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Navier-Stokes Equations and Forward-Backward Stochastic Differential Systems

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Abstract In the paper, we consider a special coupled forward-backward stochastic differential system (FBSDS) which is associated to the viscous incompressible Navier-Stokes equation and provides a probabilistic solution to the latter via the Feynman-Kac formula. With a probabilistic method, we first prove the existence and uniqueness of the solution to the FBSDS. Then under the same conditions, we verify that this solution leads to a unique local strong solution to the associated Navier-Stokes equation, and for both cases of the small Reynolds number and dimension two, we further give the global strong solutions. By truncating the time interval of the FBSDS, we approximate the Navier-Stokes equation by a new class of FBSDSs and associated partial differential equations. Basing on the relationship between our FBSDS and the Navier-Stokes equation, we also derive a probabilistic Lagrangian representation for the velocity field, which is analogous to the formulas given by Constantin and Iyer (Commun. Pure Appl. Math. LXI: 0330–0345, 2008) and Zhang (Probab. Theory Relat. Fields 148: 305–332, 2010) with the Lagrangian approach.

Keywords forward-backward stochastic differential system · Navier-Stokes equation · Feynman-Kac formula · strong solution

Mathematics Subject Classification (2000) 60H30 · 35Q30 · 76D06

1 Introduction

The standard deterministic Navier-Stokes equation describes the evolution of the velocity field of an incompressible, viscous fluid moving in a domain of $\mathbb{R}^d$ ($d = 2$ or 3 throughout this work), and takes the following form:

$$\begin{cases} \partial_t u - \dfrac{\nu}{2}\Delta u + (u \cdot \nabla)u + \nabla p + \bar{f} = 0, \quad t \geq 0; \\ \nabla \cdot u = 0, \quad u(0) = u_0, \end{cases} \tag{1}$$

where u is the d-dimensional velocity field of a fluid, p is the pressure field, $\nu \in (0, \infty)$ is the viscosity coefficient, and $\bar{f}$ is the external force which, without any loss of generality, is taken to be

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divergence free. Let $T \in (0,\infty)$ be a real sufficiently big number. If (u,p) solves the initial Cauchy problem (1), then $(\widetilde{u},\widetilde{p})$ defined by the following time-reversing transformation

$$\widetilde{u}(t,x) = -u(T-t,x), \quad \widetilde{p}(t,x) = p(T-t,x), \quad f(t,x) = \bar{f}(T-t,x), \quad \text{for } t \le T,$$

solves the following *terminal* Cauchy problem:

$$\begin{cases} \partial_t \widetilde{u} + \frac{\nu}{2}\Delta\widetilde{u} + (\widetilde{u}\cdot\nabla)\widetilde{u} + \nabla\widetilde{p} + f = 0, & t \le T; \\ \nabla\cdot\widetilde{u} = 0, \quad \widetilde{u}(T) = G := -u_0, \end{cases} \tag{2}$$

which is also called (*backward*) Navier-Stokes equation due to its equivalence to the former.

For the Navier-Stokes equation (1) or (2), the uniqueness of the weak solutions and the regularity of the solutions remain to be important open problems. Charles Fefferman in his celebrated article [22] finally commented, "Standard methods from PDE appear inadequate to settle the problem. Instead, we probably need some deep, new ideas." In the paper, we develop a probabilistic methodology to study the Navier-Stokes equation. More precisely, the Navier-Stokes equation (2) is associated to the following coupled forward-backward stochastic differential system (FBSDS):

$$\begin{cases} dX_s(t,x) = Y_s(t,x)\,ds + \sqrt{\nu}\,dW_s, \quad s\in[t,T]; \quad X_t(t,x) = x; \\ -dY_s(t,x) = \left[f(s,X_s(t,x)) + \widetilde{Y}_0(s,X_s(t,x))\right] ds - \sqrt{\nu}Z_s(t,x)\,dW_s; \\ Y_T(t,x) = G(X_T(t,x)); \\ -d\widetilde{Y}_s(t,x) = \displaystyle\sum_{i,j=1}^{d} \frac{27}{2s^3} Y_t^i Y_t^j(t,x+B_s)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)\left(B^j_s - B^j_{\frac{2s}{3}}\right) B_{\frac{s}{3}}\,ds - dM_s, \ s\in(0,\infty); \\ \widetilde{Y}_\infty(t,x) = 0, \end{cases} \tag{3}$$

where, B and W are two independent d-dimensional standard Brownian motions, Y and $\widetilde{Y}$ satisfy backward stochastic differential equations and X satisfies a forward one. The drift part of $\{Y_s(t,x), s\in[t,T]\}$ (see the third equality of the FBSDS (3)) at time s depends on $\widetilde{Y}_0$, and that of $\{\widetilde{Y}_s(t,x), s\in[t,T]\}$ depends on $Y_t(t,x+B_s)$, which make our system (3) different from the conventional coupled forward-backward stochastic differential equations (FBSDEs) (see [1,25,33,38, 41,39,47]). Furthermore, both backward stochastic differential equations (BSDEs) in the FBSDS (3) are defined on two different time-horizons $[t,T]$ and $(0,\infty)$. The H^m-solution ($m > d/2$, see Definition 2) $(X,Y,Z,\widetilde{Y}_0)$ to the FBSDS (3) will be connected to the strong solution $(\widetilde{u},\widetilde{p})$ to the Navier-Stokes equation (2) in the following manner:

$$Y_s(t,x) = \widetilde{u}(s,X_s(t,x)),\ Z_s(t,x) = \nabla\widetilde{u}(s,X_s(t,x)),\ \text{and } \widetilde{Y}_0(t,x) = \nabla\widetilde{p}(t,x),$$

for $(s,x)\in[t,T]\times\mathbb{R}^d$. For more details on the connections between the FBSDS (3) and the Navier-Stokes equation (2), the reader can skip to Theorem 5 in Section 3.3.

The FBSDEs have been connected to a system of semi-linear parabolic partial differential equations (PDEs) (see among many others [1,25,33,38,39,47]). However, Navier-Stokes equation (2) usually goes beyond that context, as it has the nonlocal constraint $\nabla\cdot\widetilde{u} = 0$. Our difficulty is two-fold: one is nonlinearity and the other is the nonlocal constraint. To attack the divergence-free constraint, we introduce the last BSDE in the infinite time interval. Indeed, taking divergence on the Navier-Stokes equation (2), we get by the free divergence condition

$$0 = \partial_t(\nabla\cdot\widetilde{u}) + \frac{\nu}{2}\Delta(\nabla\cdot\widetilde{u}) + \nabla\cdot(\widetilde{u}\cdot\nabla)\widetilde{u} + \Delta\widetilde{p} + \nabla\cdot f = \text{div div } (\widetilde{u}\otimes\widetilde{u}) + \Delta\widetilde{p},$$

which yields the following representation for the pressure field

$$\nabla\widetilde{p} = -\nabla\Delta^{-1}\text{div div } (\widetilde{u}\otimes\widetilde{u}).$$

Thus, the Navier-Stokes equations (2) can be written into the following form

$$\begin{cases} \partial_t\widetilde{u} + \frac{\nu}{2}\Delta\widetilde{u} + (\widetilde{u}\cdot\nabla)\widetilde{u} - \nabla\Delta^{-1}\text{div div } (\widetilde{u}\otimes\widetilde{u}) + f = 0, & t\le T; \\ \widetilde{u}(T) = G. \end{cases}$$

To give a probabilistic representation for the nonlocal operator $\nabla\Delta^{-1}\mathrm{div}\ \mathrm{div}$, we construct the last BSDE in the infinite time interval.

To concentrate our attention on the primary connection between the FBSDS (3) and Navier-Stokes equation (2), the FBSDS (3) is discussed from an analytic point of view only in a Hilbert space in this work, on basis of which we study further the connections in Hölder spaces in the related work [16]. By truncating the time interval of the FBSDS (3), we approximate the Navier-Stokes equation by a new class of FBSDSs and associated PDEs. Moreover, basing on the relationship between our FBSDS and the Navier-Stokes equation, we also derive a probabilistic Lagrangian representation for the velocity field, which is analogous to the formulas given by [10,48] with the Lagrangian approach.

There is a long history on the formalisms to represent solutions of PDEs as the expected functionals of stochastic processes. We only mention here those concerning a deterministic incompressible Navier-Stokes equation. The velocity field is related to the vorticity field in a linear fashion by the Biot-Savart law, and moreover, the analysis of the vorticity field is fundamental to the issues like the possible emergence of singularities (for instance, see [3,34]). For the two-dimensional case, since the vorticity obeys a Fokker-Planck type parabolic PDE, the random vortex method was formulated by Chorin [9] who used random walks and a particle limit to represent the vorticity field, and Busnello [7] used the Girsanov transformation to give a probabilistic representation of the vorticity field. The latter work was further extended by Busnello, Flandoli and Romito [8] to the three-dimensional case, where the vorticity field turns out to satisfy a parabolic PDE with an additional stretching term. As noted by Busnello, Flandoli and Romito [8], a partially similar representation formula for the vorticity field of the three-dimensional Navier-Stokes equations had been given before by Esposito et al. [20,19] but without the probabilistic representation for the Biot-Savart law. Note that a probabilistic interpretation for the Biot-Savart law was given by Busnello, Flandoli and Romito [8] and Busnello [7], where the Bismut-Elworthy-Li formula is used so that the velocity can be recovered from the vorticity through probabilistic approaches. Le Jan and Sznitman [29] interpreted the Fourier transformation of the Laplacian of the three-dimensional velocity field in terms of a backward branching process and a composition rule along the associated tree, and got a new existence theorem, and their approach was extensively studied and generalized by others (see, for instance [5,36]).

Recently, Constantin and Iyer [10,11] and Iyer [26–28] derived a stochastic representation for the incompressible Navier-Stokes equations based on stochastic Lagrangian paths and gave a self-contained proof of the existence. Later, Zhang [48] considered a backward analogue and provided short elegant proofs for the classical existence results. In fact, basing on the relationship between our FBSDS and the Navier-Stokes equation, we derive a probabilistic Lagrangian representation for the velocity field, which is analogous to the formulas given by [10,11,26–28,48] with the Lagrangian approach. We also mention that Cruzeiro and Shamarova [12] established a connection between the strong solution to the spatially periodic Navier-Stokes equations and a solution to a system of FBSDEs on the group of volume-preserving diffeomorphisms of a flat torus, and that Qiu, Tang and You [42] considered a similar non-Markovian FBSDS to ours (3) in the two-dimensional spatially periodic case, and studied the well-posedness of the corresponding backward stochastic PDEs. The list of literature on probabilistic approaches to the Navier-Stokes equations is not exhausted here, and there are many others.

The rest of this paper is organized as follows. In Section 2, we introduce notations and functional spaces, and recall auxiliary results. In Section 3, a lemma on equivalent norms and the definition of the solution to the FBSDS (3) are given first and then the FBSDS is connected to the Navier-Stokes Equation, which constitutes our main result of Theorem 5. In Section 4, we prove Theorem 5 and before the proof we present several auxiliary results which include the connection between the coupled FBSDEs and the PDEs of Burgers type. In Section 5, we give global existence of the solution for the small Reynolds number and the two-dimensional cases, respectively. By truncating the time interval of the FBSDS, we approximate the Navier-Stokes equation by a new class of FBSDSs. By truncating the time interval of the FBSDS, we approximate in Section 6 the Navier-Stokes equation by a new class of FBSDSs and associated PDEs. In Section 7, basing on the relationship between our FBSDS and the Navier-Stokes equation, we also derive a probabilistic Lagrangian representation for the velocity field, which is analogous to the formulas with the Lagrangian approach. Finally in Section 8 as an appendix, we prove Lemmas 3 and 6.

2 Preliminaries

Let $(\Omega, \bar{\mathscr{F}}, \{\bar{\mathscr{F}}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space on which are defined two d-dimensional standard Brownian motion $W = \{W_t : t \in [0,\infty)\}$ and $B = \{B_t : t \in [0,\infty)\}$ such that $\{\bar{\mathscr{F}}_t\}_{t\geq 0}$ is the natural filtration generated by W and B, and augmented by all the $\mathbb{P}$-null sets in $\bar{\mathscr{F}}$. By $\{\mathscr{F}\}_{t\geq 0}$ and $\{\mathscr{F}^B\}_{t\geq 0}$, we denote the natural filtration generated by W and B respectively, and they are both augmented by all the $\mathbb{P}$-null sets. $\mathscr{P}$ is the σ-Algebra of the predictable sets on $\Omega \times [0,T]$ associated with $\{\mathscr{F}_t\}_{t\geq 0}$.

$\mathbb{Z}$ is the set of all the integers and $\mathbb{N} = \mathbb{Z}^+$ denotes the set of the positive integers. Denote by $|\cdot|$ (respectively, $\langle \cdot, \cdot \rangle$ or $\cdot$) the norm (respectively, scalar product) in finite-dimensional Hilbert space such as $\mathbb{R}, \mathbb{R}^k, \mathbb{R}^{k\times l}$ where k, l are positive integers and

$$|x| := \left(\sum_{i=1}^k x_i^2 \right)^{1/2} \quad \text{and} \quad |y| := \left(\sum_{i=1}^k \sum_{j=1}^l y_{ij}^2 \right)^{1/2} \quad \text{for } (x,y) \in \mathbb{R}^k \times \mathbb{R}^{k\times l}.$$

For each Banach space $(\mathcal{X}, \|\cdot\|_X)$ and real $q \in [1,\infty]$, we denote by $S^q([t,\tau];\mathcal{X})$ the set of $\mathcal{X}$-valued, $\mathscr{F}_t$-adapted and càdlàg processes $\{X_s\}_{s\in[t,\tau]}$ such that

$$\|X\|_{S^q([t,\tau];\mathcal{X})} := E\Big[\sup_{s\in[t,\tau]} \|X_s\|_{\mathcal{X}}^q \Big]^{1/q} < \infty.$$

$L^q_{\mathscr{F}}(t,\tau;\mathcal{X})$ denotes the set of (equivalent classes of) $\mathcal{X}$-valued predictable processes $\{X_s\}_{s\in[t,\tau]}$ such that

$$\|X\|_{L^q_{\mathscr{F}}(t,\tau;\mathcal{X})} := E\Big[\int_t^\tau \|X_s\|_{\mathcal{X}}^q \, ds \Big]^{1/q} < \infty.$$

Both $\big(S^q([t,\tau];\mathcal{X}), \|\cdot\|_{S^q([t,\tau]\mathcal{X})}\big)$ and $\Big(L^q_{\mathscr{F}}(t,\tau;\mathcal{X}), \|\cdot\|_{L^q_{\mathscr{F}}(t,\tau;\mathcal{X})}\Big)$ are Banach spaces.

