THE BANACH SPACE OF WORKABLE CONTINGENT CLAIMS IN ARBITRAGE THEORY.

L'ESPACE DE BANACH DES ACTIFS CONTINGENTS RÉALISABLES EN THÉORIE D'ARBITRAGE

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Abstract. For a locally bounded local martingale S, we investigate the vector space generated by the convex cone of maximal admissible contingent claims. By a maximal contingent claim we mean a random variable $(H \cdot S)_\infty$, obtained as a final result of applying the admissible trading strategy H to a price process S and which is optimal in the sense that it cannot be dominated by another admissible trading strategy. We show that there is a natural, measure–independent, norm on this space and we give applications in Mathematical Finance.

Résumé. Si S est une martingale locale, localement bornée, on étudie l'espace vectoriel engendré par le cône des actifs contingents maximaux. Une variable aléatoire est un actif contingent maximal si elle peut s'écrire sous la forme $(H \cdot S)_\infty$, où la stratégie H est admissible et optimale dans le sense qu'elle n'est pas dominée par une autre stratégie admissible. Sur cet espace, on introduit une norme naturelle, invariante par changement de mesure, et on donne des applications en finance mathématique.

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1.Introduction.

A basic problem in Mathematical Finance is to see under what conditions the price of an asset, e.g. an option, is given by the expectation with respect to a so-called risk neutral measure. The existence of such a measure follows from no–arbitrage properties on the price process S of given assets, see [9], [10], [12] for the first papers on the topic and see [2] for a general form of this theory and for references to earlier papers.

Investment strategies H are described by S–integrable predictable processes and the outcome of the strategy is described by the value at infinity $(H \cdot S)_\infty$. In order to avoid doubling strategies one has to introduce lower bounds on the losses incurred by the economic agent. Mathematically this is translated by the property that $H \cdot S$ is bounded below by some constant. In this case we say that H is admissible, see [10]. It turns out that for some admissible strategies H the contingent claim $(H \cdot S)_\infty$ is not optimal in the sense that it is dominated by the outcome of another admissible strategy K. In this case there is no reason for the economic agent to follow the strategy H since at the end she can do better by following K. Let us say that H is maximal if the contingent claim $(H \cdot S)_\infty$ cannot be dominated by another outcome of an admissible strategy K in the sense that $(H \cdot S)_\infty \leq (K \cdot S)_\infty$ a.s. but $\mathbb{P}[(H \cdot S)_\infty < (K \cdot S)_\infty] > 0$.

In [2] and [4], we have used such maximal contingent claims in order to show that under the condition of No Free Lunch with Vanishing Risk, a locally bounded semi-martingale S admits an equivalent local martingale measure. In [4] we encountered a close relation between the existence of a martingale measure (not just a local martingale measure) for the process $H \cdot S$ and the maximality of the contingent claim $(H \cdot S)_\infty$. These results generalised results previously obtained by Ansel-Stricker, [1] and Jacka, [Jk]. We related this connection to a characterisation of good numéraires and to the hedging problem.

In this paper we show that the set of maximal contingent claims forms a convex cone in the space $L^0(\Omega, \mathcal{F}, \mathbb{P})$ of measurable functions and that the vector space generated by this cone can be characterised as the set of contingent claims of what we might call workable strategies. The vector space of these contingent claims, will be denoted by $\mathcal{G}$. It carries a natural norm for which it becomes a Banach space. These properties solve some arbitrage problems when constructing multi–currency models. We refer to a paper of the first named author with Shirakawa on this subject, [6].

The paper is organised as follows. The rest of this introduction is devoted to the basic notations and assumptions. Section 2 deals with the concept of acceptable contingent claims and it is shown that the set of maximal admissible contingent claims forms a convex cone. In section 3 we introduce the vector space spanned by the maximal admissible contingent claims and we show that there is a natural norm on it. The norm can also be interpreted as the maximal price that one is willing to pay for the absolute value of the contingent claim. Section 4 gives some results that are related to the geometry of the Banach space $\mathcal{G}$. In the complete market case it is an L^1–space, but we also give an example showing that it can be isomorphic to an L^∞–space. The precise interpretation of these properties in mathematical finance remains a challenging task. In section 5 we show that for a given maximal admissible contingent claim f, the set of equivalent local martingale measures $\mathbb{Q}$ such that $\mathbf{E}_{\mathbb{Q}}[f] = 0$ forms a dense subset in the set of all absolutely continuous local martingale measures. That not all equivalent local martingale measures $\mathbb{Q}$ satisfy the equality $\mathbf{E}_{\mathbb{Q}}[f] = 0$, is illustrated by a counter–example. The main theorem in section 6 states that in a certain way the space of workable contingent claims is invariant for numéraire changes. In section 7 we use finitely additive measures in order to describe the closure of the space of bounded workable contingent claims.

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The setup in this paper is the usual setup in mathematical finance. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{0\le t}$ is given. The time set is supposed to be $\mathbb{R}_+$, the other cases, e.g. finite time interval or discrete time set, can easily be imbedded in our more general approach. The filtration is assumed to satisfy the "usual conditions", i.e. it is right continuous and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$.

A price process S, describing the evolution of the discounted price of d assets, is defined on $\mathbb{R}_+ \times \Omega$ and takes values in $\mathbb{R}^{\mathrm{d}}$. We assume that the process S is locally bounded, e.g. continuous. As shown under a wide range of hypothesis, the assumption that S is a semi-martingale follows from arbitrage considerations, see [2] and references given there. We will therefore assume that the process S is a locally bounded semi-martingale. In order to avoid cumbersome notation and definitions, we will always suppose that measures are absolutely continuous with respect to $\mathbb{P}$. Stochastic integration is used to describe outcomes of investment strategies. When dealing with more dimensional processes it is understood that vector stochastic integration is used. We refer to Protter [13] and Jacod [11] for details on these matters.

Definition 1.1. *An $\mathbb{R}^{\mathrm{d}}$–valued predictable process H is called a-admissible if it is S-integrable, if $H_0 = 0$, if the stochastic integral satisfies $H \cdot S \ge -a$ and if $(H \cdot S)_\infty = \lim_{t\to\infty}(H \cdot S)_t$ exists a.s.. A predictable process H is called admissible if it is a-admissible for some a.*

Remark. We explicitly required that $H_0 = 0$ in order to avoid the contribution of the integral at zero.

The following notations will be used:

$$\mathcal{K} = \{(H \cdot S)_\infty \mid H \text{ is admissible}\}$$
$$\mathcal{K}_a = \{(H \cdot S)_\infty \mid H \text{ is a-admissible}\}$$
$$\mathcal{C}_0 = \mathcal{K} - L^0_+$$
$$\mathcal{C} = \mathcal{C}_0 \bigcap L^\infty$$

The basic theorem in Delbaen-Schachermayer [2] uses the concept of No Free Lunch with Vanishing Risk, $NFLVR$ for short. This is a rather weak hypothesis of no-arbitrage type and it is stated in terms of L^∞–convergence. The $NFLVR$ property is therefore independent of the choice of the underlying probability measure, i.e. it does not change if we replace $\mathbb{P}$ by an equivalent probability measure $\mathbb{Q}$. Only the class of negligible sets comes into play. We also recall the definition of the property of No–Arbitrage, NA for short.

Definition 1.2. *The locally bounded semi-martingale S satisfies the No–Arbitrage or NA property if*

$$\mathcal{C} \cap L^\infty_+ = \{0\}.$$

Definition 1.3. *We say that the locally bounded semi-martingale S satisfies the No Free Lunch with Vanishing Risk or $NFLVR$ property if*

$$\overline{\mathcal{C}} \cap L^\infty_+ = \{0\},$$

where the bar denotes the closure in the supnorm topology of L^∞.

The fundamental theorem of asset pricing, as in [2], can now be formulated as follows:

Theorem 1.4. *The locally bounded semi-martingale S satisfies the $NFLVR$ property if and only if there is an equivalent probability measure $\mathbb{Q}$ such that S is a $\mathbb{Q}-$local martingale. In this case the set $\mathcal{C}$ is already weak* (i.e. $\sigma(L^\infty, L^1)$) closed in L^∞.*

Remark. If $\mathbb{Q}$ is an equivalent local martingale measure for S and if the integrand or stategy H satisfies $H \cdot S \geq -a$, i.e. H is a–admissible, then by a result of Emery, [8] and Ansel-Stricker [1], the process $H \cdot S$ is still a local martingale and hence, being bounded below, is a super-martingale. It follows that the limit $(H \cdot S)_\infty$ exists a.s. and that $\mathbf{E}_{\mathbb{Q}}[(H \cdot S)_\infty] \leq 0$.

We also need the following equivalent reformulations of the property of No Free Lunch with Vanishing Risk, see [2] for more details.

Theorem 1.5. *The locally bounded semi-martingale S satisfies the No Free Lunch with Vanishing Risk Property or $NFLVR$ if for any sequence of S-integrable strategies $(H_n,\delta_n)_{n\geq 1}$ such that each H_n is a δ_n-admissible strategy and where δ_n tends to zero, we have that $(H \cdot S)_\infty$ tends to zero in probability $\mathbb{P}$.*

Theorem 1.6. *The locally bounded semi-martingale S satisfies the property $NFLVR$ if and only if*

(1) *it satisfies the property (NA)*
(2) *$\mathcal{K}_1$ is bounded in L^0, for the topology of convergence in measure.*

Theorem 1.7. *The locally bounded semi-martingale S satisfies the property $NFLVR$ if and only if*

(1) *it satisfies the property (NA)*
(2) *There is a strictly positive local martingale L, $L_0 = 1$, such that at infinity $L_\infty > 0$, $\mathbb{P}$ a.s. and such that LS is a local martingale.*

We suppose from now on that the process S is a fixed d–dimensional locally bounded semi-martingale and that it satisfies the property $NFLVR$. The set of local martingale measures is therefore, according to the previous theorems not empty. In section 7, we will also make use of finitely additive measures. So we let $\mathbf{ba}\,(\Omega, \mathcal{F}, \mathbb{P})$ be the Banach space of all finitely additive measures that are absolutely continuous with respect to $\mathbb{P}$, i.e. $\mathbf{ba}\,(\Omega, \mathcal{F}, \mathbb{P})$ is the dual of $L^\infty\,(\Omega, \mathcal{F}, \mathbb{P})$. We will use Roman letters $\mathbb{P}, \mathbb{Q}, \mathbb{Q}^0, \ldots$ for σ–additive measures and Greek letters for elements of $\mathbf{ba}$ which are not necessarily σ–additive. We say that a finitely additive measure μ is absolutely continuous with respect to the probability measure $\mathbb{P}$ if $\mathbb{P}[A] = 0$ implies $\mu[A] = 0$ for any set $A \in \mathcal{F}$.

Let us put:

$$\mathbf{M}^e = \left\{ \mathbb{Q} \mid \begin{array}{l} \mathbb{Q} \text{ is equivalent to } \mathbb{P} \\ \text{and the process } S \text{ is a } \mathbb{Q}\text{-local martingale} \end{array} \right\}$$

$$\mathbf{M} = \left\{ \mathbb{Q} \mid \begin{array}{l} \mathbb{Q} \text{ is absolutely continuous with respect to } \mathbb{P} \\ \text{and the process } S \text{ is a } \mathbb{Q}\text{-local martingale} \end{array} \right\}$$

$$\mathbf{M}^{ba} = \left\{ \mu \mid \begin{array}{l} \mu \text{ is in } \mathbf{ba}\,(\Omega, \mathcal{F}_\infty, \mathbb{P}) \\ \text{and for every element } h \in \mathcal{C}\colon \mathbf{E}_\mu[h] \le 0 \end{array} \right\}$$

We identify, as usual, absolutely continuous measures with their Radon–Nikodym derivatives. It is clear that, under the hypothesis $NFLVR$, the set $\mathbf{M}^e(\mathbb{P})$ is dense in $\mathbf{M}(\mathbb{P})$ for the norm of $L^1(\Omega, \mathcal{F}, \mathbb{P})$. This density together with Fatou's lemma imply that for random variables g that are bounded below we have the equality

$$\sup\{\mathbf{E}_\mathbb{Q}[g] \mid \mathbb{Q} \in \mathbf{M}^e\} = \sup\{\mathbf{E}_\mathbb{Q}[g] \mid \mathbb{Q} \in \mathbf{M}\}.$$

We will use this equality freely.

