{1}{3}$. From (2.12) and a simple compactness argument applied in the space $(\mathcal{E}_2^3, \eta)$ it follows that the sequence $(B_{E_n})_n$ tends to an ellipse. Thus $m_E = \frac{1}{3}$, and the claim is proved.

It is clear that if E_0 is the subspace spanned by $(1, 0, 0)$ and $(0, 1, 0)$, then $m_{E_0} = 0$. Moreover, an easy computation reveals that if $\varrho = 2^{-1/4}$, $\sigma = (18)^{-1/4}$ and E_1 is the subspace spanned by $(\varrho, \varrho, 0)$ and $(\sigma, -\sigma, -2\sigma)$, then $m_{E_1} = \frac{1}{3}$. Since the 'Grassmannian' (G_2^3, η) is connected, (2.13) is an easy consequence of the above claim. ■

The proof of the counterpart of Proposition 2.1 in the complex case is even simpler. Let (Ω, Σ, μ) be an arbitrary measure space.

Proposition 2.4. *Every 2-dimensional subspace of complex $L_4(\mu)$ embeds isometrically into complex ℓ_4^4.*

We begin with some notation and a few observations. Given $f, g \in L_4(\mu)$, we put

$$m(f,g) := \int_\Omega |f|^2 |g|^2 d\mu \qquad \text{and} \qquad n(f,g) := \Big| \int_\Omega f^2 \overline{g}^2 d\mu \Big| \ .$$

Clearly, if $\|f\|_{L_4(\mu)} = \|g\|_{L_4(\mu)} = 1$, then

$$0 \le n(f,g) \le m(f,g) \le 1 \ .$$

Next observe that if $\{f,g\}$ is an Auerbach basis for a subspace of $L_4(\mu)$ then, for any $\alpha\in\mathbb{C}$ with $|\alpha|=1$, $\{\alpha f,g\}$ is also an Auerbach basis, $m(f,g)=m(\alpha f,g)$ and $n(\alpha f,g)=n(f,g)$. Choosing α so that $\alpha^2\int_\Omega f^2\overline{g}^2 d\mu = n(f,g)$, we get for arbitrary $a,b\in\mathbb{C}$,

$$\|a(\alpha f)+bg\|^4_{L_4(\mu)} \;=\; |a|^4+|b|^4+4|a|^2|b|^2 m(f,g)+2\mathrm{Re}\,(a\overline{b})^2 n(f,g)\;. \tag{2.14}$$

The arguments to prove this are similar to those which led to (2.11) because the complex - valued functions $|f|^2\,\overline{f}$ and $|g|^2\,\overline{g}$ are just the ones which represent the biorthogonal functionals to (f,g).

¿From (2.14) we obtain:

Corollary 2.5. *Let μ_1 and μ_2 be arbitrary measures, and let $\{f_j,g_j\}$ be an Auerbach basis for a subspace E_j of complex $L_4(\mu_j)$, $j=1,2$. If $m(f_1,g_1)=m(f_2,g_2)$ and $n(f_1,g_1)=n(f_2,g_2)$, then E_1 and E_2 are isometrically isomorphic.*

We need one more lemma to prove Proposition 2.4.

Lemma 2.6. *For every choice of $0\le n\le m\le 1$ there exists an Auerbach basis $\{f,g\}$ for some 2 - dimensional subspace of complex ℓ^4_4 such that $m(f,g)=m$ and $n(f,g)=n$.*

Proof. For $0\le t,\varphi,\psi\le\pi/2$ we put

$$\begin{aligned} f_{t,\varphi} &= 2^{-1/4}\big(\sqrt{\cos\varphi}\,,\,-\sqrt{\cos\varphi}\,,\,e^{it}\sqrt{\sin\varphi}\,,\,-e^{it}\sqrt{\sin\varphi}\big)\\ g_{t,\psi} &= 2^{-1/4}\big(\sqrt{\cos\psi}\,,\,\sqrt{\cos\psi}\,,\,\sqrt{\sin\psi}\,,\,\sqrt{\sin\psi}\big)\end{aligned}$$