Define the set of multi-indices

$$\mathcal{A} := \{\alpha = (\alpha_1, \cdots, \alpha_d) : \alpha_1, \cdots, \alpha_d \text{ are nonnegative integers}\}.$$

For any $\alpha \in \mathcal{A}$ and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, denote

$$|\alpha| = \sum_{i=1}^d \alpha_i, \; x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \; D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}.$$

For differentiable transformations ϕ, ψ on $\mathbb{R}^d$, define the Jacobi matrix $\nabla \phi$ of ϕ:

$$\nabla \phi = \begin{pmatrix} \partial_{x_1}\phi^1, \partial_{x_2}\phi^1, \cdots, \partial_{x_d}\phi^1 \\ \partial_{x_1}\phi^2, \partial_{x_2}\phi^2, \cdots, \partial_{x_d}\phi^2 \\ \cdots, \;\cdots, \;\cdots, \;\cdots \\ \partial_{x_1}\phi^d, \partial_{x_2}\phi^d, \cdots, \partial_{x_d}\phi^d \end{pmatrix}$$

whose transpose is denoted by $\nabla^{\mathcal{T}} \phi$, the divergence $\text{div}\phi = \nabla \cdot \phi$, and the matrix

$$\phi \otimes \psi = \begin{pmatrix} \phi^1\psi^1, \phi^1\psi^2, \cdots, \phi^1\psi^d \\ \phi^2\psi^1, \phi^2\psi^2, \cdots, \phi^2\psi^d \\ \cdots, \;\cdots, \;\cdots, \;\cdots \\ \phi^d\psi^1, \phi^d\psi^2, \cdots, \phi^d\psi^d \end{pmatrix}.$$

Now we extend several spaces of real-valued functions to those of vector-valued functions. For a positive integer number l and k, we denote by $C_c^\infty(\mathbb{R}^l; \mathbb{R}^k)$ (respectively, $C_c^\infty(\mathcal{O}; \mathbb{R}^k)$ for each open set $\mathcal{O} \subset \mathbb{R}^l$) the set of all infinitely differentiable $\mathbb{R}^k$-valued functions with compact supports on $\mathbb{R}^l$ ($\mathcal{O}$, respectively) and by $\mathscr{D}'(\mathbb{R}^l;\mathbb{R}^k)$ the totality of all the $\mathbb{R}^k$-valued general functions with each component being Schwartz distribution. For simplicity, we write C_c^∞ and $\mathscr{D}'$ for the case $l = k = d$. On $\mathbb{R}^d$ we denote by $\mathscr{S}$ ($\mathscr{S}'$, respectively) the set of all the $\mathbb{R}^d$-valued functions whose elements are Schwartz functions (tempered distributions, respectively). We shall denote by $(\cdot,\cdot)$ not only the duality between C_c^∞ and $\mathscr{D}'$ but also the duality between $\mathscr{S}$ and $\mathscr{S}'$. Then the Fourier transform $\mathcal{F}(f)$ of $f \in \mathscr{S}$ is given by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp\left(-\sqrt{-1}\langle x,\, \xi\rangle\right) f(x)\, dx, \quad \xi \in \mathbb{R}^d,$$

and the inverse Fourier transform $\mathcal{F}^{-1}(f)$ is given by

$$\mathcal{F}^{-1}(f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \exp\left(\sqrt{-1}\langle x,\, \xi\rangle\right) f(\xi)\, d\xi, \quad x \in \mathbb{R}^d.$$

Extended to the general function space $\mathscr{S}'$, the Fourier transform defines an isomorphism from $\mathscr{S}'$ onto itself. As usual, for each $s \in \mathbb{R}$ and $f \in \mathscr{S}'$, we denote the Bessel potential $I_s(f) := (1-\Delta)^{s/2} f = \mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\mathcal{F}(f)(\xi))$.

For each positive integer l, $1 \le q \le \infty$ and $m = 0, 1, \ldots$ by $L^q(\mathbb{R}^l)$ and $H^{m,q}(\mathbb{R}^l)$ (L^q and $H^{m,q}$, with a little notional abuse), we denote the usual $\mathbb{R}^l$-valued Lebesgue and Sobolev spaces on $\mathbb{R}^d$, respectively. $H^{m,q}$ is equipped with the norm:

$$\|\phi\|_{m,q} := \begin{cases} \left(\|\phi\|_{L^q}^q + \sum_{|\alpha|=1}^m \|D^\alpha \phi\|_{L^q}^q\right)^{1/q}, \ \phi \in H^{m,q}, q \in [1,\infty); \\ \|\phi\|_{L^\infty} + \sum_{|\alpha|=1}^m \|D^\alpha \phi\|_{L^\infty}, \quad \phi \in H^{m,\infty}, \end{cases}$$

which is equivalent to the norm:

$$\|\phi\|_{m,q} := \|(1-\Delta)^{\frac{m}{2}} \phi\|_{L^q}, \ \phi \in H^{m,q}, \quad \text{for } q \in (1,\infty).$$

Both norms will not be distinguished unless there is a confusion. Furthermore, using the Bessel potentials, we define the Soblev space $H^{m,q} := I_{-m}(L^q)$ for $m \in \mathbb{Z} \setminus (0 \cup \mathbb{N})$ and $q \in (1,\infty)$. In particular, for the case of $q = 2$, $H^{m,2}$ is a Hilbert space with the inner product:

$$\langle \phi,\, \psi\rangle_m := \int_{\mathbb{R}^d} \langle I_{m/2}\phi(x),\, I_{m/2}\psi(x)\rangle\, dx, \ \phi, \psi \in H^{m,2}.$$

We define the duality between $H^{s,q}$ and $H^{r,q'}$ for $q \in (1,\infty)$ and $q' = q/(q-1)$ as:

$$\langle \phi,\, \psi\rangle_{s,r} := \int_{\mathbb{R}^d} \langle I_{s/2}\phi(x),\, I_{r/2}\psi(x)\rangle\, dx, \ \phi \in H^{s,q}, \psi \in H^{r,q'}.$$

For simplicity, we write the space H^m and the norm $\|\cdot\|_m$ for $H^{m,2}$ and $\|\cdot\|_{m,2}$, respectively.

Define

$$\mathscr{D}_\sigma := \{\phi \in C_c^\infty : \ \nabla \cdot \phi = 0\}.$$

Denote by $H_\sigma^{m,q}$ the completion of $\mathscr{D}_\sigma$ under the norm $\|\cdot\|_{m,q}$, which is a Banach subspace of $H^{m,q}$.

Now we introduce several spaces of continuous functions. For each positive integer l, nonnegative integer k and domain $\mathcal{O} \subset \mathbb{R}^d$, we denote by $C(\mathcal{O},\mathbb{R}^l)$, $C^k(\mathcal{O},\mathbb{R}^l)$ and $C^{k,\delta}(\mathcal{O},\mathbb{R}^l)$ with $\delta \in (0,1)$ the continuous function spaces equipped with the following norms respectively:

$$\|\phi\|_{C(\mathcal{O},\mathbb{R}^l)} := \sup_{x\in\mathcal{O}} |\phi(x)|, \quad \|\phi\|_{C^k(\mathcal{O},\mathbb{R}^l)} := \|\phi\|_{C(\mathcal{O},\mathbb{R}^l)} + \sum_{|\alpha|=1}^k \|D^\alpha\phi\|_{C(\mathcal{O},\mathbb{R}^l)},$$

$$\|\phi\|_{C^{k,\delta}(\mathcal{O},\mathbb{R}^l)} := \|\phi\|_{C^k(\mathcal{O},\mathbb{R}^l)} + \sum_{|\alpha|=k} \sup_{x,y\in\mathcal{O},x\neq y} \frac{|D^\alpha\phi(x) - D^\alpha\phi(y)|}{|x-y|^\delta},$$

with the convention that $C^0(\mathcal{O},\mathbb{R}^l) \equiv C(\mathcal{O},\mathbb{R}^l)$. Whenever there is no confusion, we write $C(\mathbb{R}^d)$, C^k, and $C^{k,\delta}$ for $C(\mathbb{R}^d,\mathbb{R}^l)$, $C^k(\mathbb{R}^d,\mathbb{R}^l)$ and $C^{k,\delta}(\mathbb{R}^d,\mathbb{R}^l)$, respectively. We define $C^\infty(\mathbb{R}^d) := \cap_{k=1}^\infty C^k(\mathbb{R}^d)$.

In an obvious way, we define spaces of Banach space valued functions such as $C(0,T;H^{m,q})$ and $L^r(0,T;H^{m,q})$ for $m\in\mathbb{Z}, r,q\in[1,\infty]$, and related local spaces like the following ones:

$$L^r_{\mathrm{loc}}(T_0,T;H^{m,q}) := \bigcap_{T_1\in(T_0,T]} L^r(T_1,T;H^{m,q}),$$
$$C_{\mathrm{loc}}((T_0,T];H^{m,q}) := \bigcap_{T_1\in(T_0,T]} C([T_1,T];H^{m,q}).$$

We have the following properties on Sobolev spaces, whose proof is omitted.

Lemma 1 *There holds the following assertions:*

(i) the space H^n, $n>d/2+k$, $k\in\mathbb{Z}^+\cup 0$, is continuously embedded into the space $C^{k,\delta}$ for any $\delta\in(0,(n-d/2-k)\wedge 1)$, i.e., there exists a constant $C>0$ such that

$$\|\phi\|_{C^{k,\delta}}\le C\|\phi\|_n, \quad \forall\,\phi\in H^n;$$

(ii) if $1<r<s<\infty$ and $m,n\in\mathbb{Z}$ satisfying $\frac{d}{s}-m=\frac{d}{r}-n$, then $H^{n,r}$ is continuously embedded into $H^{m,s}$, i.e., there exists a constant $C>0$ such that

$$\|\phi\|_{m,s}\le C\|\phi\|_{n,r}, \quad \forall\,\phi\in H^{n,r};$$

(iii) for $m\in\mathbb{Z}^+$, there exists a constant $C>0$ such that, for any $\phi,\psi\in L^\infty\cap H^m$,

$$\begin{aligned}\|\phi\psi\|_m \le& C\left\{\|\phi\|_{L^\infty}\|D^m\psi\|_{L^2}+\|\psi\|_{L^\infty}\|D^m\phi\|_{L^2}\right\},\\ \sum_{0\le|\alpha|\le m}\|D^\alpha(\phi\psi)-\phi D^\alpha\psi\|_{L^2}\le& C\left\{\|\nabla\phi\|_{L^\infty}\|D^{m-1}\psi\|_{L^2}+\|D^m\phi\|_{L^2}\|\psi\|_{L^\infty}\right\};\end{aligned}$$

(iv) for any $s>d/2$, H^s is a Banach algebra, i.e., there exists a constant $C>0$ such that,

$$\|\phi\psi\|_s\le C\|\phi\|_s\|\psi\|_s, \quad \forall\phi,\psi\in H^s.$$

The first two assertions of Lemma 1 are borrowed from the well-known embedding theorem in Sobolev space theory (see [46]) and the others are referred to [34, Lemma 3.4, Page 98]. For simplicity, we shall denote by $\hookrightarrow$ the embedding relationship, i.e., by $A\hookrightarrow B$ we mean that normed space $(A,\|\cdot\|_A)$ is embedded into $(B,\|\cdot\|_B)$ with a constant C such that

$$\|f\|_B\le C\|f\|_A, \ \forall f\in A.$$

Remark 1 *Note that $d=2$ or 3 throughout this work. For any $h,g\in H^2$, we have*

$$\begin{aligned}\|h\cdot g\|_1^2 =& \|h\cdot g\|_0^2+\sum_{i=1}^d\|\partial_{x_i}h\cdot g\|_0^2+\sum_{i=1}^d\|h\cdot\partial_{x_i}g\|_0^2\\ \le& C\Big\{\|h\|^2_{C(\mathbb{R}^d)}\|g\|_0^2+\|\nabla h\|_0^{2\beta}\|\nabla h\|_1^{2-2\beta}\|g\|_0^{2\beta}\|g\|_1^{2-2\beta}+\|h\|^2_{C(\mathbb{R}^d)}\|g\|_1^2\Big\}\\ &\big(\text{ using Gagliardo-Nirenberg Inequality (see [23, 32, 35])}\big)\\ \le& C\|h\|_2^2\|g\|_1^2,\end{aligned}$$

where $\beta:=1-d/4$, and H^2 is embedded into $C^{0,\delta}$ for some $\delta\in(0,1)$. In view of Lemma 1, we have for any integer $m>d/2$,

$$\|hg\|_{m-1}\le C(m,d)\|h\|_m\|g\|_{m-1}, \ \forall h\in H^m, g\in H^{m-1}.$$

Lemma 2 *There hold the following :*

$$|\langle(\phi\cdot\nabla)\psi,\,\psi\rangle_{m-1,m+1}|\le C\|\phi\|_m\|\psi\|_m^2, \ m\ge 2, \phi\in H^m_\sigma, \psi\in H^{m+1}_\sigma. \tag{4}$$

and

$$|\langle(\phi\cdot\nabla)\phi,\,\phi\rangle_m|\le C\|\nabla\phi\|_{L^\infty(\mathbb{R}^d)}\|\phi\|_m^2, \ m\ge 2, \phi\in H^{m+1}_\sigma. \tag{5}$$

Proof The first inequality (4) is referred to [30]. For the reader's convenience, we give a simple proof of the second inequality (5).