As shown in [2], Remark 5.10, the set $\mathbf{M}^e$ is weak*–dense, i.e. for the topology $\sigma\,(\mathbf{ba}, L^\infty)$, in the set $\mathbf{M}^{ba}$.

The first two sets are sets of σ–additive measures, the third set is a set of finitely additive measures. Clearly $\mathbf{M}^e \subset \mathbf{M} \subset \mathbf{M}^{ba}$ andsince S is locally bounded the set $\mathbf{M}$ is closed in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. If needed we will add the process S in parenthesis, e.g. $\mathbf{M}^e(S)$, to make clear that we are dealing with a set of local martingale measures for the process S.

2. Maximal Admissible Contingent Claims

We now give the definition of a maximal admissible contingent claim and its relation to the existence of an equivalent martingale measure. As mentioned above we always suppose that S is a d–dimensional locally bounded semi-martingale that satisfies the $NFLVR$–property.

Definition 2.1. *If $\mathcal{U}$ is a non-empty subset of L^0, then we say that a contingent claim $f \in \mathcal{U}$ is maximal in $\mathcal{U}$, if the properties $g \geq f$ a.s. and $g \in \mathcal{U}$ imply that $g = f$ a.s..*

The NA property can be rephrased as the property that 0 is maximal in $\mathcal{K}$. It is clear that if S satisfies the No-Arbitrage property, then the fact that f is maximal in $\mathcal{K}_a$ already implies that f is maximal in $\mathcal{K}$. Indeed if $g = (H \cdot S)_\infty \in \mathcal{K}$ and $g \geq f$ a.s., then $g \geq -a$. From lemma 3.5 [2] it then follows that g is $a-$admissible and hence the maximality of f in $\mathcal{K}_a$ implies that $g = f$ a.s..

Definition 2.2. *A maximal admissible contingent claim is a maximal element of $\mathcal{K}$. The set of maximal admissible contingent claims is denoted by $\mathcal{K}^{max}$. The set of maximal a–admissible contingent claims is denoted by $\mathcal{K}_a^{max}$.*

The proof of the theorem 1.4 uses the following intermediate results, [2] section 4:

Theorem 2.3. *If S is a locally bounded semi-martingale and if $(f_n)_{n\geq 1}$ is a sequence in $\mathcal{K}_1$, then*

1. *there is a sequence of convex combinations $g_n \in conv(f_n, f_{n+1}, ...)$ such that g_n tends in probability to a function g, taking finite values a.s.,*
2. *there is a maximal contingent claim h in $\mathcal{K}_1$ such that $h \geq g$ a.s..*

Corollary 2.4. *Under the hypothesis of theorem 2.3, maximal contingent claims of the closure L^0–closure $\overline{\mathcal{K}_1}$ of $\mathcal{K}_1$, are already in $\mathcal{K}_1$. By L^0–closure we mean the closure with respect to convergence in measure.*

Using a change of numéraire technique, the following result was proved in [4]. We refer also to Ansel-Stricker, [1] and Jacka, [Jk] for an earlier proof of the equivalence of (2) and (3).

Theorem 2.5. *If S is a locally bounded semi-martingale that satisfies the $NFLVR$ property then for a contingent claim $f \in \mathcal{K}$ the following are equivalent*

(1) *f is maximal admissible,*
(2) *there is an equivalent local martingale measure $\mathbb{Q} \in \mathbf{M}^e$ such that $\mathbf{E}_{\mathbb{Q}}[f] = 0$,*
(3) *if $f = (H \cdot S)_\infty$ for some admissible strategy H, then $H \cdot S$ is a uniformly integrable martingale for some $\mathbb{Q} \in \mathbf{M}^e$.*

Corollary 2.6. *Suppose that the hypothesis of theorem 2.5 is valid. If f is maximal admissible and $f = (H \cdot S)_\infty$ for some admissible strategy H, then for every stopping time T, the contingent claim $(H \cdot S)_T$ is also maximal.*

Proof. If f is maximal and $f = (H \cdot S)_\infty$ where H is a–admissible, then there is $\mathbb{Q} \in \mathbf{M}^e$ such that $\mathbf{E}_{\mathbb{Q}}[f] = 0$, i.e. $\mathbf{E}_{\mathbb{Q}}[(H \cdot S)_\infty] = 0$. Because H is admissible, the process $H \cdot S$ is, see [1], a $\mathbb{Q}$–local martingale and hence a $\mathbb{Q}$–supermartingale. Because $\mathbf{E}_{\mathbb{Q}}[(H \cdot S)_\infty] = 0$, we necessarily have that $H \cdot S$ is a $\mathbb{Q}$–uniformly integrable martingale. It follows that $\mathbf{E}_{\mathbb{Q}}[(H \cdot S)_T] = 0$ and consequently $(H \cdot S)_T$ is maximal.
q.e.d.

Remark. The corollary also shows that if $f = (H \cdot S)_\infty$ is maximal admissible, then the strategy that produces f is uniquely determined in the sense that any other admissible strategy K that produces f necessarily satisfies $H \cdot S = K \cdot S$. The following definition therefore make sense

Definition 2.7. *If H is an admissible strategy such that $f = (H \cdot S)_\infty$ is a maximal admisible contingent claim, then we say that H is a maximal admissible strategy.*

Definition 2.8. *We say that a strategy K is acceptable if there is a positive number a and a maximal admissible strategy L such that $(K \cdot S) \geq -(a + (L \cdot S))$.*

Remark. If we take a big enough, the process $V = a + L \cdot S$ stays bounded away from zero and can be used as a new numéraire. Under this new currecy unit, the process $K \cdot S$, where K is acceptable, has to be replaced by the process $\frac{K \cdot S}{V}$. The latter process is a stochastic integral with respect to the process $\left(\frac{S}{V}, \frac{1}{V}\right)$, more precisely, see [4] for the details of this calculation, $\frac{K \cdot S}{V} = (K, (K \cdot S)_- - KS_-) \cdot \left(\frac{S}{V}, \frac{1}{V}\right) = K' \cdot \left(\frac{S}{V}, \frac{1}{V}\right)$ remains bigger than a constant, i.e. the strategy $K' = (K, (K \cdot S)_- - KS_-)$ is admissible. Another way of saying that K is acceptable, is to say that K' is admissible in a new numéraire. In [4] we proved that the only numéraires that do not destroy the no arbitrage properties are the numéraires given by maximal strategies. The definition of acceptable strategies is therefore very natural. The outcomes of acceptable strategies are the numéraire invariant version of the outcomes of admissible strategies.

Lemma 2.9. *If S is a locally bounded semi-martingale that satisfies the $NFLVR$ property and if K is acceptable then $\lim_{t \to \infty} (K \cdot S)_t$ exists a.s..*

Proof. Suppose that $K \cdot S \geq -(a + L \cdot S)$ where L is admissible and maximal. Clearly we have that $K+L$ is a-admissible and hence by the results of [2] the $\lim_{t\to\infty}((K+L)\cdot S)_t$ exists a.s.. Because the $\lim_{t\to\infty}(L\cdot S)_t$ exists a.s., we necessarily have that $\lim_{t\to\infty}(K\cdot S)_t$ also exists a.s.

q.e.d.

The set of outcomes of acceptable strategies, which is a convex cone in L^0, is denoted by

$$\mathcal{J} = \{(K \cdot S)_\infty \mid K \text{ acceptable}\}.$$

We now prove some elementary properties of acceptable contingent claims. Most of these properties are generalisations of no arbitrage concepts for admissible contingent claims.

Proposition 2.10. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If K is acceptable and if $(K\cdot S)_\infty \geq 0$, then $(K\cdot S)_\infty = 0$.*

Proof. Suppose that $K \cdot S \geq -(a + L \cdot S)$ where L is admissible and maximal. Clearly we have that $K+L$ is a-admissible. But at infinity we have that $((K+L)\cdot S)_\infty \geq (L\cdot S)_\infty$ and by maximality of L we obtain the equality $((K+L)\cdot S)_\infty = (L\cdot S)_\infty$, which is equivalent to $(K\cdot S)_\infty = 0$ a.s..

q.e.d.

In the same way we prove the subsequent result.

Proposition 2.11. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If K is acceptable and $(K\cdot S)_\infty \geq -c$ for some positive real constant c, then the strategy K is already c–admissible.*

Proof. Take $\varepsilon > 0$ and let

$$T_1 = \inf\{t \mid (K\cdot S)_t < -c-\varepsilon\}.$$

We then define

$$T_2 = \inf\{t > T_1 \mid (K\cdot S)_t \geq -c\}.$$

By assumption we have that on $\{T_1 < \infty\}$ the strategy $K\mathbf{1}_{]\!]T_1,T_2]\!]}$ produces an outcome $(K\cdot S)_{T_2} - (K\cdot S)_{T_2} \geq \varepsilon$. This strategy is easily seen to be acceptable. Indeed

$$(K\mathbf{1}_{]\!]T_1,T_2]\!]})\cdot S \geq c + \varepsilon + (-a - H\cdot S)$$

for some real number a and some maximal strategy H. By the previous lemma we necessarily have that the contingent claim is zero a.s. and hence $T_1 = \infty$ a.s..

q.e.d.

We now turn again to the analysis of maximal admissible contingent claims.

Theorem 2.12. *If S is a locally bounded semi-martingale that satisfies the $NFLVR$ property, if f and g are maximal admissible contingent claims, then $f+g$ is also a maximal contingent claim. It follows that the set $\mathcal{K}^{max}$ of maximal contingent claims is a convex cone.*

Proof. Let $f = (H^1 \cdot S)_\infty$ and $g = (H^2 \cdot S)_\infty$, where H^1 and H^2 are maximal strategies and are respectively a_1 and a_2 admissible. Suppose that K is a k–admissible strategy such that $(K \cdot S)_\infty \geq f + g$. ¿From the inequalities $(K - H^2) \cdot S = K \cdot S - H^2 \cdot S \geq -k - H^2 \cdot S$, it follows that $K - H^2$ is acceptable. Since also $\big((K - H^2) \cdot S\big)_\infty \geq f \geq -a_1$, the proposition 2.11 shows that $K - H^2$ is a_1–admissible. Because f was maximal we have that $\big((K - H^2) \cdot S\big)_\infty = f$ and hence we have that $(K \cdot S)_\infty = f + g$. This shows that $f+g$ is maximal. Since the set $\mathcal{K}^{max}$ is clearly closed under multiplication with positive scalars, it follows that it is a convex cone.

q.e.d.

Corollary 2.13. *If S is a locally bounded semi-martingale that satisfies the $NFLVR$ property and if $(f_n)_{1 \leq n \leq N}$ is a finite sequence of contingent claims in $\mathcal{K}$ such that for each n there is an equivalent risk neutral measure $\mathbb{Q}^n \in \mathbf{M}^e$ with $\mathbf{E}_{\mathbb{Q}^n}[f_n] = 0$, then there is an equivalent risk neutral measure $\mathbb{Q} \in \mathbf{M}^e$ such that $\mathbf{E}_{\mathbb{Q}}[f_n] = 0$ for each $n \leq N$.*

Proof. This is a rephrasing of the theorem since by theorem 2.5, the condition on the existence of an equivalent risk neutral measure is equivalent with the maximality property.

q.e.d.

The previous theorem will be generalised to sequences (see corollary 2.16 below). We first prove the following

Proposition 2.14. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $(f_n)_{n \geq 1}$ is a sequence in $\mathcal{K}_1^{max}$, such that*

1. *The sequence $f_n \to f$ in probability*
2. *for all n we have $f - f_n \geq -\delta_n$ where δ_n is a sequence of strictly positive numbers tending to zero,*

then f is in $\mathcal{K}^{max}$ too, i.e. it is maximal admissible.