It is readily seen that $\{f_{t,\varphi},g_{t,\psi}\}$ is an Auerbach basis in its span in ℓ^4_4, and a simple computation reveals that

$$\begin{aligned} m(f_{t,\varphi},g_{t,\psi}) &= \cos(\varphi-\psi)\\ n(f_{t,\varphi},g_{t,\psi}) &= |\cos\varphi\cdot\cos\psi+e^{2it}\cdot\sin\varphi\cdot\sin\psi|\;.\end{aligned}$$

Put

$$\alpha := \arccos m \quad,\quad \varphi := \frac{\pi}{4}+\frac{\alpha}{2} \quad\text{and}\quad \psi := \frac{\pi}{4}-\frac{\alpha}{2}\;.$$

Then $0\le\alpha\le\pi/2$, $0\le(\pi/4)\pm(\alpha/2)\le\pi/2$, and $m(f_{t,\varphi},g_{t,\psi})=\cos\alpha=m$. Furthermore,

$$\begin{aligned} n(f_{t,\varphi},g_{t,\psi}) &= \Big|\cos\big(\frac{\pi}{4}+\frac{\alpha}{2}\big)\cdot\cos\big(\frac{\pi}{4}-\frac{\alpha}{2}\big)+e^{2it}\cdot\sin\big(\frac{\pi}{4}+\frac{\alpha}{2}\big)\cdot\sin\big(\frac{\pi}{4}-\frac{\alpha}{2}\big)\Big|\\ &= \Big|\frac{1+e^{2it}}{2}\cdot\cos\alpha\Big| \;=\; \cos^2 t\cdot\cos\alpha \;=\; m\cdot\cos^2 t\;.\end{aligned}$$

To get to the desired conclusion just take $t:=\arccos\sqrt{\frac{n}{m}}$. ■

The proof of Proposition 2.4 is now easy to complete.

Corollary 2.7. $N(2,4,\mathbb{C}) \;=\; 4$.

Proof. The upper estimate $N(2,4,\mathbb{C})\le 4$ follows from Proposition 2.4, while the lower estimate follows from the equality $H(2,4,\mathbb{C})=4$ (see [Kö], last paragraph on p. 54). ■

Remark. The counterpart of formula (2.8) (Lemma 2.2) is false in the complex case. From (2.14) we get:

An Auerbach basis $\{f,g\}$ for a subspace of complex $L_4(\mu)$ is a (complex) unconditional basis of norm 1 iff $n(f,g)=0$; in that case it is a (complex) symmetric basis of norm one.

3. The nonexistence of finite - dimensional Banach spaces isometrically universal for certain families of finite - dimensional Banach spaces

We begin with Klee's criterion mentioned in the introduction.

For integers k, n with $1 \le k < n$ we put

$$c^{\mathbb{R}}_{n,k} := k(n-k) \qquad \text{and} \qquad c^{\mathbb{C}}_{n,k} := 2k(n-k) \ .$$

The **topological dimension** of a metrizable space M is denoted by $\operatorname{tdim} M$ (cf. [HW], p. 24).

Klee's Criterion 3.1. *Let $\mathcal{F}$ be a family of k - dimensional Banach spaces. If there exists an n - dimensional Banach space which is isometrically universal for $\mathcal{F}$ then $\mathcal{F}$, with the topology induced from BM_k, satisfies*

$$\operatorname{tdim} \mathcal{F} \le c^{\mathbb{K}}_{n,k} \ .$$

In the Appendix we briefly describe Klee's approach to this result.

The next theorem, stated in terms of Local Theory of Banach Spaces, is the main result of this section. Definitions and basic properties related to the notion of type and cotype of a Banach space can be found e.g. in [DJT], [LT] and [T-J].