For any multi-index α such that $|\alpha| \leq m$, we have

$$D^\alpha((\phi\cdot\nabla)\phi) = (\phi\cdot\nabla)D^\alpha\phi + \sum_{0<\beta\leq\alpha} C(\alpha,\beta)(D^\beta\phi\cdot\nabla)D^{\alpha-\beta}\phi.$$

Since $\langle(\phi\cdot\nabla)D^\alpha\phi,\, D^\alpha\phi\rangle_0 = 0$, we have from Assertion (iii) of Lemma 1 that

$$\begin{aligned}\langle(\phi\cdot\nabla)\phi,\, \phi\rangle_m &= \sum_{0<|\alpha|\leq m}\sum_{0<\beta\leq\alpha} C(\alpha,\beta)\langle(D^\beta\phi\cdot\nabla)D^{\alpha-\beta}\phi,\, D^\alpha\phi\rangle_0 \\ &\leq C\sum_{i=1}^d \|(\partial_{x_i}\phi\cdot\nabla)\phi\|_{m-1}\|\phi\|_m \\ &\leq C(m,d)\|\nabla\phi\|_{L^\infty}\|\phi\|_m^2.\end{aligned}$$

3 Connection between FBSDS and the Navier-Stokes Equation

3.1 FBSDEs with coefficients in Sobolev Spaces

Assume that $\nu > 0$ and that

$$b \in C([0,T];H^m)\cap L^2(0,T;H^{m+1}), \phi\in L^1(0,T;L^2(\mathbb{R}^d)), \psi\in L^2(\mathbb{R}^d), \tag{6}$$

for some integer $m > d/2$. Consider the following FBSDE:

$$\begin{cases} dX_s(t,x) = b(s,X_s(t,x))\,ds + \sqrt{\nu}\,dW_s,\ \ s\in[t,T]; \quad X_t(t,x) = x; \\ -dY_s(t,x) = \phi(s,X_s(t,x))\,ds - \sqrt{\nu}Z_s(t,x)\,dW_s,\ \ s\in[t,T]; \quad Y_T(t,x) = \psi(X_T(t,x)). \end{cases} \tag{7}$$

Since $H^m \hookrightarrow C^{0,\delta}$ and $H^{m+1}\hookrightarrow C^{1,\delta}$ for $m > d/2$, in view of [31, Theorems 3.4.1 and 4.5.1], the *forward* SDE is well posed for each $(t,x)\in[T_0,T]\times\mathbb{R}^d$, and the unique solution in relevance to the initial data $(t,x)\in[T_0,T]\times\mathbb{R}^d$ defines a stochastic flow of homeomorphisms. Since the function ϕ is only measurable, the following lemma serves to give a clear meaning of the composition $\phi(s,X_s(t,x))$.

Lemma 3 *Assume that $m > d/2$ and $b\in C([0,T];H^m)\cap L^2(0,T;H^{m+1})$. Then there are two positive constants κ and K which only depend on $\|div\, b\|_{L^1(0,T;L^\infty)}$, such that for all $t\in[0,T], s\in [t,T], (\varphi,\eta)\in L^1(\mathbb{R}^l)\times L^2([0,T]\times\mathbb{R}^d;\mathbb{R}^l)$, and $l\in\mathbb{Z}^+$, we have*

$$\kappa\|\varphi\|_{L^1(\mathbb{R}^l)} \leq \int_{\mathbb{R}^d} E\big[|\varphi(X_s(t,x))|\big]\,dx \leq K\|\varphi\|_{L^1(\mathbb{R}^l)}, \tag{8}$$

$$\kappa\|\eta\|_{L^1([t,T]\times\mathbb{R}^l)} \leq \int_{\mathbb{R}^d}\int_t^T E\big[|\eta(s,X_s(t,x))|\big]\,dsdx \leq K\|\eta\|_{L^1([t,T]\times\mathbb{R}^l)}. \tag{9}$$

Our preceding lemma weakens the assumptions on b of [2, Theorem 14.3], where $b(t,\cdot)\equiv b(\cdot)$ is time invariant and is required to lie in the more regular space $C^1(\mathbb{R}^d)\cap W^{1,\infty}(\mathbb{R}^d)$. Since $b(t,x)$ is not necessarily uniformly Lipschitz continuous in x, the stability of X with respect to the coefficient b has to be proved very carefully and the proof of [2] has to be generalized accordingly. we give a probabilistic proof in the appendix.

Remark 2 *From Lemma 3, we see that Lebesgue's measure transported by the flow $\{X_s(t,x), s\in [t,T]\}$ results in a group of measures $\{\mu_s, s\in[t,T]\}$ satisfying for any Borel measurable set $A\subset\mathbb{R}^d$,*

$$\mu_s(A) = \int_{\mathbb{R}^d} E\left[1_A(X_s(t,x))\right]\,dx.$$

These measures are all equivalent to Lebesgue measure and the exponential rate of compression or dilation are governed by the divergence of b. This is similar to that of a system of ordinary differential equations (see [17]). On the other hand, thanks to Lemma 3, our FBSDE (7) *makes sense under assumption* (6), *i.e., the* forward *SDE is well posed for each $(t,x)\in[0,T]\times\mathbb{R}^d$ and for each $t\in[0,T]$ the BSDE is well posed for almost every $x\in\mathbb{R}^d$.*

3.2 Definition of the solution to system (3)

First, let us consider the following trivial BSDE on $[0,\infty]$:

$$-dS_t = g_t\,dt - dM_t; \quad S_\infty = 0. \tag{10}$$

In the conventional sense (see [6,13,18,40,41]), a solution of BSDE is always defined as a pair of processes (S,M) and the process M serves to guarantee the adaptedness of S. However, since the FBSDS (3) only involves $\widetilde{Y}_0$ rather than the whole process $\widetilde{Y}$, we are interested only in determining S_0 instead of the whole process S for such kind of BSDEs on $[0,\infty]$.

Different from [6,13,18,40,41], we assume that $\{g_t, t\in(0,\infty)\}$ is an $\mathbb{R}^d$-valued $\mathscr{F}_t^B$-adapted process such that

$$\int_\varepsilon^\infty E\left[|g_t|\right]\,dt<\infty,\ \forall\varepsilon>0 \text{ and } \lim_{\varepsilon\to0}\int_\varepsilon^\infty E\left[g_t\right]\,dt<\infty. \tag{11}$$

The class $\mathcal{E}$ is defined as the totality of BSDEs (10) with the drift satisfying the preceding assumption.

Each BSDE (10) lying in $\mathcal{E}$ can be written into the integral form:

$$S_t = \int_t^\infty g_s\,ds - \widetilde{M}_t,$$

where as we can not solve it in the whole time interval $[0,\infty)$ at once, we solve it with a pair $(S,\widetilde{M})$ instead of (S,M), with $E[\widetilde{M}_t|\mathscr{F}_t^B]=0$, $\forall t\in(0,\infty)$. Therefore, we have

$$S_t = E\Big[\int_t^\infty g_s\,ds\big|\mathscr{F}_t^B\Big], \quad \widetilde{M}_t=\int_t^\infty g_s\,ds - S_t,\ \forall t\in(0,\infty).$$

S_0 is defined as follows

$$S_0 := \lim_{\varepsilon\downarrow0} E\left[S_\varepsilon\right], \tag{12}$$

while $\widetilde{M}_0$ is not necessary for us to know its meaning in this work.

Now, our trivial BSDE (10) makes sense and the definition of the solution does not conflict with the common cases (see [6,13,18,40,41]). Further, we may consider more general nonlinear cases such as $g_s = g(s,S_s,\widetilde{M}_s)$ which may be used to describe more general operators like in Lemma 4 below. However, we do not seek such a generality in this paper.

Definition 1 *We say that $(X,Y,Z,\widetilde{Y}_0)$ is a solution (local solution, respectively) of the FBSDS* (3) *if for almost every $(t,x)\in[0,T]\times\mathbb{R}^d$ ($(t,x)\in(T_0,T]\times\mathbb{R}^d$ for some $T_0\in(0,T)$, respectively), the BSDE on the infinite time interval belongs to class $\mathcal{E}$, and*

$$(X_\cdot(t,x),Y_\cdot(t,x),Z_\cdot(t,x))\in S^2([t,T];\mathbb{R}^d)\times S^2([t,T];\mathbb{R}^d)\times L^2_{\mathscr{F}}(t,T;\mathbb{R}^{d\times d})$$

such that the first two stochastic differential equations of (3) *on $[0,T]$ (any subinterval $[T_1,T]$ with $T_1\in(T_0,T)$, respectively) hold almost surely.*

If both the BSDE on the infinite time interval $[0,\infty]$ and its unknown variable $\widetilde{Y}$ do not appear in the FBSDS (3), then Definition 1 becomes automatically the definition of the solutions (local solutions, respectively) for an FBSDE by abandoning all the requirements on both $\widetilde{Y_0}$ and the BSDE on the infinite time interval.

To specify the regularity of the solutions, we introduce the following definition.

Definition 2 $(X,Y,Z,\widetilde{Y}_0)$ *is called a (local, respectively) H^m-solution of the FBSDS* (3) *if it is a solution (local solution, respectively) of the FBSDS* (3) *(on some time interval $(T_0,T]$, respectively) with $\widetilde{Y}_0\in L^2(0,T;H^{m-1})$ ($\widetilde{Y}_0\in L^2_{loc}(T_0,T;H^{m-1})$, respectively) and for each $t\in[0,T]$ ($t\in(T_0,T]$, respectively) and almost every $x\in\mathbb{R}^d$, $\{Y_s(t,x),\ s\in[t,T]\}\in L^\infty_{\mathscr{F}}(t,T;\mathbb{R}^d)$.*

Remark 3 *In the FBSDS* (3), *the regularity of* $\widetilde{Y}_0$ *depends on that of* $Y_t(t,x)$. *However, Proposition 7 below shows that the regularity of* $\widetilde{Y}_0$ *dominates that of* (Y,Z). *This explains why our* H^m*-solutions of the FBSDS* (3) *only require the regularity of* $\widetilde{Y}_0$ *in Definition 2. On the other hand, Definition 2 requires the uniform boundedness of the unknown process* $Y_\cdot(t,x)$. *Indeed, in this work we only pursue the* H^m*-solution* ($m > d/2$, $d = 2$ *or* 3) *of which* $Y_\cdot(t,x)$ *is naturally uniformly bounded (see Theorem 5 below); moreover, the uniform boundedness is only required for the particular case* $m = 2$ *(see Proposition 7 below). Compared with the standard assumptions for the solvability of FBSDEs (see [1, 1, 25, 33, 39, 47]), it can be viewed as a compensation for the lack of continuity of the function* $f(s,x)$ *with respect to* x.

3.3 Connections between FBSDS and Navier-Stokes equation

Before we show the connections between the FBSDSs and Navier-Stokes equaitons, we give a probabilistic representation for an integral operator.

Lemma 4 *For* $\phi, \psi \in H^m$ *with* $m > \frac{d}{2} + 1$, *define*

$$\xi(x) := \nabla(-\Delta)^{-1} div\ div(\phi \otimes \psi)(x) = \sum_{i,j=1}^{d} \nabla(-\Delta)^{-1} \partial_{x^i}\partial_{x^j}(\phi^i(x)\psi^j(x)).$$

Then, the following BSDE :

$$\begin{cases} -d\widetilde{Y}_s(x) = \dfrac{27}{2s^3}\phi^i\psi^j(x+B_s)\left(B_s^j - B_{\frac{2s}{3}}^j\right)\left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i\right)B_{\frac{s}{3}}\,ds - dM_s, \ s \in (0,\infty) \\ \widetilde{Y}_\infty(x) = 0 \end{cases} \tag{13}$$

belongs to class $\mathcal{E}$ *for each* $x \in \mathbb{R}^d$ *and there holds*

$$\xi \in C(\mathbb{R}^d) \cap H^{m-2}, \quad \xi(x) = \widetilde{Y}_0(x),\ \forall x \in \mathbb{R}^d,$$

and

$$\|\xi\|_{m-2} \le C\|\phi \otimes \psi\|_m, \tag{14}$$

with C *a positive constant independent of* ϕ *and* ψ.

Proof For $m > d/2 + 1$, H^m is a Banach algebra embedded into $H^{2,\gamma}$ for some $\gamma > d$ and also into $C^{1,\delta}(\mathbb{R}^d)$ for some $\delta \in (0,1)$, so $\phi^i\psi \in H^m \cap H^{m,1}$ which implies $\partial_{x^i}\partial_{x^j}(\phi^i(x)\psi^j(x)) \in H^{0,\gamma}(\mathbb{R}) \cap H^{0,1}(\mathbb{R})$, $i,j = 1, \cdots, d$.

For each $\varepsilon > 0$, we have

$$\begin{aligned} &E\left[\int_\varepsilon^\infty \frac{27}{2s^3}\left|\phi^i\psi^j(x+B_s)\left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i\right)\left(B_s^j - B_{\frac{2s}{3}}^j\right)B_{\frac{s}{3}}\right| ds\right] \\ &\le \|\phi^i\psi^j\|_{L^\infty}\int_\varepsilon^\infty \frac{27}{2s^3}E\left[\left|\left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i\right)\left(B_s^j - B_{\frac{2s}{3}}^j\right)B_{\frac{s}{3}}\right|\right] ds \\ &\le C\,\|\phi \otimes \psi\|_2 \int_\varepsilon^\infty \frac{1}{s^{3/2}}\ ds < \infty. \end{aligned}$$

Thus, our BSDE (13) is well-posed on the interval $[\varepsilon, \infty]$ and we need only prove

$$\xi(x) = \lim_{\varepsilon \downarrow 0} E\left[\widetilde{Y}_\varepsilon(x)\right]. \tag{15}$$

On the other hand, we have

$$\widetilde{Y}_\varepsilon(x) = E\left[\int_\varepsilon^\infty \frac{27}{2s^3}\phi^i\psi^j(x+B_s)\left(B_{\frac{2s}{3}}^i - B_{\frac{s}{3}}^i\right)\left(B_s^j - B_{\frac{2s}{3}}^j\right)B_{\frac{s}{3}}\,ds\middle|\mathscr{F}_\varepsilon^B\right].$$