Proof. If g is a maximal contingent claim such that $g \geq f$, then we have $g - f_n \geq -\delta_n$. Since each f_n is maximal we find that $g - f_n$ is acceptable and hence δ_n–admissible by proposition 2.11. Since δ_n tends to zero, we find that the $NFLVR$ property implies that $g - f_n$ tends to zero in probabilty. This means that $g = f$ and hence f is maximal.

q.e.d.

Corollary 2.15. *If S is a locally bounded semi-martingale that satisfies the $NFLVR$*

property, if $(a_n)_{n\geq 1}$ is a sequence of strictly positive real numbers such that

$$\sum_1^\infty a_n < \infty,$$

if for each n, H^n is an a_n–admissible maximal strategy, then we have that the series

$$f = \sum_1^\infty (H^n \cdot S)_\infty$$

converges in probability to a maximal contingent claim.

Proof. Let $h_n = (H^n \cdot S)_\infty$, the partial sums $f_N = \sum_1^N h_n$ are outcomes of $\sum_1^\infty a_n$–admissible strategies. For an arbitrary element $\mathbb{Q} \in \mathbf{M}^e$ we have that

$$\mathbf{E}_\mathbb{Q}\left[(h_n + a_n)\right] \leq a_n.$$

It follows that the series of positive functions $\sum_1^\infty (h_n + a_n)$ converges in $L^1(\mathbb{Q})$ and hence the series $\sum_1^\infty h_n$ also converges in $L^1(\mathbb{Q})$. The series $f = \sum_1^\infty h_n = \lim f_n$ therefore also converges to a contingent claim f in $\mathbb{P}$. ¿From the proposition 2.14, we now deduce that f is maximal.

q.e.d.

Corollary 2.16. *If S is a locally bounded semi-martingale that satisfies the $NFLVR$ property, if $(f_n)_{n\geq 1}$ is a sequence of contingent claims in $\mathcal{K}$ such that for each n there is an equivalent risk neutral measure $\mathbb{Q}^n \in \mathbf{M}^e$ with $\mathbf{E}_{\mathbb{Q}^n}[f_n] = 0$, then there is an equivalent risk neutral measure $\mathbb{Q} \in \mathbf{M}^e$ such that $\mathbf{E}_\mathbb{Q}[f_n] = 0$ for each $n \geq 1$.*

Proof. We may without loss of generality suppose that f_n is the result of an a_n–admissible and maximal strategy where the series $\sum_1^\infty a_n$ converges. If not we replace f_n by a suitable multiple $\lambda_n f_n$, with λ_n strictly positive and small enough. The corollary 2.15 then shows that the sum $f = \sum_1^\infty f_n$ is still maximal and hence there is an element $\mathbb{Q} \in \mathbf{M}^e$ such that $\mathbf{E}_\mathbb{Q}[f] = 0$. As observed in the proof of the theorem, we have that the series $\sum_1^\infty f_n$ converges to f in $L^1(\mathbb{Q})$. For each n we already have that $\mathbf{E}_\mathbb{Q}[f_n] \leq 0$. From this it follows that for each n we need to have $\mathbf{E}_\mathbb{Q}[f_n] = 0$.

q.e.d.

Corollary 2.17. *If S is a locally bounded semi-martingale that satisfies the $NFLVR$ property, if $(f_n)_{n\geq 1}$ is a sequence of 1–admissible maximal contingent claims, if f is a random variable such that for each element $\mathbb{Q} \in \mathbf{M}^e$ we have $f_n \to f$ in $L^1(\mathbb{Q})$, then f is a 1–admissible maximal contingent claim.*

Proof. From theorem 2.3 we deduce the existence of a maximal contingent claim g such that $g \geq f$. From the previous corollary we deduce the existence of an element

$\mathbb{Q} \in \mathbf{M}^e$ such that for all n we have $\mathbf{E}_{\mathbb{Q}}[f_n] = 0$. It is straightforward to see that $\mathbf{E}_{\mathbb{Q}}[f] = 0$ and that $\mathbf{E}_{\mathbb{Q}}[g] \leq 0$. This can only be true if $f = g$, i.e. if f is 1-admissible and maximal.

q.e.d.

We now extend the "No Free Lunch with Vanishing Risk"–property which was phrased in terms of admissible startegies, to the framework of acceptable strategies. As always it is assumed that S is locally bounded and satisfies $NFLVR$.

Theorem 2.18. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. Let $f_n = (L^n \cdot S)_\infty$ be a sequence of outcomes of acceptable strategies such that $L^n \cdot S \geq -a_n - H^n \cdot S$, with H^n maximal and a_n–admissible. If* $\lim_{n\to\infty} a_n = 0$, *then* $\lim f_n = 0$ *in probability* $\mathbb{P}$.

Proof. The strategies $H^n + L^n$ are a_n–admissible and by the $NFLVR$ property of S we therefore have that $((H^n + L^n) \cdot S)_\infty$ tends to zero in probability $\mathbb{P}$. Because each H^n–admissible and $\lim_{n\to\infty} a_n = 0$ the $NFLVR$ property of S implies that $(H^n \cdot S)_\infty$ tends to zero in probability $\mathbb{P}$. It follows that also $(L^n \cdot S)_\infty$ tends to zero in probability $\mathbb{P}$.

q.e.d.

3. The Banach Space Generated by Maximal Contingent Claims

In this section we show that the subspace $\mathcal{G}$ of L^0, generated by the convex cone $\mathcal{K}^{max}$ of maximal admissible contingent claims can be endowed with a natural norm. We start with a definition.

Definition 3.1. *A predictable process H is called workable if both H and $-H$ are acceptable.*

Proposition 3.2. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. The vector space $\mathcal{G}$, or if there is danger of confusion and the price process S is important $\mathcal{G}(S)$, generated by the cone of maximal admissible contingent claims, satisfies*

$$\begin{aligned} \mathcal{G} &= \mathcal{K}^{max} - \mathcal{K}^{max} \\ &= \{(H \cdot S)_\infty \mid H \text{ is workable}\} \\ &= \mathcal{J} \cap (-\mathcal{J}) \,. \end{aligned}$$

Proof. The first statement is a trivial exercise in linear algebra. If H is workable then there are a real number a and maximal strategies L^1 and L^2 such that $-a - L^1 \cdot S \leq H \cdot S \leq a + L^2 \cdot S$. Take now $\mathbb{Q} \in \mathbf{M}^e$ such that both $L^1 \cdot S$ and $L^2 \cdot S$ are $\mathbb{Q}$–uniformly integrable martingales. The strategy $H + L^1$ is a–admissible and satisfies

$(H+L^1)\cdot S \leq a+(L^1+L^2)\cdot S$. It follows that $(H+L^1)\cdot S$ is a $\mathbb{Q}$–uniformly integrable martingale, i.e. $(H+L^1)$ is a maximal strategy. Since $H=(H+L^1)-L^1$ we obtain that $(H\cdot S)_\infty \in (\mathcal{K}^{max}-\mathcal{K}^{max})$. If conversely $H=H^1-H^2$, where both terms are maximal, then we have to show that H is workable. This is quite obvious, indeed if H^1 is a–admissible we have that $H\cdot S \geq -a - H^2\cdot S$. A similar reasoning applies to $-H$.

q.e.d.

Proposition 3.3. *Suppose that S is a locally bounded semi-martingale that satisfies the NFLVR property. If H is workable then there is an element $\mathbb{Q}\in\mathbf{M}^e$ such that the process $H\cdot S$ is a $\mathbb{Q}$–uniformly integrable martingale. Hence for every stopping time T, the random variable $(H\cdot S)_T$ is in $\mathcal{G}$. The process $H\cdot S$ is uniquely determined by $(H\cdot S)_\infty$.*

Proof. If H is workable then there are maximal admissible strategies K and K' such that $H=K-K'$. From theorem 2.5 and corollary 2.12 it follows that there is an equivalent local martingale measure $\mathbb{Q}\in\mathbf{M}^e$ such that both $K\cdot S$ and $K'\cdot S$ are $\mathbb{Q}$–uniformly integrable martingales. The rest is obvious.

q.e.d.

Proposition 3.4. *Suppose that S is a locally bounded semi-martingale that satisfies the NFLVR property. If $g\in\mathcal{G}$ satisfies $\|g^-\|_\infty<\infty$, then $g\in\mathcal{K}^{max}$.*

Proof. Put $L=(H^1-H^2)$, where H^1 and H^2 are both maximal, and so that $g=(L\cdot S)_\infty$. Since L is acceptable and $(L\cdot S)_\infty \geq -\|g^-\|_\infty$ we find by proposition 2.11 that L is admissible. For a well chosen element $\mathbb{Q}\in\mathbf{M}^e$, the process $L\cdot S$ is a uniformly integrable martingale and hence L is maximal.

q.e.d.

Corollary 3.5. *Suppose that S is a locally bounded semi-martingale that satisfies the NFLVR property. If V and W are maximal admissible strategies, if $((V-W)\cdot S)_\infty$ is uniformly bounded from below, then $V-W$ is admissible and maximal.*

Corollary 3.6. *Suppose that S is a locally bounded semi-martingale that satisfies the NFLVR property. Bounded contingent claims in $\mathcal{G}$ are characterised as*

$$\begin{aligned}\mathcal{G}^\infty = \mathcal{G}\cap L^\infty &= \mathcal{K}^{max}\cap L^\infty \\ &= \{(H\cdot S)_\infty \mid H\cdot S \text{ is bounded }\}.\end{aligned}$$

Remark. The vector space $\mathcal{G}^\infty$ should not be mixed up with the cone $\mathcal{K}\cap L^\infty$. As shown in [2] and [5], the contingent claim -1 can be in $\mathcal{K}$ but by the No Arbitrage property, the contingent claim $+1$ cannot be in $\mathcal{K}$. The vector space $\mathcal{G}^\infty$ was used in the study of the convex set $\mathbf{M}(S)$, see [2], [1] and [Jk].

Notation. *We define the following norm on the space $\mathcal{G}$:*

$$\|g\| = \inf \left\{ a \mid g = (H^1 \cdot S)_\infty - (H^2 \cdot S)_\infty \ , \ H^1 \ , \ H^2 \ \text{a–admissible and maximal} \right\}$$

The norm on the space $\mathcal{G}$ is quite natural and is suggested by its definition. It is easy to verify that $\|\cdot\|$ is indeed a norm. We will investigate the relation of this norm to other norms, e.g. L^∞ and L^1 norms.

Proposition 3.7. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $g = (H \cdot S)_\infty$ where H is workable then for every stopping time T, $g_T = (H \cdot S)_T \in \mathcal{G}$ and $\|g_T\| \leq \|g\|$.*

Proof. Follows immediately from the definition and the proof of corollary 2.6 above. q.e.d.

Proposition 3.8. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $g \in \mathcal{G}^\infty$ then as shown above $g \in \mathcal{K}_{\|g^-\|_\infty}$ and $-g \in \mathcal{K}_{\|g^+\|_\infty}$. Hence*

$$\|g\| \leq \min\left(\|g^+\|_\infty, \|g^-\|_\infty\right) \leq \max\left(\|g^+\|_\infty, \|g^-\|_\infty\right) = \|g\|_\infty.$$

The following lemma is an easy exercise in integration theory and immediately gives the relation with the L^1 norm.