Theorem 3.2. *Let X be an infinite - dimensional Banach space. Assume that $p(X) \notin 2\mathbb{N}$ or $q(X) \notin 2\mathbb{N}$, where*

$$p(X) = \sup \{1 \le p \le 2 : \ X \text{ is of type } p\}$$

$$q(X) = \inf \{2 \le q \le \infty : \ X \text{ is of cotype } q\} \ .$$

Then there is no finite - dimensional Banach space which is isometrically universal for $BM_2(X)$.

The first step in the proof of Theorem 3.2 consists in a reduction to the case of the family

$$\mathcal{T}_p := \bigcup_{m \ge 3} BM_2(L^m_p) \qquad (p \notin 2\mathbb{N}) \ .$$

Klee's Criterion 3.1 will be applied to $\mathcal{T}_p$ when $p \notin 2\mathbb{N}$, and this condition on p is in fact crucial. We need the following lemma which will lead us to

(3.1) $$\operatorname{tdim} BM_2(L^m_p) \ge m-2 \qquad \forall\, m \in \mathbb{N}, \ m \ge 3 \ .$$

For $m \in \mathbb{N}$, $m \ge 3$, we define

$$Q_m := \Big\{ a = (a_j) \in \mathbb{R}^m : \ a_1 > 0 > a_2 > \ldots > a_m ; \ \sum_j a_j = 0 ; \ \|a\|_{L^m_p} = 1 \Big\} \ ,$$

and we put, for $a \in Q_m$,

$$E_a := \operatorname{span}\{1, a\} \ .$$

We consider E_a as a subspace of L^m_p.

Lemma 3.3. *Let $0 < p < \infty$, $p \notin 2\mathbb{N}$, and $a, b \in Q_m$. If there exists an isometric isomorphism $u : E_a \to E_b$, then $a = b$ and $u = s_0 \cdot id_{E_a}$ for some $s_0 \in \mathbb{K}$ with $|s_0| = 1$.*

PROOF. Let $s_0, t_0, s_1, t_1 \in \mathbb{K}$ be such that $u(1) = s_0 1 + t_0 b$ and $u(a) = s_1 1 + t_1 b$. Since $p \notin 2\mathbb{N}$, $u(1)$ has full support ([Har], Lemma 3.4), so that $s_0 + t_0 b_j \neq 0$ for $j = 1, \dots, m$. We define a probability measure μ on the field of all subsets of $\{1, \dots, m\}$ by

$$\mu(A) := m^{-1} \cdot \sum_{j \in A} |s_0 + t_0 b_j|^p \qquad (A \subset \{1, \dots, m\}) ,$$

and an isometric isomorphism $w : L_p^m \to L_p(\mu)$ by

$$w(f) := \Big(\big[|s_0 + t_0 b_j| \cdot \operatorname{sign}(s_0 + t_0 b_j) \big]^{-1} \cdot f_j \Big)_{j=1}^m \qquad \text{for} \quad f = (f_j)_1^m \in L_p^m .$$

Accordingly, $v = wu : E_a \to L_p(\mu)$ is an isometric embedding. Since obviously $v(1) = 1$, the Equimeasurability Lemma 0.1 yields $\mu_a = \mu_x$ where

$$x = v(a) = \Big(\big[|s_0 + t_0 b_j| \cdot \operatorname{sign}(s_0 + t_0 b_j) \big]^{-1} (s_1 + t_1 b_j) \Big)_{j=1}^m .$$