Careful calculations yield that, for each $i,j,k=1,2,\cdots,d$,

$$\begin{aligned}
&E\left[\frac{27}{2s^3}\phi^i\psi^j(x+B_s)\left(B^i_{\frac{2s}{3}}-B^i_{\frac{s}{3}}\right)\left(B^j_s-B^j_{\frac{2s}{3}}\right)B^k_{\frac{s}{3}}\right]\\
=&\frac{27}{2s^3}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi^i\psi^j(x+y+z+r)y^iz^jr^k(2\pi s/3)^{-3d/2}e^{-\frac{3(|y|^2+|z|^2+|r|^2)}{2s}}\,dydzdr\\
=&-\frac{9}{2s^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi^i\psi^j(y)z^jr^k(2\pi s/3)^{-3d/2}\partial_{y^i}e^{-\frac{3(|y-x-z-r|^2+|z|^2+|r|^2)}{2s}}\,dydzdr\\
=&\frac{9}{2s^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{y^i}(\phi^i\psi^j)(y)z^jr^k(2\pi s/3)^{-3d/2}e^{-\frac{3(|y-x-z-r|^2+|z|^2+|r|^2)}{2s}}\,dydzdr\\
=&-\frac{3}{2s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{y^i}(\phi^i\psi^j)(y)r^k(2\pi s/3)^{-3d/2}e^{-\frac{3|y-x-r-z|^2}{2s}}\partial_{z^j}e^{-\frac{3(|z|^2+|r|^2)}{2s}}\,dydzdr\\
=&-\frac{3}{2s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{y^i}(\phi^i\psi^j)(y)r^k(2\pi s/3)^{-3d/2}\partial_{y^j}e^{-\frac{3|y-x-r-z|^2}{2s}}e^{-\frac{3(|z|^2+|r|^2)}{2s}}\,dzdydr\\
=&\frac{3}{2s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{y^i}\partial_{y^j}(\phi^i\psi^j)(y)r^k(4\pi s/3)^{-d/2}e^{-\frac{3|y-x-r|^2}{4s}}(2\pi s/3)^{-d/2}e^{-\frac{3|r|^2}{2s}}\,dydr\\
=&-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{y^i}\partial_{y^j}(\phi^i\psi^j)(y)(4\pi s/3)^{-d/2}e^{-\frac{3|y-x-r|^2}{4s}}(2\pi s/3)^{-d/2}\partial_{r^k}e^{-\frac{3|r|^2}{2s}}\,dydr\\
=&-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\partial_{y^i}\partial_{y^j}(\phi^i\psi^j)(y)(4\pi s/3)^{-d/2}e^{-\frac{3|r|^2}{4s}}(2\pi s/3)^{-d/2}\partial_{y^k}e^{-\frac{3|y-x-r|^2}{2s}}\,drdy\\
=&-\frac{1}{2}\int_{\mathbb{R}^d}\partial_{y^i}\partial_{y^j}(\phi^i\psi^j)(y)(2\pi s)^{-d/2}\partial_{y^k}e^{-\frac{|y-x|^2}{2s}}\,dy\\
=&\frac{1}{2s}\int_{\mathbb{R}^d}\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(y+x)y^k(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}\,dy\\
=&\frac{1}{2s}E\left[\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(x+B_s)B^k_s\right],\quad s>0.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&s^{-1}\int_{\mathbb{R}^d}|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(y+x)y^k(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}|\,dy\\
\le&C\,s^{-\frac{d}{2}-1}\|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)\|_{0,q}\sqrt{s}s^{\frac{d}{2}(1-\frac{1}{q})}\\
\le&C\,\|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)\|_{0,q}s^{-\frac{1}{2}-\frac{d}{2q}},\ q\in[1,\gamma],
\end{aligned}\tag{16}$$

and

$$\begin{aligned}
&\int_0^\infty s^{-1}E\left[\left|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(x+B_s)B^k_s\right|\right]\,ds\\
=&\int_0^\infty s^{-1}\int_{\mathbb{R}^d}|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(y+x)y^k(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}|\,dy\,ds\\
=&C\int_0^1\|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)\|_{0,\gamma}\,s^{-\frac{1}{2}-\frac{d}{2\gamma}}\,ds+C\int_1^\infty\|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)\|_{0,1}\,s^{-\frac{d+1}{2}}\,ds\\
\le&C\left(\|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)\|_{m-2}+\|\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)\|_{m-2,1}\right).
\end{aligned}\tag{17}$$

Therefore,

$$\begin{aligned}
\lim_{\varepsilon\downarrow0}E\left[\widetilde{Y}^k_\varepsilon(x)\right]&=\lim_{\varepsilon\downarrow0}\int_\varepsilon^\infty\frac{1}{2s}\int_{\mathbb{R}^d}\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(y+x)y^k(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}\,dyds\\
&=\int_0^\infty\frac{1}{2s}\int_{\mathbb{R}^d}\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(y+x)y^k(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}\,dyds\\
&\quad\text{(by (17) and Fubini Theorem)}\\
&=\int_{\mathbb{R}^d}\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(x+y)y^k\int_0^\infty\frac{1}{2s}(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}\,dsdy
\end{aligned}\tag{18}$$

$$
\begin{aligned}
&= C_d \int_{\mathbb{R}^d} \partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(x+y)\frac{y^k}{|y|^d}dy \\
&= \partial_{x^k}(-\Delta)^{-1}\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(x)
\end{aligned}
$$

which coincides with the convolution representation of the operator $\nabla(-\Delta)^{-1}$ described in [34, Page 31]. Hence, BSDE (13) belongs to class $\mathcal{E}$ and by (17), one has $\widetilde{Y}_0 \in C(\mathbb{R}^d)$ on account of the continuity of the translation operator on $L^p(\mathbb{R}^d)$, $p \in [1, \infty)$. Moreover, we have

$$
\xi(x) = \nabla(-\Delta)^{-1}\partial_{x^i}\partial_{x^j}(\phi^i(x)\psi^j(x)) = \lim_{\varepsilon\downarrow 0} E\left[\widetilde{Y}_\varepsilon(x)\right],\ \forall x \in \mathbb{R}^d.
$$

In view of (18), we further have by Minkowski inequality

$$
\begin{aligned}
\|\tilde{Y}_0^k\|_0 &= \left\| E[\tilde{Y}_1^k] + \sum_{i,j=1}^d \int_0^1 \frac{1}{2s}\int_{\mathbb{R}^d} \partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(y+\cdot)y^k(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}\,dyds \right\|_0 \\
&\le C\Bigg(\|\mathrm{div}\ \mathrm{div}(\phi\otimes\psi)\|_0 \int_0^1 \frac{1}{2s}\int_{\mathbb{R}^d} |y^k|(2\pi s)^{-d/2}e^{-\frac{|y|^2}{2s}}\,dyds \\
&\quad + \Big\| \sum_{i,j=1}^d \phi^i\psi^j \Big\|_0 E\Big[\int_1^\infty \frac{27}{2s^3}\Big| \Big(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\Big)\Big(B^j_s - B^j_{\frac{2s}{3}}\Big) B^k_{\frac{s}{3}} \Big|\, ds\Big]\Bigg) \\
&\le C\,\|\phi\otimes\psi\|_2, \qquad (19)
\end{aligned}
$$

from which we obtain the estimate (14) by taking derivatives. The proof is complete.

Remark 4 *We have $\widetilde{Y}_0 \in C(\mathbb{R}^d)$ from (17) and the continuity of the translation operator on $L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$. In fact, the operator $\boldsymbol{P} := \boldsymbol{I} - \nabla\Delta^{-1}div$ is the well-known Leray-Hodge projection onto divergence free vector fields, where $\boldsymbol{I}$ is the identity operator. Define $\boldsymbol{P}^\perp := \boldsymbol{I} - \boldsymbol{P}$. We have in Lemma 4 that $\xi = -\boldsymbol{P}^\perp(div(\phi\otimes\psi))$. Indeed, the singular integral operator $\boldsymbol{P}$ (see [34, 43]) is a bounded transformation in $H^{n,q}$ for $q \in (1,\infty)$ and $n \in \mathbb{Z}$. Note that for any $g \in H^m_\sigma$, integration-by-parts formula yields*

$$
\langle \boldsymbol{P}^\perp(div(\phi\otimes\psi)),\, g\rangle_{m-2,m} = 0.
$$

Remark 5 *For the defined ξ, there exists a scalar-valued fucntion η such that $\xi = \nabla\eta$. Indeed, we may take $\eta(x) =: (-\Delta)^{-1}\partial_{x^i}\partial_{x^j}(\phi^i\psi^j)(x)$ which, by the second order Elliptic partial differential equation theory (see [24]), lies in $H^m(\mathbb{R}^d;\mathbb{R})$.*

Remark 6 *In fact for any $\varepsilon > 0$, we have by Minkowski inequality*

$$
\begin{aligned}
\left\| E\left[\tilde{Y}_\varepsilon\right]\right\|_0 &= \left\| E\Big[\int_\varepsilon^\infty \frac{27}{2s^3}\phi^i\psi^j(\cdot+B_s)\Big(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\Big)\Big(B^j_s - B^j_{\frac{2s}{3}}\Big)B_{\frac{s}{3}}\,ds\Big]\right\|_0 \\
&\le \sum_{i,j=1}^d \|\phi^i\psi^j\|_0 E\int_\varepsilon^\infty \frac{27}{2s^3}\Big|\Big(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\Big)\Big(B^j_s - B^j_{\frac{2s}{3}}\Big)B_{\frac{s}{3}}\Big|\,ds \\
&\le \frac{27}{\sqrt{\varepsilon}}\sum_{i,j=1}^d \|\phi^i\psi^j\|_0. \qquad (20)
\end{aligned}
$$

Putting

$$
\boldsymbol{P}^\varepsilon(\phi\otimes\psi) = E\left[\tilde{Y}_\varepsilon\right], \quad \forall \phi,\psi \in H^m,\ m > d/2+1,
$$

we have

$$
\|\boldsymbol{P}^\varepsilon(\phi\otimes\psi)\|_k \le \frac{C}{\sqrt{\varepsilon}}\|\phi\otimes\psi\|_k, \quad \forall\, 0 \le k \le m,
$$

with the constant C independent of ε. Then, the operator $\boldsymbol{P}^\varepsilon$ can be seen as a regular approximation of $-\boldsymbol{P}^\perp div$. This approximation will be used when we study the numerical approximation of Navier-Stokes equation in Section 6.

Define the following operator θ: for any map g defined on $[0,\infty)\times[0,\infty)\times\mathbb{R}^d$,

$$\theta_g(t,x) := g(t,t,x), \quad (t,x)\in[0,\infty)\times\mathbb{R}^d.$$

Our main result is stated as follows.

Theorem 5 *Let $\nu>0, G\in H^m_\sigma$, and $f\in L^2(0,T;H^{m-1}_\sigma)$ with $m>d/2$. Then our FBSDS (3) admits one and only one local H^m-solution $(X,Y,Z,\widetilde{Y}_0)$ on some time interval $(T_0,T]$ with $\theta_Y\in C_{loc}((T_0,T];H^m_\sigma)\cap L^2_{loc}(T_0,T;H^{m+1}_\sigma)$ and $\theta_Z\in C_{loc}((T_0,T];H^{m-1}_\sigma)\cap L^2_{loc}(T_0,T;H^m_\sigma)$, where T_0 depends on $\|f\|_{L^2(0,T;H^{m-1})}$, ν, T and $\|G\|_m$. Moreover, there hold the following representations*

$$\theta_Z(t,\cdot)=\nabla\theta_Y(t,\cdot), Y_s(t,\cdot)=\theta_Y(s,X_s(t,\cdot)) \text{ and } Z_s(t,\cdot):=\theta_Z(s,X_s(t,\cdot)), \tag{21}$$

for $T_0<t\le s\le T$, and there exists some scalar-valued function $\widetilde{p}$ such that $\nabla\widetilde{p}=\widetilde{Y}_0$ and $(\theta_Y,\theta_Z,\widetilde{p})$ satisfies

$$\begin{aligned}\theta_Y(r,X_r(t,x)) =& G(X_T(t,x))+\int_r^T\left[f(s,X_s(t,x))+\nabla\widetilde{p}(s,X_s(t,x))\right]ds\\ &-\sqrt{\nu}\int_r^T\theta_Z(s,X_s(t,x))\,dW_s,\ T_0<t\le r\le T,\ a.e. x\in\mathbb{R}^d,\ a.s..\end{aligned}\tag{22}$$

In addition, $(\widetilde{u},\widetilde{p})$ with $\widetilde{u}:=\theta_Y$ coincides with the unique strong solution to Navier-Stokes equation:

$$\begin{cases}\partial_t\widetilde{u}+\frac{\nu}{2}\Delta\widetilde{u}+(\widetilde{u}\cdot\nabla)\widetilde{u}+\nabla\widetilde{p}+f=0, & T_0<t\le T;\\ \nabla\cdot\widetilde{u}=0, \quad \widetilde{u}(T)=G.\end{cases}\tag{23}$$

Remark 7 *In Theorem 5, we only have the connections for the strong solutions. For the case of $m\le d/2$, we could not show that $\theta_Y(t,\cdot)$ takes values in $W^{1,\infty}_{loc}$, and thus we do not know whether the* forward *SDE of the FBSDS* (3) *is well-posed or whether the assertions of Lemma 3 are still true.*

4 Proof of Theorem 5

4.1 Auxiliary results

Consider the following coupled FBSDE:

$$\begin{cases} dX_s(t,x)=[b(s,X_s(t,x))+\alpha Y_s(t,x)]\,ds+\sqrt{\nu}\,dW_s,\ s\in[t,T]; \quad X_t(t,x)=x;\\ -dY_s(t,x)=\phi(s,X_s(t,x))\,ds-\sqrt{\nu}Z_s(t,x)\,dW_s,\ s\in[t,T]; \quad Y_T(t,x)=\psi(X_T(t,x)),\end{cases}\tag{24}$$

where $\nu>0$ and α are constants.

We define solutions to the FBSDE (24).

Definition 3 *We say (X,Y,Z) is a solution (local solution, respectively) of the FBSDE* (24) *if for each $t\in[0,T]$ ($t\in(T_0,T]$ for some $T_0\in(0,T)$, respectively) and almost every $x\in\mathbb{R}^d$,*

$$(X.(t,x),Y.(t,x),Z.(t,x))\in S^2([t,T];\mathbb{R}^d)\times S^2([t,T];\mathbb{R}^d)\times L^2_{\mathscr{F}}(t,T;\mathbb{R}^{d\times d})$$

and such that the forward *SDE and BSDE on $[0,T]$ (any $[T_1,T]\subset(T_0,T)$, respectively) hold almost surely. If for each $t\in[0,T]$ ($t\in(T_0,T]$ for some $T_0\in(0,T)$, respectively) and almost every $x\in\mathbb{R}^d$, we further have*

$$Y.(t,x)\in L^\infty_{\mathscr{F}}(t,T;\mathbb{R}^d), \tag{25}$$

then (X,Y,Z) is called a bounded solution (local bounded solution, respectively).