Proposition 3.9. *If $f \in L^1(\Omega, \mathcal{F}, \mathbb{Q})$ for some probability measure $\mathbb{Q}$, if $\mathbf{E}_\mathbb{Q}[f] = 0$, if $f = g - h$, where both $\mathbf{E}_\mathbb{Q}[g] \leq 0$ and $\mathbf{E}_\mathbb{Q}[h] \leq 0$, then*

$$\begin{aligned} \|f\|_{L^1(\mathbb{Q})} &= 2\,\mathbf{E}_\mathbb{Q}[f^+] = 2\max(\mathbf{E}_\mathbb{Q}[f^+], \mathbf{E}_\mathbb{Q}[f^-]) \\ &\leq 2\max\left(\|g^-\|_\infty, \|h^-\|_\infty\right). \end{aligned}$$

Proof. The first line is obvious and shows that the obvious decomposition $f = (f^+ - \mathbf{E}[f^+]) - (f^- - \mathbf{E}[f^-])$ is best possible. So let us concentrate on the last line. If $f = g - h$ then we have the following inequalities:

$$\begin{aligned} f + \|g^-\|_\infty - \|h^-\|_\infty &= g + \|g^-\|_\infty - (h + \|h^-\|_\infty) \\ \left(f + \|g^-\|_\infty - \|h^-\|_\infty\right)^+ &\leq g + \|g^-\|_\infty \\ \left(f + \|g^-\|_\infty - \|h^-\|_\infty\right)^- &\leq h + \|h^-\|_\infty \end{aligned}$$

These inequalities together with $\mathbf{E}_\mathbb{Q}[g] \leq 0$ and $\mathbf{E}_\mathbb{Q}[h] \leq 0$, imply that

$$\|f + \|g^-\|_\infty - \|h^-\|_\infty\|_{L^1(\mathbb{Q})} \leq \|g^-\|_\infty + \|h^-\|_\infty.$$

It is now easy to see that

$$\|f\|_{L^1(\mathbb{Q})} \leq \|g^-\|_\infty + \|h^-\|_\infty + \left|\|g^-\|_\infty - \|h^-\|_\infty\right| \leq 2\max\left(\|g^-\|_\infty, \|h^-\|_\infty\right).$$

q.e.d.

Corollary 3.10. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $g \in \mathcal{G}$ then*

$$2\|g\| \geq \sup\left\{\|g\|_{L^1(\mathbb{Q})} \mid \mathbb{Q} \in \mathbf{M}\right\}$$

Proof. Take $g = (H^1 \cdot S)_\infty - (H^2 \cdot S)_\infty \in \mathcal{G}$ where H^1 and H^2 are both maximal and a–admissible. For every $\mathbb{Q} \in \mathbf{M}$ we have that $\mathbf{E}_\mathbb{Q}[(H^1 \cdot S)_\infty] \leq 0$ and $\mathbf{E}_\mathbb{Q}[(H^2 \cdot S)_\infty] \leq 0$. The lemma shows that

$$\|g\|_{L^1(\mathbb{Q})} \leq 2 \max\left(\|(H^1 \cdot S)_\infty\|_{L^1(\mathbb{Q})}, \|(H^2 \cdot S)_\infty\|_{L^1(\mathbb{Q})}\right) \leq 2a.$$

By taking the infimum over all decompositions and by taking the supremum over all elements in $\mathbf{M}$ we find the desired inequality.

q.e.d.

The next theorem shows that in some sense there is an optimal decomposition. The proof relies on theorem 2.3 above and on the technical lemma A1 in [2].

Theorem 3.11. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $g \in \mathcal{G}$ then there exist two $\|g\|$–admissible maximal strategies R and U such that $g = (R \cdot S)_\infty - (U \cdot S)_\infty$.*

Proof. Take a sequence of real numbers such that $a_n \downarrow \|g\|$. For each n we take H^n and K^n maximal and a_n–admissible such that $g = (H^n \cdot S)_\infty - (K^n \cdot S)_\infty$. From the theorem 2.3 cited above we deduce that there are convex combinations $V_n \in conv(H^n, H^{n+1}, ...)$ and $W_n \in conv(K^n, K^{n+1}, ...)$ such that $(V^n \cdot S)_\infty \to h$ and $(W^n \cdot S)_\infty \to k$. Clearly $g = h - k$, $h \geq -\|g\|$ and $k \geq -\|g\|$. However we cannot, at this stage, assert that h and/or k are maximal. Theorem 2.3 above however allows us to find a maximal strategy R such that $(R \cdot S)_\infty \geq h \geq -\|g\|$. The strategy $R - H^1 + K^1$ is acceptable and satisfies

$$((R - H^1 + K^1) \cdot S)_\infty = (R \cdot S)_\infty - g \geq h - g = k \geq -\|g\|.$$

From the proposition 2.11 above it follows that $U = R - H^1 + K^1$ is $\|g\|$–admissible and maximal. By definition of U and R we have that $g = (R \cdot S)_\infty - (U \cdot S)_\infty$.

q.e.d.

Corollary 3.12. *With the notation of the above theorem 3.11: $(R \cdot S)_\infty + \|g\| \geq g^+$ and $(U \cdot S)_\infty + \|g\| \geq g^-$. Hence we find*

$$\sup\left\{\mathbf{E}_\mathbb{Q}\left[g^+\right] \mid \mathbb{Q} \in \mathbf{M}\right\} \leq \|g\|$$
$$\sup\left\{\mathbf{E}_\mathbb{Q}\left[g^-\right] \mid \mathbb{Q} \in \mathbf{M}\right\} \leq \|g\|$$

Theorem 3.13. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $g \in \mathcal{G}$ then*

$$\begin{aligned}\|g\| &= \sup\left\{\mathbf{E}_{\mathbb{Q}}\left[g^+\right] \mid \mathbb{Q}\in\mathbf{M}\right\} = \sup\left\{\mathbf{E}_{\mathbb{Q}}\left[g^+\right] \mid \mathbb{Q}\in\mathbf{M}^e\right\}\\ &= \sup\left\{\mathbf{E}_{\mathbb{Q}}\left[g^-\right] \mid \mathbb{Q}\in\mathbf{M}\right\} = \sup\left\{\mathbf{E}_{\mathbb{Q}}\left[g^-\right] \mid \mathbb{Q}\in\mathbf{M}^e\right\}\end{aligned}$$

Proof. Put $\beta = \sup\{\mathbf{E}_{\mathbb{Q}}[g^+] \mid \mathbb{Q}\in\mathbf{M}\}$, where the random variable g is decomposed as $g = (H^1\cdot S)_\infty - (H^2\cdot S)_\infty$ with H^1 and H^2 maximal. From [4], corollary 10 to theorem 9, we recall that there is a maximal strategy K^1 such that $g^+ \le \beta + (K^1\cdot S)_\infty$, implying that K^1 is β–admissible. The strategy $K^2 = K^1 - H^1 + H^2$ is also β–admissible and by proposition 2.11 therefore maximal. Since $K^1 - K^2 = H^1 - H^2$ we obtain that $\|g\| \le \beta$. Since the opposite inequality is already shown in corollary 3.12, we therefore proved the theorem.

q.e.d.

Remark. If $\mathbb{Q}$ is a martingale measure for the process $(H^1 - H^2)\cdot S$, then of course $\mathbf{E}_{\mathbb{Q}}[g^+] = \mathbf{E}_{\mathbb{Q}}[g^-]$. But not all elements in the set $\mathbf{M}$ are martingale measures for this process and hence the equality of the suprema does not immediately follow from martingale considerations.

Theorem 3.14. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. The norm of the space $\mathcal{G}$ is also given by the formula*

$$2\|g\| = \sup\{\mathbf{E}_{\mathbb{Q}}[|g|] \mid \mathbb{Q}\in\mathbf{M}\} = \sup\{\mathbf{E}_{\mathbb{Q}}[|g|] \mid \mathbb{Q}\in\mathbf{M}^e\}.$$

Proof. As in the previous result, for a contingent claim $g = (H^1\cdot S)_\infty - (H^2\cdot S)_\infty$ where H^1 and H^2 are maximal admissible, let us put:

$$\begin{aligned}\beta &= \sup\{\mathbf{E}_{\mathbb{Q}}[|g|] \mid \mathbb{Q}\in\mathbf{M}\}\\ &\le \sup\left\{\mathbf{E}_{\mathbb{Q}}\left[g^+\right] \mid \mathbb{Q}\in\mathbf{M}\right\} + \sup\left\{\mathbf{E}_{\mathbb{Q}}\left[g^-\right] \mid \mathbb{Q}\in\mathbf{M}\right\}\\ &= 2\|g\|\end{aligned}$$

From [4] it follows that there is a maximal strategy K, such that $|g| \le \beta + (K\cdot S)_\infty$. This inequality shows that

$$\begin{aligned}\beta + ((K\cdot S)_\infty) &\ge (H^1\cdot S)_\infty - (H^2\cdot S)_\infty\\ \beta + ((K\cdot S)_\infty) &\ge (H^2\cdot S)_\infty - (H^1\cdot S)_\infty.\end{aligned}$$

As in previous result we obtain that $K - H^1 + H^2$ and $K - H^2 + H^1$ are β–admissible and maximal. Since $2(H^1 - H^2) = (K - H^2 + H^1) - (K - H^1 + H^2)$, we obtain the inequality $\|2\,g\| \le \beta$.

q.e.d.

Corollary 3.15. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $g \in \mathcal{G}$, then there is a sequence of elements $\mathbb{Q}_n \in \mathbf{M}^e$ such that*

(1) $\mathbf{E}_{\mathbb{Q}_n}[g^+] \to \sup\{\mathbf{E}_{\mathbb{Q}}[g^+] \mid \mathbb{Q} \in \mathbf{M}\}$
(2) $\mathbf{E}_{\mathbb{Q}_n}[g^-] \to \sup\{\mathbf{E}_{\mathbb{Q}}[g^-] \mid \mathbb{Q} \in \mathbf{M}\}$
(3) $\mathbf{E}_{\mathbb{Q}_n}[|g|] \to \sup\{\mathbf{E}_{\mathbb{Q}}[|g|] \mid \mathbb{Q} \in \mathbf{M}\}$

Proof. It suffices to take a sequence that satisfies the third line.

q.e.d.

Remark and Example. For a contingent claim $f \in \mathcal{K}^{max}$ we do not necessarily have that

$$\|f\| = \inf\{a \mid f \in \mathcal{K}_a\}.$$

Indeed take a process S such that there is only one risk neutral measure $\mathbb{Q}$. In this case the norm on the space $\mathcal{G}$ is (half) the $L^1(\mathbb{Q})$ norm. As is well known the market is complete (see e.g. [2]) and $\mathcal{G} = \{f \mid f \in L^1(\mathbb{Q}),\ \mathbf{E}_{\mathbb{Q}}[f] = 0\}$. It follows that $\mathcal{K}_a = \{f \mid f \in L^1(\mathbb{Q}),\ \mathbf{E}_{\mathbb{Q}}[f] = 0,\ f \geq -a\}$. This cone may contain contingent claims with $\|f^-\|_\infty = a$ and with arbitrary small $L^1(\mathbb{Q})$–norm.

This example also shows that the space $\mathcal{G}$, which in this example is a hyperplane in L^1, can be isomorphic to an L^1 space. It also shows that the cone $\mathcal{K}^{max}$ is not necessarily closed. Indeed the cone $\mathcal{K}^{max}$ contains all contingent claims $f \in L^\infty$ with the property $\mathbf{E}_{\mathbb{Q}}[f] = 0$. This set is dense in $\mathcal{G} = \{f \mid f \in L^1(\mathbb{Q}),\ \mathbf{E}_{\mathbb{Q}}[f] = 0\}$. However we have the following result.

Proposition 3.16. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. The cones $\mathcal{K}_a^{max}$ are closed in the space $\mathcal{G}$.*

Proof. Take a sequence f_n in $\mathcal{K}_a^{max}$ and tending to f for the norm of $\mathcal{G}$. Since clearly $f \geq -a$, the contingent claim f is the outcome of an admissible and by corollary 2.17, also of a maximal strategy.

q.e.d.

4. Some results on the topology of $\mathcal{G}$.

We now show that the space $\mathcal{G}$ is complete. This is of course very important if one wants to apply the powerful tools of functional analysis. The proof uses theorems 3.11 and corollary 2.15 above and in fact especially corollary 2.15 suggests that the space is complete. After the proof of the theorem we will give some examples in order to show what kind of space $\mathcal{G}$ can be.