But $\mu_x = m^{-1} \sum_{j=1}^m |s_0 + t_0 b_j|^p \delta_{x_j}$ and $\mu_a := m^{-1} \sum_{j=1}^m \delta_{a_j}$, hence $1 = |s_0 + t_0 b_j|^p$ for all $j = 1, \dots, m$. Since the b_j's are real,

$$1 = |s_0 + t_0 b_j|^2 = |s_0|^2 + b_j^2 |t_0|^2 + 2 b_j \big[(\operatorname{Re} s_0)(\operatorname{Re} t_0) + (\operatorname{Im} s_0)(\operatorname{Im} t_0) \big]$$

for all $j = 1, \dots, m$. Hence, if $1 \le j < k \le m$, then

$$0 = (b_j^2 - b_k^2)|t_0|^2 + 2(b_j - b_k) \big[(\operatorname{Re} s_0)(\operatorname{Re} t_0) + (\operatorname{Im} s_0)(\operatorname{Im} t_0) \big]$$

and so $b_j \neq b_k$ implies

$$0 = |t_0|^2 (b_j + b_k) + 2 \big[(\operatorname{Re} s_0)(\operatorname{Re} t_0) + (\operatorname{Im} s_0)(\operatorname{Im} t_0) \big] .$$

Since $\sum_{j=1}^m b_j = 0$, summation over all pairs (j, k) with $1 \le k < j \le n$ yields

$$m(m-1) \big[(\operatorname{Re} s_0)(\operatorname{Re} t_0) + (\operatorname{Im} s_0)(\operatorname{Im} t_0) \big] = 0 .$$

This in turn leads to $0 = |t_0|^2 (b_j + b_k)$ for $1 \le j < k \le m$. Taking $j = 2$ and $k = 3$, we get $t_0 = 0$ from $b_2 < 0$, $b_3 < 0$, and so $u(1) = s_0 1$. Since u is an isometry, $|s_0| = 1$ follows, and so $x_j = (s_1 + t_1 b_j)/s_0$ for each $j = 1, \dots, m$.

On the other hand, since $\mu_a = \mu_x$, there is a permutation π of $\{1, \dots, m\}$ such that

$$a_j = x_{\pi(j)} = \frac{s_1 + t_1 b_{\pi(j)}}{s_0} \quad \text{for} \quad j = 1, \dots, m .$$

Using our hypothesis on a and b, we get by summation

$$0 = \sum_{j=1}^m a_j = \frac{s_1}{s_0} + t_1 \cdot \sum_{j=1}^m b_{\pi(j)} = \frac{s_1}{s_0} ,$$

whence $s_1 = 0$, and so $a_j = (t_1/s_0) b_\pi(j)$ for all $j = 1, \ldots, m$. But $\|a\|_{L_p^m} = \|b\|_{L_p^m}$, so that $|t_1/s_0| = 1$. Since $m \geq 3$, some $j_0 \in \{1, \ldots, m\}$ must be such that $j_0 > 1$ and $\pi(j_0) > 1$, which forces $a_{j_0} < 0$ and $b_{\pi(j_0)} < 0$. ¿From $a_{j_0} = (t_1/s_0) b_{\pi(j_0)}$ we infer that t_1/s_0 is positive, thus $t_1 = s_0$. It follows that $u(a) = s_1 + t_1 b = s_0 b$, and this allows us to conclude that $u = s_0 \cdot id_{E_a}$.■

Corollary 3.4. *If $1 \leq p < \infty$ is not an even integer, then*

$$\operatorname{tdim} BM_2(L_p^m) \;\geq\; m - 2 \qquad \forall\, m \in \mathbb{N},\; m \geq 3\ .$$

PROOF. For each $a \in Q_m$, let $\phi_m(a) \in BM_2(L_p^m)$ be the class generated by all subspaces of L_p^m which are isometrically isomorphic to E_a. We claim that the resulting map $\phi_m : Q_m \to BM_2(L_p^m)$ is continuous.