Lemma 6 *Let $b \in C([T_0,T];H^m) \cap L^2(T_0,T;H^{m+1}), \phi \in L^2(T_0,T;H^{m-1})$, and $\psi \in H^m$, with $m > d/2$. Then, for each $t \in [T_0,T)$ and almost every $x \in \mathbb{R}^d$, the following FBSDE:*

$$\begin{cases} dX_s(t,x) = b(s,X_s(t,x))\,ds + \sqrt{\nu}\,dW_s,\ T-\varepsilon \le t \le s \le T; \quad X_t(t,x) = x; \\ -dY_s(t,x) = \phi(s,X_s(t,x))\,ds - \sqrt{\nu}Z_s(t,x)\,dW_s,\ s \in [t,T]; \quad Y_T(t,x) = \psi(X_T(t,x)) \end{cases} \tag{26}$$

admits one and only one bounded solution (X,Y,Z), and for this solution, we have

$$\theta_Y \in C_{loc}((T_0,T];H^m) \cap L^2_{loc}(T_0,T;H^{m+1}), \tag{27}$$

and for each $t \in [T_0,T]$, almost all $x \in \mathbb{R}^d$ and all $r \in [t,T]$

$$\theta_Y(r,X_r(t,x)) = \psi(X_T(t,x)) + \int_r^T \phi(s,X_s(t,x))\,ds - \sqrt{\nu}\int_r^T \theta_Z(s,X_s(t,x))\,dW_s,\ a.s., \tag{28}$$

$$\theta_Z(t,x) = \nabla\theta_Y(t,x),\ (Y_r(t,x),Z_r(t,x)) = (\theta_Y,\theta_Z)(r,X_r(t,x)),\ a.s.. \tag{29}$$

Moreover, for any $t \in [T_0,T]$

$$\|\theta_Y(t)\|_m^2 + \nu\int_t^T \|\theta_Z(s)\|_m^2\,ds = \|\theta_Y(T)\|_m^2 + 2\int_t^T \langle\phi(s)+\theta_Z b(s),\,\theta_Y(s)\rangle_{m-1,m+1}\,ds. \tag{30}$$

Lemma 6 might exist elsewhere, but we have not found it. For the reader's convenience, a proof is sketched in the appendix.

Proposition 7 *Assume that $\psi \in H^m$, $b \in C([0,T];H^m) \cap L^2(0,T;H^{m+1})$, and $\phi \in L^2(0,T;H^{m-1})$ with $m > d/2$. Then the FBSDE (24) admits a unique locally bounded solution in some time interval $(T_0,T]$ with T_0 depending on $\|\psi\|_m$, $\|b\|_{C([0,T];H^m)}$, $\|\phi\|_{L^2(0,T;H^{m-1})}$, α, ν and T. If $\alpha = 0$, $T_0 = 0$. Moreover,*

$$\theta_Y \in C_{loc}((T_0,T];H^m) \cap L^2_{loc}(T_0,T;H^{m+1}) \text{ and } \theta_Z \in L^2_{loc}(T_0,T;H^m)$$

satisfy (28) and for any $t \in (T_0,T]$, there hold the following energy equality:

$$\begin{aligned} &\|\theta_Y(t)\|_m^2 + \nu\int_t^T \|\theta_Z(s)\|_m^2\,ds \\ &= \|\psi\|_m^2 + 2\int_t^T \langle\theta_Z(b+\alpha\theta_Y)(s),\,\theta_Y(s)\rangle_{m-1,m+1}\,ds + 2\int_t^T \langle\phi(s),\,\theta_Y(s)\rangle_{m-1,m+1}\,ds, \end{aligned} \tag{31}$$

and for almost every $x \in \mathbb{R}^d$ and all $s \in [t,T]$,

$$\theta_Z(t,x) = \nabla\theta_Y(t,x),\ (Y_s(t,x),Z_s(t,x)) = (\theta_Y,\theta_Z)(s,X_s(t,x)),\ a.s.. \tag{32}$$

In addition, if $m > d/2+1$, the locally bounded solution on the time interval $(T_0,T]$ is the unique local solution as well.

Remark 8 *In Proposition 7, we see θ_Y and θ_Z are all deterministic functions. Therefore, for each semimartingale $\{X'_s(t,x), s \in [t,T]\}$ of the form*

$$X'_s(t,x) = x + \int_t^s \varphi_r(t,x)\,dr + \int_t^s \sqrt{\nu}\,dW_r, \quad T_0 < t \le s \le T$$

with $\{\varphi_s(t,x), s \in [t,T]\}$ being bounded and predictable, it is interesting to understand $(\theta_Y,\,\theta_Z)(s, X'_s(t,x))$ in the FBSDE framework. Indeed, define the following equivalent probability measure:

$$\begin{aligned} \frac{d\mathbb{Q}^{t,x}}{d\mathbb{P}} := \exp\Bigg(&\frac{1}{\sqrt{\nu}}\int_t^T \big[(b+\alpha\theta_Y)(s,X'_s(t,x)) - \varphi_s(t,x)\big]\,dW_s \\ &- \frac{1}{2\nu}\int_t^T |(b+\alpha\theta_Y)(s,X'_s(t,x)) - \varphi_s(t,x)|^2\,ds\Bigg). \end{aligned}$$

Then in view of Girsanov theorem, there is a standard Brownian motion $(W', \mathbb{Q}^{t,x})$ such that

$$X'_s(t,x) = x + \int_t^r (b+\alpha\theta_Y)(r, X'_r(t,x))\,dr + \int_t^s \sqrt{\nu}\,dW'_r,\ T_0 < t \le s \le T.$$

Then, we obtain

$$\begin{aligned}\theta_Y(\tau, X'_\tau(t,x)) &= \psi(X'_T(t,x)) + \int_\tau^T \phi(s, X'_s(t,x))\,ds - \sqrt{\nu}\int_\tau^T \theta_Z(s, X'_s(t,x))\,dW'_s \\ &= \psi(X'_T(t,x)) + \int_\tau^T \Big\{ \big[\phi(s, X'_s(t,x)) + \theta_Z(b+\alpha\theta_Y)\big](s, X'_s(t,x)) - Z\varphi_s(t,x)\Big\}\,ds \\ &\quad - \sqrt{\nu}\int_\tau^T \theta_Z(s, X'_s(t,x))\,dW_s.\end{aligned}$$

Proof (Proof of Proposition 7)

By Lemma 6, it remains for us to consider the case $\alpha \neq 0$.

Step 1. We shall prove the existence of the solution. Choose a sequence $(b_n, \phi_n, \psi_n) \in C_c^\infty(\mathbb{R}^{1+d}) \times C_c^\infty(\mathbb{R}^{1+d}) \times C_c^\infty(\mathbb{R}^d)$ such that

$$\begin{aligned}&\lim_{n\to\infty} \Big\{ \|b_n - b\|_{C([0,T];H^m)} + \|b_n - b\|_{L^2(0,T;H^{m+1})} + \|\phi_n - \phi\|_{L^2(0,T;H^{m-1})} + \|\psi_n - \psi\|_m \Big\} = 0, \\ &\|b_n\|_{C([0,T];H^m)} \le C\|b\|_{C([0,T];H^m)},\ \|\phi_n\|_{L^2(0,T;H^{m-1})} \le C\|\phi\|_{L^2(0,T;H^{m-1})},\ \ \|\psi_n\|_m \le C\|\psi\|_m,\end{aligned}$$

and $\|b_n\|_{L^2(0,T;H^{m+1})} \le C\|b\|_{L^2(0,T;H^{m+1})}$, where C is a universal constant being independent of n. By the existing FBSDE theory (for instance, see [33]), for each n, the FBSDE (24) with (b,ϕ,ψ) being replaced by smooth triple (b_n, ϕ_n, ψ_n) admits a local solution (X^n, Y^n, Z^n) on some time interval $(\tau, T]$ such that $(\theta_{Y^n}, \theta_{Z^n})$ satisfies (32) and (28) associated with (b_n, ϕ_n, ψ_n). Moreover, we have

$$\theta_{Z^n}(t,x) = \nabla\theta_{Y^n}(t,x), Y^n_s(t,x) = \theta_{Y^n}(s, X^n_s(t,x)),\ Z^n_s(t,x) = \theta_{Z^n}(s, X^n_s(t,x)),$$

and by Lemma 6,

$$\begin{aligned}&\|\theta_{Y^n}(s)\|_m^2 + \nu\int_s^T \|\theta_{Z^n}(r)\|_m^2\,dr \\ &= \|\psi_n\|_m^2 + \int_s^T 2\left(\langle \theta_{Z^n}(b_n + \alpha\theta_{Y^n})(r),\, \theta_{Y^n}(r)\rangle_{m-1,m+1} + \langle (\phi_n(r),\, \theta_{Y^n}(r)\rangle_{m-1,m+1}\right)\,dr \\ &\le \|\psi_n\|_m^2 + C\int_s^T \Big(\|(b_n + \alpha\theta_{Y^n})(r)\|_m \|\theta_{Z^n}(r)\|_{m-1}\|\theta_{Y^n}(r)\|_{m+1} + \|\phi_n(r)\|_{m-1}\|\theta_{Y^n}(r)\|_{m+1}\Big)dr \\ &\le C(\nu)\Big\{ \int_s^T \Big(\|b(r)\|^2_{L^\infty(0,T;H^m)} + 1\Big)\|\theta_{Y^n}(r)\|_m^2\,dr + \alpha^2\int_s^T \|\theta_{Y^n}(r)\|_m^4\,dr + \int_s^T \|\phi(r)\|_{m-1}^2\,dr \Big\} \\ &\quad + \frac{\nu}{2}\int_s^T \|\theta_{Z^n}(r)\|_m^2\,dr + C\|\psi\|_m^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\theta_{Y^n}(s)\|_m^2 + \nu\int_s^T \|\theta_{Z^n}(r)\|_m^2\,dr &\le C(\nu,\phi,\psi) + C(\nu,b)\left(\int_s^T \|\theta_{Y^n}(r)\|_m^2\,dr + \alpha^2\int_s^T \|\theta_{Y^n}(r)\|_m^4\,dr\right) \\ &\le C(\phi,\psi,\nu,b,\alpha,T) + C(\nu,b)\alpha^2\int_s^T \|\theta_{Y^n}(r)\|_m^4\,dr. \qquad (33)\end{aligned}$$

Hence, there is a constant $\tau_0 \in [0,T)$ depending only on $\|\phi_n\|_{L^2(0,T;H^{m-1})}$, $\|\psi_n\|_m$, $\|b_n\|_{C([0,T];H^m)}$, T, ν and α, such that for $C_0 := C(\phi,\, \psi,\, \nu,\, b,\, \alpha,\, T)$, $\forall s \in (\tau_0, T]$,

$$\sup_{r\in[s,T]} \|\theta_{Y^n}(r)\|_m^2 + \nu\int_s^T \|\theta_{Z^n}(r)\|_m^2\,dr \le \frac{C_0}{1 - \alpha^2 C_0 C(\nu,b)(T-s)}. \qquad (34)$$

As a consequence, we are allowed to choose a uniform existing time interval $(\tau, T]$ for all (X^n, Y^n, Z^n), $n \in \mathbb{Z}$.

Put

$$(X^{nk}, Y^{nk}, Z^{nk}, b_{nk}, \phi_{nk}, \psi_{nk}) = (X^n, Y^n, Z^n, b_n, \phi_n, \psi_n) - (X^k, Y^k, Z^k, b_k, \phi_k, \psi_k).$$

Then for each fixed $\varepsilon \in (0, T-\tau)$, we have, for any $s \in (\tau+\varepsilon, T]$

$$\begin{aligned}
&\|\theta_{Y^{nk}}(s)\|_m^2 + \nu \int_s^T \|\theta_{Z^{nk}}(r)\|_m^2 \, dr \\
= &\|\psi_{nk}\|_m^2 + \int_s^T 2\left(\langle \theta_{Z^n} b_{nk}(r) + \theta_{Z^{nk}} b_k(r),\, \theta_{Y^{nk}}(r)\rangle_{m-1,m+1} + \langle \phi_{nk}(r),\, \theta_{Y^{nk}}(r)\rangle_{m-1,m+1}\right) dr \\
&+ \int_s^T 2\alpha \langle \theta_{Z^n}\theta_{Y^{nk}}(r) + \theta_{Z^{nk}}\theta_{Y^k}(r),\, \theta_{Y^{nk}}(r)\rangle_{m-1,m+1} \, dr \\
\le &\|\psi_{nk}\|_m^2 + \frac{\nu}{2}\int_s^T (\|\theta_{Z^{nk}}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2)\, dr \\
&+ C(\nu)\Big\{ \alpha^2 \int_s^T \left(\|\theta_{Y^{nk}}(r)\|_m^2 \|\theta_{Y^n}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2 \|\theta_{Y^k}(r)\|_m^2\right) dr \\
&+ \int_s^T \left(\|\phi_{nk}(r)\|_{m-1}^2 + \|b_{nk}(r)\|_m^2 \|\theta_{Y^n}(r)\|_m^2 + \|b_k(r)\|_m^2 \|\theta_{Y^{nk}}(r)\|_m^2\right) dr \Big\} \\
\le &\|\psi_{nk}\|_m^2 + \frac{\nu}{2}\int_s^T (\|\theta_{Z^{nk}}(r)\|_m^2 + \|\theta_{Y^{nk}}(r)\|_m^2)\, dr \\
&+ C\int_0^T \left(\|b_{nk}(r)\|_m^2 + \|\phi_{nk}(r)\|_{m-1}^2 + \|\theta_{Y^{nk}}(r)\|_m^2\right) dr.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sup_{s\in[T_0+\varepsilon,T]} \|\theta_{Y^{nk}}(s)\|_m^2 + \nu \int_{T_0+\varepsilon}^T \|\theta_{Z^{nk}}(r)\|_m^2 \, dr \\
\le &C\left(\|\psi_{nk}\|_m^2 + \int_0^T \left(\|b_{nk}(r)\|_m^2 + \|\phi_{nk}(r)\|_{m-1}^2\right) dr \right) \longrightarrow 0 \text{ as } n,k\to\infty,
\end{aligned} \tag{35}$$

where the constant C is independent of n and k. Then $\{(\theta_{Y^n}, \theta_{Z^n}), n \in \mathbb{Z}^+\}$ is a Cauchy sequence in $C([\tau+\varepsilon, T]; H^m) \times L^2(\tau+\varepsilon, T; H^m)$, whose limit is denoted by $(\zeta, \nabla\zeta)$. Moreover, through a bootstrap argument, we can extend the existing interval to a maximal one denoted by $(T_0, T]$ with T_0 depending on $\|\psi\|_m$, $\|b\|_{C([0,T];H^m)}$, $\|\phi\|_{L^2(0,T;H^{m-1})}$, α, ν and T.