Theorem 4.1. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. The space $\mathcal{G}, \|\cdot\|$ is complete, i.e. it is a Banach space.*

Proof. We have to show that each Cauchy sequence converges. This is equivalent to the statement that every series of contingent claims whose norms form a convergent series, actually converges. So we start with a sequence $(g_n)_{n\geq 1}$ in $\mathcal{G}$ such that $\sum_{n\geq 1} \|g_n\| < \infty$. For each n we take according to theorem 3.11 above, two $\|g_n\|$–admissible maximal strategies H^n and L^n such that $g_n = (H^n \cdot S)_\infty - (L^n \cdot S)_\infty$. Since $\sum_{n\geq 1} \|g_n\|$ converges, proposition 2.14 above shows that $h = \sum_{n\geq 1}(H^n \cdot S)_\infty$ and $l = \sum_{n\geq 1}(L^n \cdot S)_\infty$ converge and define the maximal contingent claims h and l. Put now $g = h - l$, clearly an element of the space $\mathcal{G}$. We still have to show that the series actually converge to g for the norm defined on $\mathcal{G}$. But this is obvious since

$$g - \sum_{n=1}^{n=N} g_n = \left(\sum_{n>N} (H^n \cdot S)_\infty - (\sum_{n>N} (L^n \cdot S)_\infty\right)$$

and each term on the right hand side defines, according to corollary 2.15, a maximal contingent claim that is generated by a $\sum_{n>N} \|g_n\|$–admissible strategy. This remainder series tends to zero which completes the proof of the theorem.

q.e.d.

Theorem 4.2. *Suppose that S is a locally bounded semi-martingale that satisfies the NFLVR property. If $(f_n)_{n\geq 1}$ is a sequence that converges in $\mathcal{G}$ to an contingent claim f and if for each n, $f_n = (H^n \cdot S)_\infty$ with H^n workable, then there is an element $\mathbb{Q} \in \mathbf{M}^e$ such that all $H^n \cdot S$ are uniformly integrable $\mathbb{Q}$–martingales as well as a workable strategy H such that the martingales $H^n \cdot S$ converge in $L^1(\mathbb{Q})$ to the martingale $H \cdot S$.*

Proof. Take $\mathbb{Q} \in \mathbf{M}^e$ such that all $(H^n \cdot S)_{n\geq 1}$ are $\mathbb{Q}$–uniformly integrable martingales. Such a probability exists by corollary 2.16. The rest is obvious and follows from the inequality $\|g\| \geq \|g\|_{L^1(\mathbb{Q})}$.

q.e.d.

Theorem 4.3. *Suppose that S is a locally bounded semi-martingale that satisfies the NFLVR property. If $(f_n)_{n\geq 1}$ is a sequence tending to f in the space $\mathcal{G}$, then there are maximal admissible contingent claims $(g_n, h_n)_{n\geq 1}$ in $\mathcal{K}^{max}$ such that $f_n = g_n - h_n$ and such that $g_n \to g \in \mathcal{K}^{max}$, $h_n \to h \in \mathcal{K}^{max}$, both convergences hold for the norm of $\mathcal{G}$.*

Proof. We first show that the statement of the theorem holds for a well chosen subsequence $(n_k)_{k\geq 1}$. Afterwards we will fill in the remaining gaps.

The subsequence n_k is chosen so that for all $N \geq n_k$ we have $\|f - f_N\| \leq 2^{-k-1}$. It follows that $\|f_{n_{k+1}} - f_{n_k}\| \leq 2^{-k}$, for all k. We take, according to theorem 3.11, contingent claims in $\mathcal{K}^{max}$, denoted by $(\psi_k, \varphi_k)_{k\geq 1}$ such that

$$f_{n_1} = \psi_1 - \varphi_1$$
$$f_{n_{k+1}} - f_{n_k} = \psi_k - \varphi_k.$$

and such that ψ_k and φ_k are 2^{-k} admissible for $k \geq 2$. Let $g_{n_k} = \sum_{l=1}^k \psi_l$ and $h_{n_k} = \sum_{l=1}^k \varphi_l$. By corollary 2.15 and the reasoning in the proof of theorem 4.1, these sequences converge in the norm of $\mathcal{G}$ to respectively g and h. Furthermore $f_{n_k} = g_{n_k} - h_{n_k}$ and hence $f = g - h$.

We now fill in the gaps $]n_k, n_{k+1}[$. For $n_k < n < n_{k+1}$ we choose maximal 2^{-k}–admissible contingent claims ρ_n and σ_n such that $f_n - f_{n_k} = \rho_n - \sigma_n$. To complete the proof we just have to check the obvious fact that $g_n = g_{n_k} + \rho_n$ and $h_n = h_{n_k} + \sigma_n$ satisfy the requirements of the theorem.

q.e.d.

We will now discuss an example that serves as an illustration of what can go wrong in an incomplete market.

Example 4.4 The example is a slight modification of the example of [3], see also [14]. We start with a two dimensional standard Brownian motion (B, W), with its natural filtration $(\mathcal{F}_t)_{t\geq 0}$. For the price process S we take a stochastic volatility process defined as $dS_t = (2 + \arctan(W_t))dB_t$. It is clear that the natural filtration of S is precisely $(\mathcal{F}_t)_{t\geq 0}$. Furthermore it is easy to see that the set of stochastic integrals with respect to S is the same as the set of stochastic integrals with respect to B. We will use this fact without further notice. We define $L = \mathcal{E}(B)$ and $Z = \mathcal{E}(W)$, where $\mathcal{E}$ denotes the stochastic exponential. The stopping times τ and σ are defined as $\tau = \inf\{t \mid L_t \leq 1/2\}$ and $\sigma = \inf\{t \mid Z_t \geq 2\}$. The process X is defined as $X = L^{\tau\wedge\sigma}$. The measure $\mathbb{Q}$ is nothing else but $d\mathbb{Q} = Z_{\tau\wedge\sigma} d\mathbb{P}$. For the process X and the measure $\mathbb{Q}$, the following hold

1. The process X is continuous, strictly positive, also $X_\infty > 0$ a.s. and $X_0 = 1$, it is a local martingale for $\mathbb{P}$, i.e. $\mathbb{P} \in \mathbf{M}^e$
2. Under $\mathbb{P}$, the process X is a strict local martingale, i.e. $\mathbf{E}_\mathbb{P}[X_\infty] < 1$
3. for each $t < \infty$ the stopped process X^t is a $\mathbb{P}$–uniformly integrable martingale
4. there is an equivalent probability measure $\mathbb{Q} \in \mathbf{M}^e$ for which X becomes a $\mathbb{Q}$–uniformly integrable martingale.

We refer to [3] for the proof of these statements.

Let us now verify some additional features.

Proposition 4.5. *In the setting of the above example, the space $\mathcal{G}^\infty$ is not dense in $\mathcal{G}$. In fact even the closure of L^∞ for the norm $\|g\| = \frac{1}{2}\sup\{\|g\|_{L^1(\mathbb{Q})} \mid \mathbb{Q} \in \mathbf{M}^e\}$, does not contain $\mathcal{G}$ as a subset.*

Proof. For each $t \leq \infty$ we clearly have that $f_t = X_t - 1 \in \mathcal{G}$. Suppose now that the contingent claim f_∞ is in the closure of the space L^∞ for the norm $\|g\| = \frac{1}{2}\sup\{\|g\|_{L^1(\mathbb{Q})} \mid \mathbb{Q} \in \mathbf{M}^e\}$. For $\varepsilon = -\mathbf{E}[f_\infty]/4 > 0$ we can find g bounded such that for all $\mathbb{Q} \in \mathbf{M}^e$ we have $\|g - f_\infty\| \leq \varepsilon$. For the measure $\mathbb{P}$ we find $\mathbf{E}_\mathbb{P}[|f_\infty - g|] \leq \varepsilon$ and hence for each $t \leq \infty$ we have, by taking conditional expectations,

$$\mathbf{E}_\mathbb{P}[|f_t - \mathbf{E}_\mathbb{P}[g \mid \mathcal{F}_t]|] \leq \varepsilon.$$

In particular, since $\mathbf{E}[f_t] = 0$ for each $t < \infty$, we have $\mathbf{E}_{\mathbb{P}}[g] = \mathbf{E}[\mathbf{E}_{\mathbb{P}}[g \mid \mathcal{F}_t]] \geq -\varepsilon$. This in turn implies that $\mathbf{E}_{\mathbb{P}}[f_\infty] \geq -2\varepsilon$ a contradiction to the choice of ε.

q.e.d.

Theorem 4.6. *In the setting of the above example, the Banach space $\mathcal{G}$ contains a subspace isometric to l^∞. In other words there is an isometry $u\colon l^\infty \longrightarrow \mathcal{G}$. Moreover u can be chosen such that $u(l^\infty) \subset \mathcal{G}^\infty$.*

Proof. We start with a partition of Ω into a sequence of pairwise disjoint sets, defined by the process W. More precisely we put $A_1 = \{W_1 \in]-\infty, 1]\}$ and for $n \geq 2$ we put $A_n = \{W_1 \in]n-1, n]\}$. Let M be the stochastic exponential $M = \mathcal{E}(B - B^1)$ and let the stopping time T be defined as

$$T = \inf\{t \mid M_t \geq 2\}$$

The sequence that will we will use to construct the subspace isometric to l^∞ is defined as

$$f_n = 2\,(M_T - 1)\,\mathbf{1}_{A_n}.$$

For each n and each $\varepsilon > 0$ there is a real number $\alpha(n,\varepsilon)$ depending only on ε and n such that the random variable $\phi(n,\varepsilon) = \alpha(n,\varepsilon)\mathbf{1}_{A_k} + \varepsilon\mathbf{1}_{\cup_{m\neq k} A_m}$ is strictly positive and defines a density for a measure $d\mathbb{Q}_{n,\varepsilon} = \phi(n,\varepsilon)d\mathbb{P}$ which is necessarily in $\mathbf{M}^e$, since the random variable $\phi(n,\varepsilon)$ can be written as a stochastic integral with respect to W. It is clear that $\mathbb{Q}_{n,\varepsilon}[A_n] \geq 1-\varepsilon$. This shows that for each n, $\sup\{\mathbb{Q}[A_n] \mid \mathbb{Q} \in \mathbf{M}^e\} = 1$.

Clearly each f_n is a 2–admissible maximal contingent claim. Since for each measure $\mathbb{Q} \in \mathbf{M}^e$ we have $\mathbb{Q}[f_n = 2 \mid \mathcal{F}_1] = \mathbb{Q}[f_n = -2 \mid \mathcal{F}_1] = \frac{1}{2}\mathbf{1}_{A_n}$ we obtain that $\|f_n\|_{L^1(\mathbb{Q})} = 2\mathbb{Q}(A_n)$, hence for the $\mathcal{G}$–norm we find $\|f_n\| = 1$.

We now show that for each $x \in l^\infty$ we can define a contingent claim

$$u(x) = \sum_{k\geq 1} x_k f_k \in \mathcal{G}.$$

If $x = (x_k)_{k\geq 1}$ is an element of l^∞ and if m is a natural number, we denote by x^m the element defined as $x^m_k = x_k$ if $k \leq m$ and $x^m_k = 0$ otherwise. Let us already put $u(x^m) = \sum_{k=1}^m x^m_k f_k$. Now if x is a positive element in l^∞ then the sequence $(u(x^m))_{m\geq 1}$ is a sequence converging in $L^1(\mathbb{Q})$ to a contingent claim $u(x) = \sum_{k=1}^\infty x_k f_k$ and this for each $\mathbb{Q} \in \mathbf{M}^e$. By corollary 2.17 and theorem 2.5, the random variable $u(x)$ is in $\mathcal{G}$. For arbitrary x we split into the positive and the negative part. This defines a linear mapping from l^∞ into $\mathcal{G}$. For each $\mathbb{Q} \in \mathbf{M}^e$ we have that $u(x) = \sum_{k=1}^\infty x_k f_k$, where the sum actually converges in $L^1(\mathbb{Q})$. Let us now calculate the norm of $u(x)$. For an arbitrary measure $\mathbb{Q} \in \mathbf{M}^e$ we find

$$\|u(x)\|_{L^1(\mathbb{Q})} \leq \int \sum_{k=1}^{\infty} |x_k| \|f_k\|_{L^1(\mathbb{Q})}$$

and hence we have

$$\|u(x)\| \leq \sup_k |x_k|.$$

Take now for $\varepsilon > 0$ given, an index k such that $\sup_k |x_k| > \|x\|_\infty - \varepsilon$. Take the measure $\mathbb{Q}_{k,\varepsilon}$ as above.