To see why this is so, fix $a, b \in Q_m$ and consider the isomorphism $u_{a,b} : E_a \to E_b$ defined by

$$u_{a,b}(s1 + ta) \;:=\; s1 + tb\ .$$

Note that $u_{a,b}^{-1} = u_{b,a}$. Taking the inner product with respect to normalized counting measure (denoted by $(\,\cdot\,|\,\cdot\,)$), using the orthogonality of the functions 1 and a and applying Hölder's inequality (with $p^* = \frac{p}{p-1}$), we get

$$|t| \cdot \|a\|_{L_2^m}^2 \;=\; \bigl|\bigl(s1 + ta\big|a\bigr)\bigr| \;\leq\; \|s1 + ta\|_{L_p^m} \cdot \|a\|_{L_{p^*}^m}\ .$$

Let $D = D\bigl(m\bigr)$ and $d = d\bigl(m\bigr)$ be the norms of the formal identities $L_2^m \to L_{p^*}^m$ and $L_2^m \to L_p^m$, respectively. In particular, $\|a\|_{L_{p^*}^m} \leq D\|a\|_{L_2^m}$ and $1 = \|a\|_{L_p^m} \leq d\|a\|_{L_2^m}$. Combining these three inequalities we easily obtain

$$|t| \;\leq\; C \cdot \|s1 + ta\|_{L_p^m}$$

with $C = D \cdot d$. Thus

$$\|s1 + ta - u_{a,b}(s1 + ta)\|_{L_p^m} \;=\; |t| \cdot \|a - b\|_{L_p^m} \;\leq\; C \cdot \|a - b\|_{L_p^m} \cdot \|s1 + ta\|_{L_p^m}$$

and so $\|u_{a,b}(s1 + ta)\|_{L_p^m} \leq \bigl(1 + C \cdot \|a - b\|_{L_p^m}\bigr) \cdot \|s1 + ta\|_{L_p^m}$. Consequently,

$$\|u_{a,b} : E_a \to E_b\| \;\leq\; 1 + C \cdot \|a - b\|_{L_p^m}\ .$$

Similarly, $\|u_{b,a} : E_b \to E_a\| \leq 1 + C \cdot \|a - b\|_{L_p^m}$, hence

$$d(E_a, E_b) \;\leq\; \bigl(1 + C \cdot \|a - b\|_{L_p^m}\bigr)^2\ .$$

This proves continuity of ϕ_m.

By Lemma 3.3, ϕ_m is one-to-one and so it induces a homeomorphism on every compact subset of Q_m.

Now observe that Q_m is a real Euclidean manifold depending on m parameters each taking values in an open interval, the parameters being constrained by the two conditions $\sum_{j=1}^m |a_j|^p = m$ and $\sum_{j=1}^m a_j = 0$. It follows that every point in Q_m has a compact neighbourhood U such that $\operatorname{tdim} U = m - 2$. Since ϕ_m induces a homeomorphism on U, we conclude that

$$m - 2 \;=\; \operatorname{tdim} U \;=\; \operatorname{tdim} \phi_m(U) \;\leq\; \operatorname{tdim} BM_2(L_p^m)\ . \qquad ■$$

REMARK. *By the results in* [Be] *and* [Kl], *the assertion of Corollary 3.4 is true for* $p = \infty$ *as well.*

PROOF OF THEOREM 3.2. By deep results of B. Maurey and G. Pisier [MP] and of J.L. Krivine [Kr], every infinite-dimensional Banach space X contains, for every $m \in \mathbb{N}$ and $\varepsilon > 0$, m-dimensional subspaces E_ε^m and F_ε^m such that $d(E_\varepsilon^m, L_{p(X)}^m) < 1 + \varepsilon$ and $d(F_\varepsilon^m, L_{q(X)}^m) < 1 + \varepsilon$.