On the other hand, consider the following FBSDE:

$$\left\{\begin{aligned}
dX_s(t,x) &= [b(s, X_s(t,x)) + \alpha\zeta(s, X_s(t,x))]\, ds + \sqrt{\nu}\, dW_s; \quad X_t(t,x) = x; \\
-dY_s(t,x) &= \phi(s, X_s(t,x))\, ds - \sqrt{\nu} Z_s(t,x)\, dW_s,\ s \in [t,T]; \quad Y_T(t,x) = \psi(X_T(t,x)),
\end{aligned}\right. \tag{36}$$

which admits a unique local solution (X, Y, Z) on $(T_0, T]$. From Lemma 6, we have (27), (28) and (29). Letting $k \to \infty$ in (35), we have $\zeta(t,x) = \theta_Y(t,x)$ and $\nabla\zeta(t,x) = \theta_Z(t,x)$ for *a.e.* $(t,x) \in (T_0, T] \times \mathbb{R}^d$. Furthermore, from Lemma 6, we deduce that the triple $(X_s(t,x),\ \zeta(s, X_s(t,x)),\ \nabla\zeta(s, X_s(t,x)))$ solves the FBSDE (24) and satisfies all the assertions of this proposition except the uniqueness, which is left to the next step.

Step 2. We now verify the uniqueness. For simplicity, we assume $\tau = T_0$. Let (X, Y, Z) be any locally bounded solution of (24) on $(T_0, T]$. For each $t \in (T_0, T]$ and almost every $x \in \mathbb{R}^d$, define the following equivalent probability measure:

$$\begin{aligned}
d\mathbb{Q}^{t,x} := \exp\Bigg(&-\frac{1}{\sqrt{\nu}} \int_t^T [b(s, X_s(t,x)) + \alpha Y_s(t,x)]\, dW_s \\
&- \frac{1}{2\nu}\int_t^T |b(s, X_s(t,x)) + \alpha Y_s(t,x)|^2\, ds \Bigg) d\mathbb{P}.
\end{aligned}$$

Then the FBSDE (24) reads

$$\begin{cases} dX_s(t,x) = \sqrt{\nu}\, dW'_s, \ s \in [t,T]; \quad X_t(t,x) = x; \\ -dY_s(t,x) = \Big[\phi(s, X_s(t,x)) + Z_s(t,x)\big(b(s,X_s(t,x)) + \alpha Y_s(t,x)\big)\Big]\, ds - \sqrt{\nu} Z_s(t,x)\, dW'_s; \\ Y_T(t,x) = \psi(X_T(t,x)), \end{cases}$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion.

Define

$$\widetilde{Y}^n_s(t,\cdot) = \theta_{Y^n}(s, X_s(t,\cdot)) \text{ and } \widetilde{Z}^n_s(t,\cdot) = \theta_{Z^n}(s, X_s(t,\cdot)). \tag{37}$$

As $m > d/2$ and $H^m \hookrightarrow C^{0,\delta}(\mathbb{R}^d)$ for some $\delta \in (0,1)$, there is a constant N^t such that

$$\sup_n \left(\sup_{s\in[t,T], x\in\mathbb{R}^d} \big|\theta_{Y^n}(s,x)\big| + \int_t^T \sup_{x\in\mathbb{R}^d} \big|\theta_{Z^n}(s,x)\big|\, ds \right) \le N^t.$$

By Remark 8, we have for almost all $x \in \mathbb{R}^d$,

$$\begin{aligned} \widetilde{Y}^n_s(t,x) =& \psi_n(X_T(t,x)) + \int_s^T \big[\widetilde{Z}^n_r(t,x)\big(b_n(r, X_r(t,x)) + \alpha \widetilde{Y}^n_r(t,x)\big) + \phi_n(r, X_r(t,x))\big]\, dr \\ & - \sqrt{\nu} \int_s^T \widetilde{Z}^n_r(t,x)\, dW'_r, \ t \le s \le T. \end{aligned}$$

Then Itô's formula yields

$$\begin{aligned} &|\widetilde{Y}^n_s(t,x) - Y_s(t,x)|^2 + \nu \int_s^T |\widetilde{Z}^n_r(t,x) - Z_r(t,x)|^2\, dr \\ =& |\psi_n(X_T(t,x)) - \psi(X_T(t,x))|^2 - 2\int_s^T \langle \widetilde{Y}^n_r(t,x) - Y_r(t,x),\, (\widetilde{Z}^n_r - Z_r)(t,x)\, dW'_r\rangle \\ &+ 2\int_s^T \langle \widetilde{Y}^n_r(t,x) - Y_r(t,x),\, \widetilde{Z}^n_r(t,x)\big(b_n(r,X_r(t,x)) + \alpha\widetilde{Y}^n_r(t,x)\big) - Z_r(t,x)\big(b(r,X_r(t,x)) + \alpha Y_r(t,x)\big) \\ &\quad + \phi_n(r, X_r(t,x)) - \phi(r, X_r(t,x))\rangle\, dr. \end{aligned}$$

Note that both $(\widetilde{Y}^n_\cdot(t,x), \widetilde{Z}^n_\cdot(t,x))$ and $((Y_\cdot(t,x), Z_\cdot(t,x)))$ belong to $S^2([t,T];\mathbb{R}^d) \times L^2_{\mathscr{F}}(t,T;\mathbb{R}^d \times \mathbb{R}^d)$ and moreover, there exists a constant $K^{t,x}$ such that

$$\sup_{s\in[t,T]} |Y_s(t,x)| \le K^{t,x}, a.s..$$

Then, we have

$$\begin{aligned} & E_{\mathbb{Q}^{t,x}}\left[|\widetilde{Y}^n_s(t,x) - Y_s(t,x)|^2 + \nu \int_s^T |\widetilde{Z}^n_r(t,x) - Z_r(t,x)|^2\, dr \right] \\ =&\ E_{\mathbb{Q}^{t,x}}[|\psi_n(X_T(t,x)) - \psi(X_T(t,x))|^2] + 2E_{\mathbb{Q}^{t,x}}\Big[\int_s^T \langle \widetilde{Y}^n_r(t,x) - Y_r(t,x), \\ & \widetilde{Z}^n_r(t,x)\big(b_n(r,X_r(t,x)) + \alpha\widetilde{Y}^n_r(t,x)\big) - Z_r(t,x)\big(b(r,X_r(t,x)) + \alpha Y_r(t,x)\big) \\ & + \phi_n(r,X_r(t,x)) - \phi(r,X_r(t,x))\rangle\, dr \Big] \\ \le& \frac{\nu}{2} E_{\mathbb{Q}^{t,x}}\Big[\int_s^T |\widetilde{Z}^n_r(t,x) - Z_r(t,x)|^2\, dr\Big] + C(m,d)\|\psi_n - \psi\|_m^2 \\ & + CE_{\mathbb{Q}^{t,x}}\Big[\int_s^T \big(|\phi_n(r,X_r(t,x)) - \phi(r,X_r(t,x))|^2 + N^t\|b_n - b\|^2_{C([0,T];H^m)} \\ & \qquad + |Y^n_r(t,x) - Y_r(t,x)|^2(1 + \|b(r)\|_m^2 + \|\theta_{Z^n}(r)\|_{C(\mathbb{R}^d)} + |Y_r(t,x)|^2)\big)\, dr \\ \le& \frac{\nu}{2} E_{\mathbb{Q}^{t,x}}\Big[\int_s^T |\widetilde{Z}^n_r(t,x) - Z_r(t,x)|^2\, dr\Big] + C(m,d)\|\psi_n - \psi\|_m^2 \end{aligned}$$

$$+CE_{\mathbb{Q}^{t,x}}\Big[\int_s^T \big(|\phi_n(r,X_r(t,x))-\phi(r,X_r(t,x))|^2+\|b_n-b\|^2_{C([0,T];H^m)}$$
$$+|Y_r^n(t,x)-Y_r(t,x)|^2(1+\|\theta_{Z^n}(r)\|_{C(\mathbb{R}^d)})\big)\,dr.$$

Then, from Gronwall Inequality, we have

$$\begin{aligned}&\sup_{s\in[t,T]} E_{\mathbb{Q}^{t,x}}\Big[|\widetilde{Y}_s^n(t,x)-Y_s(t,x)|^2\Big]+\nu E_{\mathbb{Q}^{t,x}}\Big[\int_t^T |\widetilde{Z}_s^n(t,x)-Z_s(t,x)|^2 ds\Big]\\ &\le C\Big\{E_{\mathbb{Q}^{t,x}}\Big[\int_s^T \big(|\phi_n(r,X_r(t,x))-\phi(r,X_r(t,x))|^2\big)\,dr\Big]\\ &\quad+\|\psi_n-\psi\|_m^2+\|b-b_n\|^2_{C([0,T];H^m)}\Big\}\longrightarrow 0,\end{aligned}$$

where the constant C depends only on N^t, $K^{t,x}$, T, $\|b\|_{C([0,T];H^m)}$, m, d, ν and α, and is independent of n.

Thus, in view of (37), we conclude that

$$Y_s(t,x)=\zeta(s,X_s(t,x)) \text{ and } Z_s(t,x)=\nabla\zeta(s,X_s(t,x)).$$

Therefore, any locally bounded solution of the FBSDE (24) on $(T_0,T]$ must have the form described as above. Now, let (X,Y,Z) and $(\bar{X},\bar{Y},\bar{Z})$ be any two locally bounded solutions of the FBSDE (24) on $(T_0,T]$. By previous argument we have

$$\begin{aligned}Y_s(t,x)&=\zeta(s,X_s(t,x)),\;\; Z_s(t,x)=\nabla\zeta(s,X_s(t,x)),\\ \bar{Y}_s(t,x)&=\zeta(s,X_s(t,x)),\;\; \bar{Z}_s(t,x)=\nabla\zeta(s,X_s(t,x)).\end{aligned}\tag{38}$$

Hence $X.(t,x)$ and $\bar{X}.(t,x)$ satisfy the same forward SDE with the same initial value. Thus we must have $X\equiv\bar{X}$, a.s., which in turn shows that $(Y,Z)=(\bar{Y},\bar{Z})$, a.s..

Step 3. For $m>d/2+1$, to prove that the unique locally bounded solution on $(T_0,T]$ is also the unique local solution, it is sufficient to prove that the locally bounded solution constructed in **Step 1**

$$(X_s(t,x),\zeta(s,X_s(t,x)),\nabla\zeta(s,X_s(t,x)))_{T_0<t\le s\le T}$$

is the unique local solution to the FBSDE (24) on $(T_0,T]$ as well.

Since $m>d/2+1$ and $H^{m-1}\hookrightarrow C^{0,\delta}(\mathbb{R}^d)$ for some $\delta\in(0,1)$, our BSDE is well-posed for each $x\in\mathbb{R}^d$. Let (X,Y,Z) be any solution of (24) on $[t,T]$ with $t\in(T_0,T]$. For every $x\in\mathbb{R}^d$, define

$$\widetilde{Y}_s^n(t,x)=\theta_{Y^n}(s,X_s(t,x)) \text{ and } \widetilde{Z}_s^n(t,x)=\theta_{Z^n}(s,X_s(t,x)).\tag{39}$$

By Remark 8, we have

$$\begin{aligned}\widetilde{Y}_s^n(t,x)&=\psi_n(X_T(t,x))-\sqrt{\nu}\int_s^T \widetilde{Z}_r^n(t,x)\,dW_r\\ &\quad+\int_s^T\big[\widetilde{Z}_r^n(t,x)\big(b_n(r,X_r(t,x))+\alpha\widetilde{Y}_r^n(t,x)-b(r,X_r(t,x))-\alpha Y_r(t,x)\big)\\ &\qquad+\phi_n(r,X_r(t,x))\big]\,dr,\;\; t\le s\le T,\;\text{ a.e. } x\in\mathbb{R}^d.\end{aligned}$$

Then Itô's formula yields

$$\begin{aligned}&|\widetilde{Y}_s^n(t,x)-Y_s(t,x)|^2+\nu\int_s^T |\widetilde{Z}_r^n(t,x)-Z_r(t,x)|^2\,dr\\ &=|\psi_n(X_T(t,x))-\psi(X_T(t,x))|^2-2\int_s^T\langle\widetilde{Y}_r^n(t,x)-Y_r(t,x),\,(\widetilde{Z}_r^n-Z_r)(t,x)\,dW_r\rangle\\ &\quad+2\int_s^T\langle\widetilde{Y}_r^n(t,x)-Y_r(t,x),\,\widetilde{Z}_r^n(t,x)\big(b_n(r,X_r(t,x))+\alpha\widetilde{Y}_r^n(t,x)-b(r,X_r(t,x))-\alpha Y_r(t,x)\big)\\ &\qquad+\phi_n(r,X_r(t,x))-\phi(r,X_r(t,x))\rangle\,dr.\end{aligned}$$

Note that both $(\widetilde{Y}^n_\cdot(t,x), \widetilde{Z}^n_\cdot(t,x))$ and $((Y_\cdot(t,x), Z_\cdot(t,x)))$ belong to $S^2([t,T];\mathbb{R}^d) \times L^2_{\mathscr{F}}(t,T;\mathbb{R}^d \times \mathbb{R}^d)$ and moreover that there exists a positive constant K_t such that

$$\sup_n \sup_{s\in[t,T]} |\widetilde{Z}^n_s(t,x)| \le C\|\theta_{Z^n}\|_{C([t,T];H^{m-1})} \le K_t, \; a.s., \; \forall x \in \mathbb{R}^d.$$

Thus, we have

$$\begin{aligned}
&E\left[|\widetilde{Y}^n_s(t,x) - Y_s(t,x)|^2 + \nu \int_s^T |\widetilde{Z}^n_r(t,x) - Z_r(t,x)|^2 \, dr\right] \\
&= E[|\psi_n(X_T(t,x)) - \psi(X_T(t,x))|^2] + 2E\bigg[\int_s^T \langle \widetilde{Y}^n_r(t,x) - Y_r(t,x), \\
&\quad \widetilde{Z}^n_r(t,x)\big(b_n(r,X_r(t,x)) + \alpha\widetilde{Y}^n_r(t,x) - b(r,X_r(t,x)) - \alpha Y_r(t,x)\big) \\
&\quad\; + \phi_n(r,X_r(t,x)) - \phi(r,X_r(t,x))\rangle \, dr\bigg] \\
&\le C(K_t,T,\alpha,\nu,d,m)E\bigg[\int_s^T |\widetilde{Y}^n_r(t,x) - Y_r(t,x)|(|\widetilde{Y}^n_r(t,x) - Y_r(t,x)| \\
&\quad\; + \|b(r) - b_n(r)\|_m + \|\phi_n(r) - \phi(r)\|_{m-1}) \, dr\bigg] + C\|\psi_n - \psi\|^2_m.
\end{aligned}$$

Using Gronwall inequality, we have

$$\begin{aligned}
&\sup_{s\in[t,T]} E\left[|\widetilde{Y}^n_s(t,x) - Y_s(t,x)|^2\right] + \nu E\left[\int_t^T |\widetilde{Z}^n_s(t,x) - Z_s(t,x)|^2 \, ds\right] \\
&\le C\{\|\psi_n - \psi\|^2_m + \|b - b_n\|^2_{C([0,T];H^m)} + \|\phi_n - \phi\|^2_{L^2(0,T;H^{m-1})}\} \longrightarrow 0.
\end{aligned} \tag{40}$$

Thus, in view of (39), we conclude that

$$Y_s(t,x) = \zeta(s, X_s(t,x)) \text{ and } Z_s(t,x) = \nabla\zeta(s, X_s(t,x)).$$

Analogous to the arguments of **Step 2**, we verify that the triple constructed in **Step 1**

$$(X_s(t,x), \zeta(s,X_s(t,x)), \nabla\zeta(s,X_s(t,x)))_{T_0<t\le s\le T}$$

is the unique local solution to the FBSDE (24) on $(T_0,T]$ as well. The proof is complete.