We find that

$$\begin{aligned}\|u(x)\|_{L^1(\mathbb{Q}_{k,\varepsilon})} &\geq \int_{A_k} |u(x)|\phi(k,\varepsilon)\, d\mathbb{P} \\ &\geq \alpha(k,\varepsilon)|x_k|2\mathbb{P}[A_k].\end{aligned}$$

Since clearly $\alpha(k,\varepsilon)\mathbb{P}[A_k] \geq 1-\varepsilon$ we find that

$$\|u(x)\|_{L^1(\mathbb{Q}_{k,\varepsilon})} \geq (\|x\|_\infty - \varepsilon)\, 2(1-\varepsilon).$$

Because $\varepsilon > 0$ was arbitrary we find that

$$\|u(x)\| = \|x\|_\infty.$$

The linear mapping is therefore an isometry. Furthermore it is easily seen that for each $x \in l^\infty$ we have $u(x) \in \mathcal{G}^\infty$.

q.e.d.

Theorem 4.7. *In the setting of the above example, there is a contingent claim f in $\mathcal{G}$ such that for each $\mathbb{Q} \in \mathbf{M}^e$ we have $\mathbf{E}_\mathbb{Q}[f] = 0$, but such that f is not in the closure of $\mathcal{G}^\infty$.*

Proof. We will make use of the notation and proof of the preceding theorem. So we take the same sequence $(A_n)_{n\geq 1}$ as above. This time we introduce stopping times

$$T_n = \inf\{t \mid M_t \geq n+1\}$$

and functions

$$f_n = (M_{T_n} - 1)\, \mathbf{1}_{A_n}.$$

Exactly as in the previous proof one shows that the contingent claim $f = \sum_{n=1}^\infty f_n$ is in $\mathcal{G}$ and has norm 1. Suppose now that h is a bounded variable in $\mathcal{G}$. We will show that $\|f-h\| \geq 1$. For each n we take an element $\mathbb{Q}_n \in \mathbf{M}^e$ such that $\mathbb{Q}[A_n] \geq 1 - \frac{1}{n}$; such an element surely exists. Because $\mathbb{Q}[f_n = n \mid \mathcal{F}_n] = \frac{1}{n}\mathbf{1}_{A_n}$ we find for $n > \|h\|_\infty$, that

$$\begin{aligned}\mathbf{E}_{Q_n}[(f-h)^+] &\geq \left(1-\frac{1}{n}\right)\frac{1}{n}(n - \|h\|_\infty) \\ &\geq \left(1-\frac{1}{n}\right)\left(1 - \frac{\|h\|_\infty}{n}\right)\end{aligned}$$

From theorem 3.14 we can now deduce that the distance of f to $\mathcal{G}^\infty$ is precisely equal to 1.

q.e.d.

This completes the discussion of the example 4.4.

Example 4.8 This is an example showing that the space $\mathcal{G}$ can be one-dimensional, whereas the set $\mathbf{M}^e$ remains very big. For this we take a finite time set $[0,1]$,and we take $\Omega = [0,1]$ with the Lebesgue measure. For $t < 1$, we put $\mathcal{F}_t$ equal to the σ–algebra generated by the zero sets with respect to Lebesgue–measure. For $t = 1$ we put $\mathcal{F}_t$ equal to the σ–algebra of all Lebesque measurable sets. The price process is defined as $S_t = 0$ for $t < 1$ and $S_1(\omega) = \omega - 1/2$. Of course $\mathcal{G} = span(S_1)$. The set $\mathbf{M}^e$ is the set $\{f \mid f > 0\ ,\ \int_0^1 (t-1/2)f(t)\,dt = 0\}$. This set is big in the sense that it is not relatively weakly compact in $L^1[0,1]$.

Example 4.9 This example shows that the space $\mathcal{G}$ can actually be isomorphic to an L^∞ space. The example is constructed is the same spirit as the previous one. We take $[0,2]$ as the time set and $\Omega = [-1,1]\times[-1,1]$ with the two dimensional Lebesgue measure. Let g_1, respectively g_2 be the first and second coordinate projection defined on Ω. For $t < 1$ the σ–algebra $\mathcal{F}_t$ is the σ–algebra generated by the zero sets, for $1 \le t < 2$ we have $\mathcal{F}_t = \sigma(\mathcal{F}_0, g_1)$ and $\mathcal{F}_2 = \sigma(\mathcal{F}_1, g_2)$, which is also the σ–algebra of Lebesgue measurable subsets of Ω. The process S is defined as $S_t = 0$ for $t < 1$, $S_t = g_1$ for $1 \le t < 2$ and $S_2 = g_1 + g_2$. We remark that the filtration is generated by the process S.

Clearly $(H\cdot S)_2 \in \mathcal{G}$ if and only if it is of the form $(H\cdot S)_2 = \alpha g_1 + h\,g_2$, where h is $\mathcal{F}_1$ measurable and bounded. This implies that $\mathcal{G}$ can be identified with $I\!\!R\times L^\infty(\Omega,\mathcal{F}_1,\mathbb{P})$. We will not calculate the norm of the space $\mathcal{G}$, but instead we will use the closed graph theorem to see that this the norm is equivalent to the norm defined as $\|(\alpha,h)\| = |\alpha| + \|h\|_\infty$. It follows that $\mathcal{G}$ is isomorphic to an L^∞–space.

Example 4.10 The following example is in the same style as the process S has exactly one jump. But this time the behaviour of the process S before the jump is such that the space $\mathcal{G}$ is not of L^∞–type.

We start with the one dimensional Brownian Motion W, starting at zero and with its natural filtration $(\mathcal{H}_t)_{0\le t\le 1}$. At time $t = 1$ we add a jump g uniformly distributed over the interval $[-1,1]$ and independent of the Brownian Motion W. So the price process becomes $S_t = W_t$ for $t < 1$ and $S_1 = W_1 + g$. The filtration becomes, up to null sets, $\mathcal{F}_t = \mathcal{H}_t$ for $t < 1$ and $\mathcal{F}_1 = \sigma(\mathcal{H}_1, g)$. For simplicity we assume that this process is defined on the probability space $\Omega\times[-1,+1]$ where Ω is the trajectory space of Brownian Motion, equipped with the usual Wiener measure $\mathbb{P}$ and where we take the uniform distribution m on $[-1,+1]$ as the second factor. The measure is therefore $\mathbb{P}\times m$.

The set of equivalent local martingales measures can also be characterised. Since

Brownian Motion has only one local martingale measure we see that for each $\mathbb{Q} \in \mathbf{M}^e$ and for each $t < 1$ we have that $\mathbb{Q} = \mathbb{P}$ on the σ–algebra $\mathcal{F}_t = \mathcal{H}_t$. Therefore also $\mathbb{Q} = \mathbb{P}$ on $\mathcal{H}_1$. From the existence theorem of conditional distributions, or the desintegration theorem of measures, we then learn that $\mathbb{Q}$ is necessarily of the form $\mathbb{Q}[d\omega \times dx] = \mathbb{P}[d\omega]\mu_\omega[dx]$, where μ is a probability kernel $\mu\colon \Omega \times \mathcal{B}[-1,+1] \longrightarrow [0,1]$, measurable for $\mathcal{H}_1$. In order for $\mathbb{Q}$ to be a local martingale measure μ should satisfy $\int_{[-1,+1]} x\,\mu_\omega(dx) = 0$ for almost all ω. In order to be equivalent to $\mathbb{P} \times m$, a.s. the measure μ_ω should be equivalent to m. This can easily be seen by using the density of $\mathbb{Q}$ with respect to $\mathbb{P} \times m$.

If H is a predictable strategy then it is clear that it is predictable with respect to the filtration of the Brownian Motion. A strategy H is therefore S–integrable if and only if $\int_0^1 H_t^2\,dt < \infty$ a.s.. It follows that a necessary condition for a predictable process H to be 1–admissible is $H \cdot W \geq -1$. We can change the value of H at time 1 without perturbing the integral $H \cdot W$. In order to obtain a characterisation of 1–admissible integrands for S, we only need a condition on H_1 in order to have, in addition, that $(H \cdot S)_1 \geq -1$. The outcome at time 1 is $(H \cdot S)_1 = (H \cdot W)_1 + H_1\,g$ and this is almost surely bigger than -1 if and only if $|H_1| \leq 1 + (H \cdot W)_1$ almost surely. If we are looking for 1–admissible maximal contingent claims the condition on H becomes

(1) $H \cdot W$ is a uniformly integrable martingale for $\mathbb{P}$ and $f = (H \cdot W)_1 \geq -1$
(2) $|H_1| \leq 1 + f$

From this it follows that a random variable k is in $\mathcal{G}$ if and only if it is of the form

$$k = f_1 - f_2 + g\ (h_1 - h_2)$$

where

(1) f_1, f_2, h_1, h_2 are $\mathcal{H}_1$ measurable
(2) $f_1, f_2 \geq -a$ for some positive real number a
(3) $\mathbf{E}_\mathbb{P}[f_1] = \mathbf{E}_\mathbb{P}[f_1] = 0$
(4) $|h_1| \leq a + f_1$ and $|h_2| \leq a + f_2$

If we want to find a better description we observe that if f is $\mathcal{H}_1$ measurable, integrable and positive then we can take $f_1 = f_2 = f - \mathbf{E}_\mathbb{P}[f]$ and hence the condition on h_1 and h_2 becomes $|h_1|, |h_2| \leq f$. It follows that the space $\mathcal{G}$ is the space of all functions k of the form

$$f + g\,h$$

where

$$\mathbf{E}_\mathbb{P}[f] = 0 \text{ and where } h, f \text{ are both } \mathcal{H}_1\text{–measurable and integrable.}$$

The norm on the space $\mathcal{G}$ can be calculated using theorem 3.14 above and using the characterisation of the measures in $\mathbf{M}^e$. We find

$$2\|f + g\,h\| = \sup_\mu \mathbf{E}_\mathbb{P}\left[\int_{[-1,+1]} |f + x\,h|\,\mu_\omega(dx)\right].$$

For given ω the measure $\mu_\omega(dx)$ on $[-1,+1]$ that maximises $\int_{[-1,+1]} |f + x\,h|\,\mu_\omega(dx)$ and that satisfies $\int_{[-1,+1]} x\,\mu_\omega(dx) = 0$ is according to balayage arguments (repeated application of Jensen's inequality) the measure that gives mass $1/2$ to both -1 and $+1$. This measure does not satisfy the requirements since it is not equivalent to the measure m on $[-1,1]$. But an easy approximation arguement shows nevertheless that

$$2\|f + g\,h\| = \mathbf{E}_{\mathbb{P}}\left[\frac{|f+h| + |f-h|}{2}\right].$$

This can be rewritten as

$$2\|f + g\,h\| = \mathbf{E}_{\mathbb{P}}\left[\max\left(|f|\,,|h|\right)\right].$$

This equality shows that $\mathcal{G}$ is isomorphic to an L^1–space.

5. The value of maximal admissible contingent claims on the set $\mathbf{M}^e$.

As shown in example 4.4, maximal contingent claims f may have different expected values for different measures in $\mathbf{M}^e$. In [3] we showed that under rather general conditions such a phenomenon is generic for incomplete markets. More precisely we have

Theorem 5.1. *([3], theorem 3.1) Suppose that S is a continuous d-dimensional semi-martingale with the NFLVR property. If there is a continuous local martingale W such that $\langle W, S\rangle = 0$ but $d\langle W, W\rangle$ is not singular to $d\langle S, S\rangle$, then for each R in $\mathbf{M}^e$, there is a maximal contingent claim $f \in \mathcal{K}_1$ such that $\mathbf{E}_R[f] < 0$.*

The preceding theorem brings up the question whether for given $f \in \mathcal{K}^{max}$, the set of measures $\mathbb{Q} \in \mathbf{M}^e$ such that $\mathbf{E}_{\mathbb{Q}}[f] = 0$ is big.