Assume now that there exists a finite-dimensional Banach space, say G, which is isometrically universal for all 2-dimensional subspaces of X. Then, for every fixed integer $m \geq 3$ and every $\varepsilon > 0$,

$$BM_2(E_\varepsilon^m) \subset BM_2(X) \subset BM_2(G) .$$

Since dim $G < \infty$, $BM_2(G)$ is compact, and so we get

$$BM_2(L_{p(X)}^m) \subset BM_2(G)$$

by passing with ε to zero. Similarly,

$$BM_2(L_{q(X)}^m) \subset BM_2(G) .$$

An appeal to Klee's Criterion 3.1 yields

$$\max\{\operatorname{tdim} BM_2(L_{p(X)}^m), \operatorname{tdim} BM_2(L_{q(X)}^m)\} \leq \operatorname{tdim} BM_2(G) \leq c_{N,2}^{\mathbb{K}} ,$$

where $N = \dim G$. If either $p(X)$ or $q(X)$ is not an even integer, then we get a contradiction with Corollary 3.4. ■

Corollary 3.5. *Under the assumptions of Theorem 3.2, if* $k = 3, 4, \ldots,$ *then there is no finite-dimensional Banach space which is isometrically universal for all* k*-dimensional subspaces of* X.

The proof of Theorem C is now straightforward.

PROOF OF THEOREM C. Since the indices $p(X)$ and $q(X)$ are isomorphic invariants, we infer that if a Banach space X is isomorphic either to L_p or to ℓ_p, then $p(X) = p(L_p) = p(\ell_p)$ and $q(X) = q(L_p) = q(\ell_p)$. Now it follows from classical results of Orlicz (see e.g. [LT], p. 73) that $p(X) = p$ if $1 \leq p \leq 2$, and $q(X) = p$ if $2 \leq p \leq \infty$. So the conclusion of Theorem C follows from Theorem 3.2 because $p \notin 2\mathbb{N}$. ■

REMARK. The concept of isometrically universal spaces generalizes in a natural fashion to quasi-Banach spaces. In such a setting, Theorem C can be extended to $0 < p < 1$. For this we use the fact that if $0 < p < 1$, then L_p contains a subspace isometric to ℓ_1, the latter taken in the metric induced by $(\|\cdot\|_{\ell_1})^p$ (compare e.g. with [DF], 24.5, Corollary 1).

The next proposition shows that the existence of a finite-dimensional Banach space which is isometrically universal for all k-dimensional subspaces of a given infinite-dimensional Banach space is rather exceptional from the viewpoint of isomorphic theory of Banach spaces.

Proposition 3.6. *Every infinite-dimensional Banach space is isomorphic to a Banach space Y such that there is no finite-dimensional Banach space which is isometrically universal for $BM_2(Y)$.*

PROOF. Let Z be a subspace of X which admits a finite-dimensional decomposition $(Z_k)_{k\in\mathbb{N}}$ such that $\dim Z_k = k$ and $d(Z_k, L_2^k) < 1 + 2^{-(k+1)}$ for each $k \in \mathbb{N}$. The existence of such a space Z goes back to the work of V.I. Gurariĭ [Gu]; it follows from Dvoretzky's Theorem on spherical sections (see e.g. [DJT], 19.1). Pick scalars p_k so that $2 < p_k < 2 + 2^{-k}$ and $d(L_2^k, L_{p_k}^k) < 1 + 2^{-k-1}$, $k \in \mathbb{N}$. Then, for each k, there is an isomorphism $u_k : Z_k \to L_{p_k}^k$ such that

$$\|z\| \le \|u_k(z)\|_{L_{p_k}} \le (1 + 2^{-k+1})\|z\| \qquad (z \in Z_k) .$$

Define a new norm on Z by

$$|\!|\!| z |\!|\!| := \max\{\|z\|, \sup_k \|u_k(z)\|\} \qquad \text{for} \quad z = \sum_{k=1}^{\infty} z_k \in Z .$$

Since $(Z_k)_{k\in\mathbb{N}}$ is a finite-dimensional decomposition of Z, there is a constant $C > 0$ such that

$$\|z\| \le |\!|\!| z |\!|\!| \le C \cdot \|z\| \quad \text{for all} \quad z \in Z . \tag{3.2}$$