Remark 9 *In Step 1, we use the existing FBSDE theory (for instance, see [33]) to guarantee the existence of the solution (X^n,Y^n,Z^n). In fact, we can prove the existence of (X^n,Y^n,Z^n) in a similar way to Step 1 of the following proof for Theorem 5 and this would make our proof self-contained. In Step 2, the equation* (28) *plays a crucial role in the proof of uniqueness. Indeed, the FBSDE* (24) *is usually associated to the deterministic vector field θ_Y which satisfies* (28) *instead of being a classical solution to some parabolic PDE in [33]. Note that equation* (28) *is probabilistic and that according to Lemma 3, it makes sense for our Sobolev coefficients. Therefore, our method here helps to probabilistically solve more general coupled FBSDEs.*

On the other hand, in view of the whole proof of Proposition 7, we have

$$T_0 \le 0 \vee \left[T - \frac{1}{\alpha^2 C_0 C(\nu,b)}\right]$$

and also, if $T\alpha^2C_0C(\nu,b) < 1$ in (34)*, then the local bounded solution is actually a global solution on the whole interval* $[0,T]$.

Remark 10 *If $\widetilde{Y}_0(\cdot,\cdot)$ of the FBSDS* (3) *lies in $L^2(T_0,T;H^{m-1})$, then θ_Y of* (3) *is deterministic and belongs to $L^2(T_0,T;H^{m+1})$. Therefore, by Lemma 3 and Proposition 7, Definition 2 and Remark 3 make senses.*

From Proposition 7, we have the following characterization of an H^m-solution to the FBSDS (3) for $m > d/2$, whose proof is omitted.

Corollary 8 *Under assumptions of Theorem 5, $(X,Y,Z,\widetilde{Y}_0)$ is an (local, respectively) H^m-solution of the FBSDS (3) (on some time interval $(T_0,T]$, respectively) if and only if $(X,Y,Z,\widetilde{Y}_0)$ is a solution of the FBSDS (3) with $\theta_Y\in C([0,T];H^m)\cap L^2(0,T;H^{m+1})$ ($\theta_Y\in C_{loc}((T_0,T];H^m)\cap L^2_{loc}(T_0,T;H^{m+1})$, respectively) and*

$$\theta_Z(t,\cdot)=\nabla\theta_Y(t,\cdot), Y_s(t,\cdot)=\theta_Y(s,X_s(t,\cdot)) \text{ and } Z_s(t,\cdot):=\theta_Z(s,X_s(t,\cdot)),\ t\le s\le T.$$

.

4.2 Proof of Theorem 5

First, for each $v\in C([0,T];H^m_\sigma)\cap L^2(0,T;H^{m+1}),\zeta\in C([0,T];H^m)\cap L^2(0,T;H^{m+1})$, consider the following FBSDS:

$$\begin{cases} dX_s(t,x)=v(s,X_s(t,x))\,ds+\sqrt{\nu}\,dW_s,\ s\in[t,T];\quad X_t(t,x)=x;\\ -dY_s(t,x)=\Big[f(s,X_s(t,x))+\widetilde{Y}_0(s,X_s(t,x))\Big]\,ds-\sqrt{\nu}Z_s(t,x)\,dW_s;\\ Y_T(t,x)=G(X_T(t,x));\\ -d\widetilde{Y}_s(t,x)=\dfrac{27}{2s^3}v^i\zeta^j(t,x+B_s)\left(B^i_{\frac{2s}{3}}-B^i_{\frac{s}{3}}\right)\left(B^j_s-B^j_{\frac{2s}{3}}\right)B_{\frac{s}{3}}\,ds-\,dM_s,\ s\in(0,\infty);\\ \widetilde{Y}_\infty(t,x)=0. \end{cases} \tag{41}$$

By BSDE theory (see [21,37]) and Lemma 4, the FBSDS (41) admits a unique solution ($X^{v,\zeta}$, $Y^{v,\zeta}$, $Z^{v,\zeta}$, $\widetilde{Y}_0^{v,\zeta}$) and in view of Lemma 4 and Remark 4, there holds that

$$Y_0(t,x)=-\mathbf{P}^\perp(\mathrm{div}(v(t,x)\otimes\zeta(t,x)))=-\mathbf{P}^\perp((v(t,x)\cdot\nabla)\zeta(t,x)),$$

where we have used the fact that $\mathrm{div}(v)=0$. By Lemma 6, we have $(\theta_{Y^{v,\zeta}},\theta_{Z^{v,\zeta}})\in C([0,T];H^m)\times L^2(0,T;H^m)$ and $\theta_{Z^{v,\zeta}}=\nabla\theta_{Y^{v,\zeta}}$.

For any $\zeta_i\in C([0,T];H^m)\cap L^2(0,T;H^{m+1})$, $i=1,2$, put

$$(\delta\theta_{Y^{v,\zeta}},\,\delta\theta_{Z^{v,\zeta}},\,\delta\zeta):=(\theta_{Y^{v,\zeta_1}}-\theta_{Y^{v,\zeta_2}},\,\theta_{Z^{v,\zeta_1}}-\theta_{Z^{v,\zeta_2}},\,\zeta_1-\zeta_2).$$

By Lemma 6, we have

$$\begin{aligned} &\|\delta\theta_{Y^{v,\zeta}}(s)\|_m^2+\nu\int_s^T\|\delta\theta_{Z^{v,\zeta}}(r)\|_m^2\,dr\\ &=\int_s^T 2\langle\delta\theta_{Z^{v,\zeta}}v(r),\,\delta\theta_{Y^{v,\zeta}}(r)\rangle_{m-1,m+1}\,dr-\int_s^T 2\langle\mathbf{P}^\perp((v\cdot\nabla)\delta\zeta)(r),\,\delta\theta_{Y^{v,\zeta}}(r)\rangle_{m-1,m+1}\,dr\\ &\le\frac{\nu}{2}\int_s^T(\|\delta\theta_{Y^{v,\zeta}}(r)\|_m^2+\|\delta\theta_{Z^{v,\zeta}}(r)\|_m^2)\,dr\\ &\quad+C(\nu)\left(\int_s^T\left(\|v(r)\|_m^2\|\delta\theta_{Y^{v,\zeta}}(r)\|_m^2+\|v(r)\|_m^2\|\delta\zeta(r)\|_m^2\right)\,dr.\right) \end{aligned}$$

Using Gronwall inequality, we obtain

$$\sup_{s\in[t,T]}\|\delta\theta_{Y^{v,\zeta}}(s)\|_m^2+\int_t^T\|\delta\theta_{Z^{v,\zeta}}(r)\|_m^2\,dr\le C(T-t)\|\delta\zeta\|^2_{C([t,T];H^m)} \tag{42}$$

with the constant C depending on $\nu,\|v\|_{C([0,T];H^m)}$ and T. Then, by the contraction mapping principle we can choose a small enough positive constant $\epsilon\le T$ depending only on $\nu,\|v\|_{C([0,T];H^m)}$ and T, such that there exists a unique function $\bar\zeta\in C([T-\varepsilon,T];H^m)\cap L^2(T-\varepsilon,T;H^{m+1})$ satisfying

$$(\theta_{Y^{v,\bar\zeta}},\theta_{Z^{v,\bar\zeta}})=\left(\bar\zeta,\nabla\bar\zeta\right)\ \text{ in } C([T-\epsilon,T];H^m)\times L^2(T-\epsilon,T;H^m).$$

Then by Lemmas 4 and 6, we have for almost all $x\in\mathbb{R}^d$,

$$\theta_{Z^{v,\bar\zeta}}(t,x)=\nabla\theta_{Y^{v,\bar\zeta}}(t,x),\left(Y_r^{v,\bar\zeta}(t,x),Z_r^{v,\bar\zeta}(t,x)\right)=\left(\theta_{Y^{v,\bar\zeta}},\theta_{Z^{v,\bar\zeta}}\right)(r,X_r(t,x)),a.s.,$$

$$
\begin{aligned}
\theta_{Y^{v,\bar\zeta}}(r, X_r(t,x)) = &G(X_T(t,x)) - \sqrt{\nu}\int_r^T \theta_{Z^{v,\bar\zeta}}(s, X_s(t,x))\, dW_s \\
&+ \int_r^T \Big[f(s, X_s(t,x)) - \mathbf{P}^{\perp}((v\cdot\nabla)\theta_{Y^{v,\bar\zeta}})(s, X_s(t,x))\Big]\, ds,\ a.s..
\end{aligned}
$$

For each $(t,x)\in[T-\varepsilon, T)\times\mathbb{R}^d$, define the following equivalent probability measure:

$$
d\mathbb{Q}^{t,x} := \exp\left(-\frac{1}{\sqrt{\nu}}\int_t^T v(s, X_s(t,x))\, dW_s - \frac{1}{2\nu}\int_t^T |v(s, X_s(t,x))|^2\, ds\right) d\mathbb{P}.
$$

Then the FBSDS (41) reads

$$
\begin{cases}
dX_s(t,x) = \sqrt{\nu}\, dW'_s,\ \ s\in[t,T]; \quad X_t(t,x) = x;\\
-dY_s(t,x) = \left[f + \theta_Z v + \widetilde{Y}_0\right](s, X_s(t,x))\, ds - \sqrt{\nu} Z_s(t,x)\, dW'_s;\\
Y_T(t,x) = G(X_T(t,x));\\
-d\widetilde{Y}_s(t,x) = \dfrac{27}{2s^3} v^i \theta_Y^j(t, x+B_s)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)\left(B^j_s - B^j_{\frac{2s}{3}}\right) B_{\frac{s}{3}}\, ds - dM_s, \quad s\in(0,\infty);\\
\widetilde{Y}_\infty(t,x) = 0,
\end{cases}
\tag{43}
$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion. Then taking the divergence operator on both sides of the above third and fourth equalities, we conclude that

$$
\theta_{Y^{v,\bar\zeta}} \in C([t,T]; H^m_\sigma)\cap L^2(t,T; H^{m+1}_\sigma).
$$

On the other hand, by Lemmas 2 and 6, we have

$$
\begin{aligned}
&\|\theta_{Y^{v,\bar\zeta}}(s)\|_m^2 + \nu\int_s^T \|\theta_{Z^{v,\bar\zeta}}(r)\|_m^2\, dr\\
&= \|G\|_m^2 + \int_s^T 2\left(\langle (v\cdot\nabla)\theta_{Y^{v,\bar\zeta}}(r),\, \theta_{Y^{v,\bar\zeta}}(r)\rangle_{m-1,m+1} + \langle f(r),\, \theta_{Y^{v,\bar\zeta}}(r)\rangle_{m-1,m+1}\right) dr\\
&\quad - \int_s^T 2\langle \mathbf{P}^{\perp}\left((v\cdot\nabla)\theta_{Y^{v,\bar\zeta}}\right)(r),\, \theta_{Y^{v,\bar\zeta}}(r)\rangle_{m-1,m+1}\, dr\\
&\quad \text{(in view of Remark 4)}\\
&= \|G\|_m^2 + \int_s^T 2\left(\langle (v\cdot\nabla)\theta_{Y^{v,\bar\zeta}}(r),\, \theta_{Y^{v,\bar\zeta}}(r)\rangle_{m-1,m+1} + \langle f(r),\, \theta_{Y^{v,\bar\zeta}}(r)\rangle_{m-1,m+1}\right) dr\\
&\le \|G\|_m^2 + C\left(\int_s^T \|v(r)\|_m \|\theta_{Y^{v,\bar\zeta}}(r)\|_m^2\, dr + \frac{1}{\nu}\int_s^T \|f(r)\|_{m-1}^2\, dr\right)\\
&\quad + \frac{\nu}{2}\int_s^T \left(\|\theta_{Y^{v,\bar\zeta}}(r)\|_m^2 + \|\theta_{Z^{v,\bar\zeta}}(r)\|_m^2\right) dr
\end{aligned}
$$

which together with the Gronwall inequality implies

$$
\begin{aligned}
&\sup_{s\in[t,T]} \|\theta_{Y^{v,\bar\zeta}}(s)\|_m^2 + \frac{\nu}{2}\int_t^T \|\theta_{Z^{v,\bar\zeta}}(r)\|_m^2\, dr\\
&\le C(\nu, T)\left(\|f\|^2_{L^2(0,T;H^{m-1})} + \|G\|_m^2\right) e^{(\|v\|_{C([t,T];H^m)}+\nu)(T-t)}.
\end{aligned}
\tag{44}
$$

Hence, through a bootstrap argument, we conclude that there exists a unique function $\widetilde{\zeta}\in C([0,T]; H^m_\sigma)$ satisfying $(\theta_{Y^{v,\widetilde\zeta}}, \theta_{Z^{v,\widetilde\zeta}}) = (\widetilde{\zeta}, \nabla\widetilde{\zeta})$ in $C([0,T]; H^m_\sigma)\times L^2(0,T; H^m_\sigma)$, and again by Lemma 6, we conclude that

$$
(X^v, Y^v, Z^v, \widetilde{Y}^v_0) := (X^{v,\widetilde\zeta}, Y^{v,\widetilde\zeta}, Z^{v,\widetilde\zeta}, \widetilde{Y}_0^{v,\widetilde\zeta})
$$

is the unique H^m-solution of the following FBSDS:

$$\begin{cases} dX_s(t,x) = v(s, X_s(t,x))\,ds + \sqrt{\nu}\,dW_s,\ \ s\in[t,T]; \quad X_t(t,x)=x; \\ -dY_s(t,x) = \left[f(s,X_s(t,x)) + \widetilde{Y}_0(s,X_s(t,x))\right]\,ds - \sqrt{\nu}Z_s(t,x)\,dW_s; \\ Y_T(t,x) = G(X_T(t,x)); \\ -d\widetilde{Y}_s(t,x) = \dfrac{27}{2} v^i Y_t^j(t,x+B_s)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)\left(B^j_s - B^j_{\frac{2s}{3}}\right) B_{\frac{s}{3}} s^{-3}\,ds - \,dM_s,\ \ s\in(0,\infty); \\ \widetilde{Y}_\infty(t,x) = 0. \end{cases} \tag{45}$$