Theorem 5.2. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If f is a maximal contingent claim i.e. $f \in \mathcal{K}^{max}$, then the mapping*

$$\begin{array}{rccc} \phi\colon & \mathbf{M}(S) & \longrightarrow & I\!R \\ & \mathbb{Q} & \longrightarrow & \mathbf{E}_{\mathbb{Q}}[f] \end{array}$$

is lower semi-continuous for the weak topology $\sigma\left(L^1(\mathbb{P}), L^\infty(\mathbb{P})\right)$. In particular the set $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}; \mathbf{E}_{\mathbb{Q}}[f] = 0\}$ is a G_δ set (with respect to the weak and therefore also for the strong topology) in $\mathbf{M}$. Furthermore this set is convex and $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}^e; \mathbf{E}_{\mathbb{Q}}[f] = 0\}$ is strongly dense in $\mathbf{M}$. In particular as $\mathbf{M}$ is a complete metric space with respect to the strong topology of $L^1(\mathbb{P})$, the set $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}; \mathbf{E}_{\mathbb{Q}}[f] = 0\}$ is of second category.

Proof. The lower semi-continuity is a consequence of Fatou's lemma and the fact that for convex sets weak and strong closedness are equivalent.

The convexity follows from $\mathbf{E}_{\mathbb{Q}}[f] \le 0$ for every $\mathbb{Q} \in \mathbf{M}$.

By the convexity of the set $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}^e; \mathbf{E}_{\mathbb{Q}}[f] = 0\}$, it only remains to be shown that the set $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}^e; \mathbf{E}_{\mathbb{Q}}[f] = 0\}$ is norm dense in $\mathbf{M}^e$, the latter being norm dense in $\mathbf{M}$.

Take $\mathbb{Q}^0 \in \mathbf{M}^e$ such that $\mathbf{E}_{\mathbb{Q}^0}[f] = 0$. Since f is maximal such a measure exists. Since f is maximal there is a strategy H such that $H \cdot S$ is a $\mathbb{Q}^0$ uniformly integrable martingale and such that $f = (H \cdot S)_\infty$. We may suppose that the process $V = 1 + H \cdot S$ remains bounded away from zero.

Take now $\mathbb{Q} \in \mathbf{M}^e$ and let Z be the cadlag martingale defined by

$$Z_t = \mathbf{E}\left[\frac{d\mathbb{Q}}{d\mathbb{Q}^0} \mid \mathcal{F}_t\right].$$

For each n, a natural number, we define the stopping time

$$T_n = \inf\{t \mid Z_t > n\}.$$

Clearly the process VZ is a $\mathbb{Q}^0$–local martingale and being positive it is a supermartingale. Therefore we have that $V_{T_n} Z_{T_n}$ is in $L^1(\mathbb{Q}^0)$. It follows that $(VZ)^{T_n} \le nV + V_{T_n} Z_{T_n}$ and hence the process $(VZ)^{T_n}$ is a uniformly integrable martingale. Therefore $\mathbf{E}_{\mathbb{Q}^0}[V_{T_n} Z_{T_n}] = 1$ and the measure $\mathbb{Q}^n$ defined as $d\mathbb{Q}^n = Z_{T_n} d\mathbb{Q}^0$ satisfies $\mathbf{E}_{\mathbb{Q}^n}[V_{T_n}] = 1$. Since $\mathbf{E}_{\mathbb{Q}^n}[V_\infty] = \mathbf{E}_{\mathbb{Q}^n}[V_{T_n}] = 1$ we clearly have $\mathbb{Q}^n \in \{\mathbb{R} \mid \mathbb{R} \in \mathbf{M}^e; \mathbf{E}_{\mathbb{R}}[f] = 0\}$. Since $\mathbb{Q}^n$ tends to $\mathbb{Q}$ in the L^1–norm, the proof of the theorem is completed.

q.e.d.

Corollary 5.3. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $\mathcal{V}$ is a separable subspace of $\mathcal{G}$, then the convex set*

$$\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}^e; \mathbf{E}_{\mathbb{Q}}[f] = 0 \text{ for all } f \in \mathcal{V}\}$$

is dense in $\mathbf{M}$ with respect to the norm topology of $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. We may and do suppose that there is sequence of maximal contingent claims in $\mathcal{V}$, $(f_n)_{n\ge1}$ such that the sequence $\{f_n - f_m \mid n \ge 1; m \ge 1\}$ is dense in $\mathcal{V}$, occasionally we enlarge the space $\mathcal{V}$. Obviously

$$\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}; \mathbf{E}_{\mathbb{Q}}[f] = 0 \text{ for all } f \in \mathcal{V}\} = \{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}; \mathbf{E}_{\mathbb{Q}}[f_n] = 0 \text{ for all } n \ge 1\}.$$

For each n the set $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}; \mathbf{E}_{\mathbb{Q}}[f_n] = 0\}$ is a norm dense and (for the norm topology) a G_δ set in $\mathbf{M}$. Since $\mathbf{M}$ is a complete space for the L^1-norm, we may apply Baire's category theorem. Therefore the intesection over all n, $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M};$ for all n : $\mathbf{E}_{\mathbb{Q}}[f_n] = 0\}$ is still a dense G_δ set of $\mathbf{M}$. Because, by corollary 2.16, the set $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}^e;$ for all n $\mathbf{E}_{\mathbb{Q}}[f_n] = 0\}$ is non–empty, an easy argument using conex combinations yields that $\{\mathbb{Q} \mid \mathbb{Q} \in \mathbf{M}^e; \mathbf{E}_{\mathbb{Q}}[f] = 0 \text{ for all } f \in \mathcal{V}\}$ is dense in $\mathbf{M}$.

q.e.d.

Corollary 5.4. *If S is a continuous d-dimensional semi-martingale with the NFLVR property, if there is a continuous local martingale W such that $\langle W, S\rangle = 0$ but $d\langle W, W\rangle$ is not singular to $d\langle S, S\rangle$, then $\mathcal{G}$ is not a separable space.*

Proof. This follows from the previous corollary and from Theorem 5.1 above.

q.e.d.

6. The space $\mathcal{G}$ under a numéraire change.

If we change the numéraire, e.g. we change from one reference currency to another, what will happen with the space $\mathcal{G}$? Referring to [4] and especially the proofs of theorem 11 and 13 therein, we expect that there is an obvious transformation which should be the mathematical translation of the change of currency. More precisely we want the contingent claims of $\mathcal{G}$ to be multiplied with the exchange ratio between the two currencies. This section will give some precise information on this problem.

We start with the investigation of how the set of equivalent martingale mesaures is changed.

Suppose that V is a strictly positive process of the form $V = H\cdot S + 1$ where $1 + (H\cdot S)_\infty$ is strictly positive and where $(H \cdot S)_\infty$ is maximal admissible. Suppose also that the process $\frac{1}{V}$ is locally bounded. This hypothesis allows us to use, without restriction, the theory developed sofar. With each element $\mathbb{R}$ of $\mathbf{M}(S)$ we asssociate the measure $\tilde{\mathbb{R}}$ defined by $d\tilde{\mathbb{R}} = V_\infty d\mathbb{R}$. Of course this measure is not a probability measure since we do not necessarily have that $\mathbf{E}_\mathbb{R}[V_\infty] = 1$. But from theorem 5.2 above it follows however that the set $G = \{\mathbb{Q} \in \mathbf{M}(S) \mid \mathbf{E}_\mathbb{Q}[V_\infty] = 1\}$ is a dense G_δ set of $\mathbf{M}(S)$. Likewise the set $\tilde{G} = \left\{\tilde{\mathbb{Q}} \in \mathbf{M}\left(\frac{S}{V}, \frac{1}{V}\right) \mid \mathbf{E}_{\tilde{\mathbb{Q}}}[\frac{1}{V_\infty}] = 1\right\}$ is a dense G_δ set of $\mathbf{M}\left(\frac{S}{V}, \frac{1}{V}\right)$. The following theorem is obvious.

Theorem 6.1. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. With the above notations, the relation $d\tilde{\mathbb{R}} = V_\infty d\mathbb{R}$, defines a bijection between the sets G and $\tilde{G}$.*

In the following theorem we make use of the notation introduced in theorem 3.2. The space $\mathcal{G}(S)$ is the space of workable contingent claims that is constructed with the d–dimensional process S, the space $\mathcal{G}\left(\frac{S}{V}, \frac{1}{V}\right)$ is the space of workable contingent claims constructed with the $d+1$ dimensional process $\left(\frac{S}{V}, \frac{1}{V}\right)$.

Theorem 6.2. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. Suppose that V is a strictly positive process of the form $V = H \cdot S + 1$ where $1 + (H \cdot S)_\infty$ is strictly positive and where $(H \cdot S)_\infty$ is maximal admissible. Suppose that the process $\frac{1}{V}$ is locally bounded. The mapping*

$$\begin{array}{rccc} \varphi\colon & \mathcal{G}(S) & \longrightarrow & \mathcal{G}(\frac{S}{V}, \frac{1}{V}) \\ & g & \longrightarrow & \frac{g}{V_\infty} \end{array}$$

defines an isometry between $\mathcal{G}(S) = \mathcal{G}(S, 1)$ *and* $\mathcal{G}(\frac{S}{V}, \frac{1}{V})$.

Proof. Suppose $V = H \cdot S + 1$ where $1 + (H \cdot S)_\infty$ is strictly positive and where $(H \cdot S)_\infty$ is maximal admissible. Take an admissible, with respect to the process S, strategy K. The process $\frac{K \cdot S}{V}$ is the outcome of the strategy $K' = (K, (K \cdot S)_- - KS_-)$, , see also the remark following definition 2.8 above and [4]. From theorem 2.5 above it follows that there is an element $\mathbb{Q} \in \mathbf{M}^e$ such that $\mathbf{E}_\mathbb{Q}[(K \cdot S)_\infty] = 0$ and such that $\mathbf{E}_\mathbb{Q}[V_\infty] = 1$. The measure $\tilde{\mathbb{Q}}$ defined as $d\tilde{\mathbb{Q}} = V_\infty \, d\mathbb{Q}$ is therefore an element of $\mathbf{M}\left(\frac{S}{V}, \frac{1}{V}\right)$ such that $\mathbf{E}_{\mathbb{Q}'}[\frac{K \cdot S}{V_\infty}] = 0$. It follows that the contingent claim $\frac{1}{V_\infty} - 1$ is maximal and admissible for the process $\left(\frac{S}{V}, \frac{1}{V}\right)$ and hence the contingent claim $\frac{K \cdot S}{V_\infty}$ is workable. It follows that the mapping φ maps $\mathcal{K}^{max}$, and hence also $\mathcal{G}(S)$, into $\mathcal{G}\left(\frac{S}{V}, \frac{1}{V}\right)$.

If we apply the numéraire $\frac{1}{V}$ to the system $\left(\frac{S}{V}, \frac{1}{V}\right)$ we find the $d+1$–dimensional process (S, V). However because V is given by a stochastic integral with respect to S, we have that $\mathcal{G}(S, V) = \mathcal{G}(S)$. It follows that the mapping that associates with each element $k \in \mathcal{G}\left(\frac{S}{V}, \frac{1}{V}\right)$, the element kV_∞ maps $\mathcal{G}\left(\frac{S}{V}, \frac{1}{V}\right)$ into $\mathcal{G}(S)$. The mapping φ is clearly bijective.