By [Pe], Proposition 1, the norm $|\!|\!| \cdot |\!|\!|$ extends to the norm on X, denoted by the same symbol, for which (3.2) is preserved for all $z \in X$. Put $Y = (X, |\!|\!| \cdot |\!|\!|)$. Then Y is isomorphic to X, and the subspace $(Z_k, |\!|\!| \cdot |\!|\!|)$ is isometrically isomorphic to $L_{p_k}^k$. Thus $\mathrm{tdim}\, BM_2((Z_k, |\!|\!| \cdot |\!|\!|)) \ge k - 2$ by Corollary 3.4, and an inessential modification of the proof of Theorem 3.2 yields the desired conclusion. ■

Being more careful one can show that, given $\varepsilon > 0$, the constant C in (3.2) can be replaced by $1 + \varepsilon$.

Appendix: A Guide to a Proof of Klee's Criterion 3.1

We begin by introducing some notation. Let us denote by

- $\mathcal{C}^n$ the family of all compact absolutely convex bodies in $\mathbb{K}^n$,
- G_k^n the Grassmannian of k-dimensional linear subspaces of $\mathbb{K}^n$ $(1 \le k \le n)$,
- $\mathcal{C}_k^n$ the collection of all $C \subset \mathbb{K}^n$ such that $C = B \cap H$ for some $B \in \mathcal{C}^n$ and some $H \in G_k^n$,
- η the Hausdorff distance between compact subsets of $\mathbb{K}^n$ induced by the standard Euclidean metric in $\mathbb{K}^n$,
- γ the metric on G_k^n defined by $\gamma(H_1, H_2) := \eta(H_1 \cap U, H_p \cap U)$ where $U = B_{\ell_2^n}$ is the standard Euclidean unit ball in $\mathbb{K}^n$,
- GL_n the group of nonsingular linear transformations of $\mathbb{K}^n$,
- $\tilde{\mathcal{C}}^n$ the family of equivalence classes of affinely equivalent members of $\mathcal{C}^n$.

Here $B, C \in \mathcal{C}^n$ are said to be **affinely equivalent** if $C = \sigma(B)$ for some $\sigma \in GL_n$.

We consider two metrics on $\tilde{\mathcal{C}}_n$. The first one is again the Banach - Mazur metric $\delta(\,\cdot\,,\,\cdot\,)$; recall that it is derived in the canonical fashion from the pseudometric on $\mathcal{C}^n$ given by

$$\delta(B_1, B_2) := \log d(B_1, B_2) \quad \text{where} \quad d(B_1, B_2) := \inf t$$

for $B_1, B_2 \in \mathcal{C}^n$; the infimum is taken over all $t \geq 1$ and $\sigma \in GL_n$ such that $B_1 \subset \sigma(B_2) \subset t \cdot B_1$.

The second is the **Macbeath metric** $\beta(\,\cdot\,,\,\cdot\,)$ which is derived analogously from the pseudometric on $\mathcal{C}^n$ given by

$$\beta(B_1, B_2) := \log b(B_1, B_2) \quad \text{where} \quad b(B_1, B_2) := \inf \frac{\mathrm{vol}\,(\sigma_1(B_1))}{\mathrm{vol}\,(B_2)} \cdot \frac{\mathrm{vol}\,(\sigma_2(B_2))}{\mathrm{vol}\,(B_1)} \;,$$

for $B_1, B_2 \in \mathcal{C}^n$; here $\mathrm{vol}\,(\,\cdot\,)$ denotes the n - dimensional volume function, and the infimum extends over all $\sigma_1, \sigma_2 \in GL_n$ such that $B_2 \subset \sigma_1(B_1)$ and $B_1 \subset \sigma_2(B_2)$.

We rely on a number of facts.