Let $G = \{\mathbb{Q} \in \mathbf{M}(S) \mid \mathbf{E}_\mathbb{Q}[V_\infty] = 1\}$ and $\tilde{G} = \left\{\tilde{\mathbb{Q}} \in \mathbf{M}\left(\frac{S}{V}, \frac{1}{V}\right) \mid \mathbf{E}_{\tilde{\mathbb{Q}}}[\frac{1}{V_\infty}] = 1\right\}$. S ince both sets are dense in respectively $\mathbf{M}(S)$ and $\mathbf{M}\left(\frac{S}{V}, \frac{1}{V}\right)$, it is clear that for every element $g \in \mathcal{G}(S)$,

$$\begin{aligned}
2\|g\| &= \sup\left\{\mathbf{E}_\mathbb{Q}[|g|] \mid \mathbb{Q} \in \mathbf{M}(S)\right\} \\
&= \sup\left\{\mathbf{E}_\mathbb{Q}[|g|] \mid \mathbb{Q} \in G\right\} \\
&= \sup\left\{\mathbf{E}_{\tilde{\mathbb{Q}}}\left[\frac{|g|}{V_\infty}\right] \mid \tilde{\mathbb{Q}} \in \tilde{G}\right\} \\
&= \sup\left\{\mathbf{E}_{\tilde{\mathbb{Q}}}\left[\frac{|g|}{V_\infty}\right] \mid \tilde{\mathbb{Q}} \in \mathbf{M}\left(\frac{S}{V}, \frac{1}{V}\right)\right\} = 2\|\frac{g}{V_\infty}\|
\end{aligned}$$

This shows that φ is also an isometry.

q.e.d.

Remark. The previous theorem shows that $\mathcal{G}$ is a numéraire invariant space provided we only accept numéraire changes induced by maximal admissible contingent claims.

7. The closure of $\mathcal{G}^\infty$ and related problems.

In this section we will study the contingent claims of $\mathcal{K}^{max}$ that are in the closure of $\mathcal{G}^\infty$. The characterisation is done using either uniform convergence over the set $\mathbf{M}^e$ or using the set $\mathbf{M}^{ba}$. Before we start the programme, we first recall some notions from integration theory with respect to finitely additive measures; we refer to Dunford-Schwartz [7] for details.

Let μ be a finitely additive measure that is in $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$. A measurable function f (we continue to identify functions that are equal $\mathbb{P}$ a.s.), defined on Ω is called μ–measurable if for each $\varepsilon > 0$ there is a bounded measurable function g such that $\mu\{\omega \mid |f(\omega) - g(\omega)| > \varepsilon\} < \varepsilon$. The reader can check that since $\mathcal{F}$ is a sigma–algebra, this definition coincides with the definition 10, p106 in [7]. We say that a μ–measurable function f is μ–integrable if and only if there is sequence $(g_n)_{n\geq 1}$ of bounded measurable functions such that g_n converges in μ–measure to f and such that $\mathbf{E}_\mu[|g_n - g_m|]$ tends to zero if n, m tend to ∞. In this case one defines $\mathbf{E}_\mu[f] = \lim_{n\to\infty} \mathbf{E}_\mu[g_n]$ as the μ–integral $\mathbf{E}_\mu[f]$ of f. In case f is bounded from below the μ–integrability of f implies via the dominated convergence theorem, valid also for finitely additive measures, that $\mathbf{E}[f - f \wedge n]$ tends to zero as n tends to ∞. Contingent claims g of $\mathcal{G}^\infty$ are μ–integrable for all $\mu \in \mathbf{M}^{ba}$ and moreover we trivially have $\mathbf{E}_\mu[g] = 0$ since $\mathbf{E}_\mathbb{Q}[g] = 0$ for all $\mathbb{Q} \in \mathbf{M}^e$.

Proposition 7.1. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If $f \in \mathcal{K}^{max}$ and $\mu \in \mathbf{M}^{ba}$, then f is μ–integrable and $\mathbf{E}_\mu[f] \leq 0$. Also $\mu[f \geq n] \leq \frac{4\|f\|}{n}$, a uniform bound over $\mu \in \mathbf{M}^{ba}$. In particular for each $\mu \in \mathbf{M}^{ba}$ and each $f \in \mathcal{K}^{max}$ we find that $f \wedge n$ tends to f in μ–measure and $\mathbf{E}_\mu[f \wedge n]$ tends to $\mathbf{E}_\mu[f]$ as n tends to infinity.*

Proof. We only have to prove the statement for contingent claims f that are 1–admissible and maximal. So suppose that f is such a contingent claim. By the optional stopping theorem, or by the maximal inequality for supermartingales, we find that for all $\mathbb{Q} \in \mathbf{M}^e$, we have that $\mathbb{Q}[f \geq n] \leq \frac{1}{n}$. The set $\mathbf{M}^e$ is $\sigma(\mathbf{ba}, L^\infty)$ dense in $\mathbf{M}^{ba}$ (see [2] remark 5.10), hence we obtain that $\mu[f \geq n] \leq \frac{1}{n}$ for all n. Since $\mu[f - f \wedge n > 0] \leq \mu[f \geq n] \leq \frac{1}{n}$, the measurability follows for functions f that are 1–admissible and maximal. The general case follows by splitting f as $f = g - h$ where each g and h are $\|f\|$–admissible and by the fact that $\{|f| > n\} \subset \{|g| > \frac{n}{2}\} \cup \{|h| > \frac{n}{2}\}$.

To see that for $\mu \in \mathbf{M}^{ba}$, the integral $\mathbf{E}_\mu[f]$ exists and is negative, let us first observe that for all n and all $\mathbb{Q} \in \mathbf{M}^e$ we have that $\mathbf{E}_\mathbb{Q}[f \wedge n] \leq 0$. This implies that for all n, necessarily, $\mathbf{E}_\mu[f \wedge n] \leq 0$. The sequence $\mathbf{E}_\mu[f \wedge n]$ is increasing and bounded above, so it converges and since $f \wedge n$ tends to f in μ–measure the $\lim \mathbf{E}_\mu[f \wedge n]$ is necessarily the integral of f with respect to μ. It follows that also $\mathbf{E}_\mu[f] \leq 0$.

q.e.d.

In the same style we can prove that $f \in \mathcal{K}^{max}$ is the limit of a sequence obtained by stopping. If f is of the form $f = (H \cdot S)_\infty$ for some S–integrable admissible process H, let for $n \geq 1$:

$$T_n = \inf\{t \mid (H \cdot S)_t > n\}.$$

Proposition 7.2. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If f is 1–admissible and maximal and if $\mu \in \mathbf{M}^{ba}$, then f_{T_n} tends to f in μ–measure.*

Proof. Simply remark that for each $\mathbb{Q} \in \mathbf{M}^e$, we have $\mathbb{Q}[T_n < \infty] \le \frac{1}{n}$.

q.e.d.

Theorem 7.3. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. If f is in the closure $\overline{\mathcal{G}^\infty}$ of $\mathcal{G}$, then $\mathbf{E}_\mu[f] = 0$ for each $\mu \in \mathbf{M}^{ba}$.*

Proof. Take $(f^n)_{n\ge1}$ a sequence of bounded contingent claims in $\mathcal{G}$ that tends to f for the topology of $\mathcal{G}$. This means that $\sup\{\|f - f^n\|_{L^1(\mathbb{Q})} \mid \mathbb{Q} \in \mathbf{M}^e\}$ tends to zero. In particular the sequence $(f^n)_{n\ge1}$ is a Cauchy sequence in $\mathcal{G}$ and hence for all $\mu \in \mathbf{M}^{ba}$ we have that $\mathbf{E}_\mu[|f^n - f^m|]$ tends to zero as n, m tend to infinity. Since, as easily seen, the sequence $(f^n)_{n\ge1}$ tends to f in μ–measure, we obtain that f is μ–integrable and $\mathbf{E}_\mu[f] = \lim_{n\to\infty} \mathbf{E}_\mu[f^n] = 0$.

q.e.d.

Proposition 7.4. *Suppose that S is a locally bounded semi-martingale that satisfies the $NFLVR$ property. Suppose $f \in \mathcal{K}^{max}$ and $f = (H \cdot S)_\infty$ for a maximal strategy H. If for each $\mu \in \mathbf{M}^{ba}$ the function f satisfies $\mathbf{E}_\mu[f] = 0$, then for each stopping time T and each $\mu \in \mathbf{M}^{ba}$, the function f_T is μ–integrable and satisfies $\mathbf{E}_\mu[f_T] = 0$.*

Proof. We already showed that f_T is in $\mathcal{G}$ and hence is μ–integrable for all $\mu \in \mathbf{M}^{ba}$ and that $\mathbf{E}_\mu[f_T] \le 0$ for all μ in $\mathbf{M}^{ba}$.

Let us prove the opposite inequality. The sequence $\mathbf{E}_\mu[f \wedge n]$ of continuous functions on $\mathbf{M}^{ba}$ tends increasingly to 0. As follows from Dini's theorem, we have that for each $\delta > 0$ there is a number n such that $\mathbf{E}_\mathbb{Q}[f \wedge n] > -\delta$ for all $\mathbb{Q} \in \mathbf{M}^e$. But for each $\mathbb{Q} \in \mathbf{M}^e$ we have that $\mathbf{E}_\mathbb{Q}[f \mid \mathcal{F}_T] = f_T$ and hence that $\mathbf{E}_\mathbb{Q}[f \wedge n \mid \mathcal{F}_T] \le f_T \wedge n$. This implies that for all $\mathbb{Q} \in \mathbf{M}^e$ and for all n large enough, we have $\mathbf{E}_\mathbb{Q}[f_T \wedge n] > -\delta$. We therefore obtain that $\mathbf{E}_\mu[f_T] \ge 0$. Since the converse inequality was already shown we obtain $\mathbf{E}_\mu[f_T] = 0$.

q.e.d.

The converse of theorem 7.3 is less trivial and we need the extra assumption that S is continuous.

Theorem 7.5. *Suppose that S is continuous and satisfies the $NFLVR$ property. Suppose that $f \in \mathcal{K}^{max}$ and suppose also that for each $\mu \in \mathbf{M}^{ba}$ we have $\mathbf{E}_\mu[f] = 0$, then $f \in \overline{\mathcal{G}^\infty}$*

Proof. Let H be a maximal acceptable strategy such that $(H \cdot S)_\infty = f$. For each $n \ge 1$ put $T_n = \inf\{t \mid |(H \cdot S)_t| > n\}$ the first time the process $H \cdot S$ exits the interval $[-n, +n]$. Clearly $f^n = (H \cdot S)_{T_n}$ defines a sequence in $\mathcal{G}^\infty$ and we will show that f^n tends to f in the topology of $\mathcal{G}$. Because $-\mathbf{E}_\mu[f \wedge n]$ tends decreasingly to 0 for n tending to infinity we infer from Dini's theorem and theorem 5.2 that $\inf\{\mathbf{E}_\mu[f \wedge n] \mid \mu \in \mathbf{M}^{ba}\}$ tends to zero. It follows that $\sup\{\mathbf{E}_\mathbb{Q}[f - f \wedge n] \mid \mathbb{Q} \in \mathbf{M}^e\}$

tends to zero as n tends to infinity. Because $(f - f^n)^+ = (f - n)^+ = (f - f \wedge n)$ we see that also $\sup\{\mathbf{E}_{\mathbb{Q}}[(f - f^n)^+] \mid \mathbb{Q} \in \mathbf{M}^e\}$ tends to zero as n tends to infinity. By theorem 3.13 this means that f^n tends to f for the norm on $\mathcal{G}$.

q.e.d.

Remark. The continuity assumption was only needed to obtain bounded contingent claims and could be replaced by the assumption that the jumps of $H \cdot S$ were bounded.

Example 4.4, addendum In the following corollary we use the same notation as in section 4, example 4.4 and Theorem 4.7. Recall that the contingent claim $f = \sum_1^\infty f_n$ satisfies $\mathbf{E}_{\mathbb{Q}}[f] = 0$ for all $\mathbb{Q} \in \mathbf{M}$.

Corollary 7.6. *The function $f = \sum_1^\infty f_n$ is in $\mathcal{K}^{max}$ but its integral with respect to $\mu \in \mathbf{M}^{ba}$ is not always zero.*

Proof. Indeed if it were, then f would be in $\overline{\mathcal{G}^\infty}$.

q.e.d.

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