(I) The metric space $(\tilde{\mathcal{C}}^n, \delta)$ is isometric to BM_n under the metric used before; the isometry assigns to each equivalence class of $B \in \mathcal{C}^n$ the equivalence class of Banach spaces isometrically isomorphic to $(\mathbb{K}^n, B) := (\mathbb{K}^n, \|\cdot\|_B)$; here $\|\cdot\|_B$ is the gauge functional of B.

(II) The metric space $(\tilde{\mathcal{C}}^n, \beta)$ is compact; cf. [Mc].

(III) For $B_1, B_2 \in \mathcal{C}^n$, we have $\beta(B_1, B_2) \leq 2n\delta(B_1, B_2)$. Therefore the spaces BM_n, alias $(\tilde{\mathcal{C}}^n, \delta)$, and $(\tilde{\mathcal{C}}^n, \beta)$ are canonically homeomorphic.

Each $B \in \mathcal{C}^n$ gives rise to the n - dimensional Banach space $E = (\mathbb{K}^n, B)$ which we use to define the map $f_E : G^n_k \to \mathcal{C}^n_k$ by

$$f_E(H) := H \cap B \qquad \text{for} \quad H \in G^n_k \;.$$

Let $q : \mathcal{C}^n_k \to \tilde{\mathcal{C}}^k$ be the map which associates with each $B \in \mathcal{C}^n_k$ its affine equivalence class. The crucial observation is that the composition

(IV) $\tilde{f}_E := q \circ f_E : (G^n_k, \gamma) \longrightarrow (\tilde{\mathcal{C}}^k, \beta)$ is Lipschitzian.

This is a consequence of two results:

(IVa) $f_E : (G^n_k, \gamma) \longrightarrow (\mathcal{C}^n_k, \eta)$ is Lipschitzian

and

(IVb) $q : (\mathcal{C}^n_k, \eta) \longrightarrow (\tilde{\mathcal{C}}^k, \beta)$ is locally Lipschitzian.

As for (IVa), use [Kl], Lemma 2.2, combined with [Kl], Proposition 2.1, whereas (IVb) can be obtained by using [Kl], Theorem 2.7 in a special case.

Write now $\mathrm{hdim}\,(M, \varrho)$ for the Hausdorff dimension (see [HW], p. 107) of a metric space (M, ϱ). Directly from the definition of that dimension we get:

(V) If (M_1, ϱ_1) is another metric space and $f : (M, \varrho) \to (M_1, \varrho_1)$ is Lipschitzian, then

$$\mathrm{hdim}\,(M_1, \varrho_1) \leq \mathrm{hdim}\,(M, \varrho) \;.$$

On the other hand we have, for every separable metric space (M, ϱ),

(VI) $\operatorname{tdim}(M, \varrho) \leq \operatorname{hdim}(M, \varrho)$

(see [HW], Ch.VII, and [Sz]). Finally (cf. [Kl], Corollary 3.2 for the real case),

(VII) $\operatorname{hdim}(G_k^n, \gamma) = c_{n,k}^{\mathbb{K}}$.

Klee's result can now be proved by assembling these informations in an appropriate way.

PROOF OF KLEE'S CRITERION 3.1. Let $\mathcal{F} \subset BM_k$ be a family of Banach spaces and suppose that E is an n-dimensional Banach space which is isometrically universal for $\mathcal{F}$. Then $\mathcal{F} \subset \tilde{f}_E(G_k^n)$, and so

$$\begin{aligned}
\operatorname{tdim} \mathcal{F} &\leq \operatorname{tdim}\,(\tilde{f}_E(G_k^n), \delta) = \operatorname{tdim}\,(\tilde{f}_E(G_k^n), \beta) && \text{by (III)} \\
&\leq \operatorname{hdim}\,(\tilde{f}_E(G_k^n), \beta) && \text{by (VI)} \\
&\leq \operatorname{hdim}\,(G_k^n, \gamma) && \text{by (IV) and (V)} \\
&= c_{n,k}^{\mathbb{K}} \ . && \text{by (VII)}
\end{aligned}$$

■

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