Choose two positive real numbers R and ε ($\varepsilon < T$) whose values are to be determined later, and define

$$U_R^\varepsilon := \Big\{ u \in C([T-\varepsilon,T]; H^m_\sigma) \cap L^2(T-\varepsilon,T;H^{m+1}_\sigma) : \\ \|u\|^2_{C([T-\varepsilon,T];H^m_\sigma)} + \frac{\nu}{2}\|\nabla u\|^2_{L^2(T-\varepsilon,T;H^{m+1}_\sigma)} \le R^2 \Big\}.$$

For any $v\in U_R^\varepsilon$, there holds the following estimate by (44):

$$\sup_{s\in[T-\varepsilon,T]} \|\theta_{Y^v}(s)\|^2_m + \frac{\nu}{2}\int_{T-\varepsilon}^T \|\theta_{Z^v}(s)\|^2_m\,dr \le\ C(\nu, \|f\|^2_{L^2(0,T;H^{m-1})}, \|G\|^2_m, T)e^{R\varepsilon}. \tag{46}$$

Choosing R to be big enough and ε to be small enough, we have

$$\sup_{s\in[T-\varepsilon,T]} \|\theta_{Y^v}(s)\|^2_m + \frac{\nu}{2}\int_{T-\varepsilon}^T \|\theta_{Z^v}(r)\|^2_m\,dr \le R^2.$$

On the other hand, for any $v_1, v_2 \in U_R^\varepsilon$, setting

$$(\delta\theta_{Y^v}, \delta\theta_{Z^v}, \delta v) := (\theta_{Y^{v_1}} - \theta_{Y^{v_2}}, \theta_{Z^{v_1}} - \theta_{Z^{v_2}}, v_1 - v_2),$$

we have

$$\begin{aligned} &\|\delta\theta_{Y^v}(s)\|^2_m + \nu\int_s^T \|\delta\theta_{Z^v}(r)\|^2_m\,dr \\ &= 2\int_s^T (\langle(\delta v\cdot\nabla)\theta_{Y^{v_1}}(r),\ \delta\theta_{Y^v}(r)\rangle_{m-1,m+1} + \langle(v_2\cdot\nabla)\delta\theta_{Y^v}(r),\ \delta\theta_{Y^v}(r)\rangle_{m-1,m+1})\ dr \\ &\le C(\nu)\left(\int_s^T \|\delta v(r)\|^2_m \|\theta_{Y^{v_1}}(r)\|^2_m\,dr + \int_s^T \|v_2(r)\|^2_m\|\delta\theta_{Y^v}(r)\|^2_m\,dr\right) \\ &\quad + \frac{\nu}{2}\int_s^T (\|\delta\theta_{Y^v}(r)\|^2_m + \|\delta\theta_{Z^v}(r)\|^2_m)\,dr, \end{aligned}$$

which together with the Gronwall-Bellman inequality, implies

$$\sup_{s\in[T-\varepsilon,T]} \|\delta\theta_{Y^v}(s)\|^2_m + \frac{\nu}{2}\int_{T-\varepsilon}^T \|\delta\theta_{Z^v}(r)\|^2_m\,dr \le\ C(\nu)R^2 e^{R^2T}\varepsilon\|\delta v\|^2_{C([T-\varepsilon,T];H^m_\sigma)}.$$

Therefore, if we choose ε to be small enough, the solution map $\Psi : v \mapsto \theta_{Y^v}$ is a contraction mapping on the complete metric space U_R^ε and then through a bootstrap argument, we obtain a unique function $\bar{u} \in C_{loc}((T_0,T]; H^m_\sigma) \cap L^2_{loc}(T_0,T;H^{m+1}_\sigma)$ satisfying $(\theta_{Y^{\bar{u}}}, \theta_{Z^{\bar{u}}}) = (\bar{u}, \nabla\bar{u})$ on $(T_0,T]\times\mathbb{R}^d$ with T_0 depending on $\nu, T, \|G\|_m$ and $\|f\|_{L^2(0,T;H^{m-1})}$. By Proposition 7, Corollary 8 and the contraction mapping principle,

$$(X,Y,Z,\widetilde{Y}_0) := (X^{\bar{u}}, Y^{\bar{u}}, Z^{\bar{u}}, \widetilde{Y}_0^{\bar{u}})$$

is the unique local H^m-solution of the FBSDS (3) and there holds (21).

From Remarks 4 and 5, we deduce that there exists some $p \in L^2_{\mathrm{loc}}(T_0, T; H^m_\sigma)$ such that $\widetilde{Y}_0 = \nabla p$. For each $t \in (T_0, T]$ a.e. $x \in \mathbb{R}^d$, define the following equivalent probability $Q^{t,x}$:

$$d\mathbb{Q}^{t,x} := \exp\left(-\frac{1}{\sqrt{\nu}}\int_t^T \theta_Y(s, X_s(t,x))\,dW_s - \frac{1}{2\nu}\int_t^T |\theta_Y(s, X_s(t,x))|^2\,ds\right) d\mathbb{P}. \tag{47}$$

Then we have

$$\begin{cases} dX_s(t,x) = \sqrt{\nu}\,dW'_s,\ \ s \in [t,T]; \quad X_t(t,x) = x; \\ -dY_s(t,x) = [(\theta_Y \cdot \nabla)\theta_Y + f + \nabla p]\,(s, X_s(t,x))\,ds - \sqrt{\nu}\nabla\theta_Y(s, X_s(t,x))\,dW'_s; \\ Y_T(t,x) = G(X_T(t,x)), \end{cases}$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion. For any $\zeta \in C_c^\infty(\mathbb{R}) \otimes C_c^\infty(\mathcal{O})$, Itô's formula yields that

$$\zeta(s, X_s(t,x)) = \zeta(T, X_T(t,x)) - \int_s^T (\partial_r + \frac{\nu}{2}\Delta)\zeta(r, X_r(t,x))dr - \int_s^T \nabla\zeta(r, X_r(t,x))dW'_r,$$

and thus,

$$\begin{aligned} &E_{\mathbb{Q}^{t,x}}[\langle\theta_Y,\,\zeta\rangle(t,x)] + \nu E\left[\int_t^T \langle\nabla\zeta,\,\nabla\theta_Y\rangle(s, X_s(t,x))\,ds\right] \\ =& E_{\mathbb{Q}^{t,x}}\left[\int_t^T (\langle -\partial_s\zeta - \frac{\nu}{2}\Delta\zeta,\ \theta_Y >\rangle + \langle\zeta,\ (\theta\cdot\nabla)\theta + \nabla p + f\rangle)(s, X_s(t,x))\,ds \right. \\ &\left. + \langle\zeta(T, X_T(t,x)),\, G(x_T(t,x))\rangle\right]. \end{aligned}$$

Integrating both sides of the last equality with respect to x, we have

$$\langle\zeta(t),\,\theta_Y(t)\rangle_0 = \langle\zeta(T),\,G\rangle_0 + \int_t^T \big[-\langle\partial_s\zeta(s),\,\theta_Y(s)\rangle_0 + \langle\zeta(s),\,\frac{\nu}{2}\Delta\theta_Y(s) + (\theta_Y\cdot\nabla)\theta_Y(s)\rangle_0\big]\,ds$$

Hence, (θ_Y, p) is a strong solution to Navier-Stokes equation (23) (see [44,45]). Because of the reversibility of the above procedure and the uniqueness of the H^m-solution of the FBSDS (3), we prove the uniqueness of the strong solution for Navier-Stokes equation (23) as well. The proof is complete.

Remark 11 *In the above proof, Proposition 7 plays an important role in characterizing the solution of the FBSDS* (3) *(see Corollary 8). This characterization together with the contraction mapping principle serves to guarantee the existence and uniqueness of the local H^m-solution of the FBSDS* (3). *On the other hand, in a similar way to the above proof, we can prove that θ_Y of Proposition 7 is in fact the unique local strong solution of the following PDE*

$$\partial_t u + \frac{\nu}{2}\Delta u + \big((b + \alpha u)\cdot\nabla\big)u + \phi = 0,\ \ t \le T; \quad u(T) = \psi, \tag{48}$$

which is the well-known Burgers equation if $\alpha = 1$ and $b \equiv 0$.

5 Global results

5.1 The case of small Reynolds numbers

We work on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/(L \times \mathbb{Z}^d)$ where $L > 0$ is a fixed length scale. Denote by $(H^{n,q}_\sigma(\mathbb{T}^d;\mathbb{R}^d), \|\cdot\|_{n,q;\mathbb{T}^d})$ the $\mathbb{R}^d$-valued Sobolev space on $\mathbb{T}^d$, each element of which is divergence free. For $q=2$, write $(H^n_\sigma(\mathbb{T}^d;\mathbb{R}^d), \|\cdot\|_{n;\mathbb{T}^d})$ for simplicity. Let $G\in H^m_\sigma(\mathbb{T}^d;\mathbb{R}^d)$ of zero mean for $m>d/2$. Consider

$$\begin{cases} dX_s(t,x) = Y_s(t,x)\,ds + \sqrt{\nu}\,dW_s,\ s\in[t,T]; \quad X_t(t,x)=x; \\ -dY_s(t,x) = \widetilde{Y}_0(s,X_s(t,x))\,ds - \sqrt{\nu}Z_s(t,x)\,dW_s,\ s\in[t,T]; \quad Y_T(t,x)=G(X_T(t,x)); \\ -d\widetilde{Y}_s(t,x) = \dfrac{27}{2s^3}Y^i_tY^j_t(t,x+B_s)\left(B^i_{\frac{2s}{3}}-B^i_{\frac{s}{3}}\right)\left(B^j_s-B^j_{\frac{2s}{3}}\right)B_{\frac{s}{3}}\,ds - dM_s, \quad s\in(0,\infty); \\ \widetilde{Y}_\infty(t,x)=0, \end{cases} \tag{49}$$

where the external forcing is not introduced for simplicity. In a similar way to Theorem 5, the FBSDS (49) admits a unique local H^m-solution on some interval $(T_0,T]$. Moreover, we have

$$\begin{aligned} \|\theta_Y(t)\|^2_{m;\mathbb{T}^d} + \nu\int_t^T \|\theta_Z(s)\|^2_{m;\mathbb{T}^d}\,ds &= \|G\|^2_{m;\mathbb{T}^d} + 2\int_t^T \langle \mathbf{P}(\theta_Z(s)\theta_Y(s)),\,\theta_Y(s)\rangle_{m-1,m+1;\mathbb{T}^d}\,ds \\ &\le \|G\|^2_{m;\mathbb{T}^d} + C\int_t^T \|\theta_Y(s)\|_{m;\mathbb{T}^d}\|\theta_Z(s)\|_{m-1;\mathbb{T}^d}\|\theta_Y(s)\|_{m+1;\mathbb{T}^d}\,ds \\ &\le \|G\|^2_{m;\mathbb{T}^d} + \widetilde{C}\int_t^T \|\theta_Y(s)\|_{m;\mathbb{T}^d}\|\theta_Z(s)\|^2_{m;\mathbb{T}^d}\,ds, \end{aligned}$$

where we have used the fact that by Poincaré inequality and the scaling properties,

$$\|\theta_Y(s,\cdot)\|_{m;\mathbb{T}^d} \le CL\|\nabla\theta_Y(s,\cdot)\|_{m;\mathbb{T}^d}, \quad s\in[t,T],$$

with $\theta_Y(s,\cdot)$ being mean zero and the constant $\widetilde{C}$ being independent of L. Thus,

$$\|\theta_Y(t)\|^2_{m;\mathbb{T}^d} + \int_t^T (\nu - \widetilde{C}L\|\theta_Y(s)\|_{m;\mathbb{T}^d})\|\theta_Z(s)\|^2_{m;\mathbb{T}^d}\,ds \le \|G\|^2_{m;\mathbb{T}^d}.$$

If we take the Reynolds number $R:=L\nu^{-1}\|G\|_{m;\mathbb{T}^d} < \widetilde{C}^{-1}$, then for this local solution $(X,Y,Z,\widetilde{Y}_0)$ we always have

$$\|\theta_Y(t)\|_{m;\mathbb{T}^d} \le \|G\|_{m;\mathbb{T}^d},\ t\in(T_0,T].$$

Using bootstrap arguments, the local solution can be extended to be a global one. In summary, we have

Theorem 9 *Assume that $G\in H^m_\sigma(\mathbb{T}^d;\mathbb{R}^d)$ ($m>d/2$) is mean zero. the FBSDS* (49) *admits one and only one local H^m-solution $(X,Y,Z,\widetilde{Y}_0)$ on some time interval $(T_0,T]$ with $\theta_Y\in H^m_\sigma(\mathbb{T}^d)$ being spacial mean zero. Moreover, there exists a positive constant R_0 ($=\frac{1}{\widetilde{C}}$ as above) such that if the Reynolds number $R<R_0$, our local H^m-solution can be extended to be a time global one and for this global H^m-solution we have*

$$\|\theta_Y(t)\|_{m;\mathbb{T}^d} \le \|G\|_{m;\mathbb{T}^d},\ \textit{for any } t\in[0,T].$$

5.2 The two-dimensional case

Consider the two dimensional case. For simplicity, we assume $f=0$ and $m\ge 3$. Then under the assumptions of Theorem 5, let $(X,Y,Z,\widetilde{Y}_0)$ be the local H^m-solution of the FBSDS (3) on the time interval $(T_0,T]$. Define the vorticity field:

$$\vartheta_Y := \text{Curl}\ \theta_Y := \partial_{x_1}\theta^2_Y - \partial_{x_2}\theta^1_Y,$$

which is scalar-valued. Consider the following FBSDE:

$$\begin{cases} dX_s(t,x) = \theta_Y(s,X_s(t,x))ds + \sqrt{\nu}dW_s; \quad X_t(t,x)=x; \\ d\bar{Y}_s(t,x) = \sqrt{\nu}\bar{Z}_sdW_s; \quad \bar{Y}_T(t,x) = (\text{Curl } G)(X_T(t,x)), \quad T_0<t\le s\le T, x\in\mathbb{R}^2. \end{cases} \tag{50}$$

By Proposition 7 and Theorem 5, we have

$$\vartheta_Y(t,x) = \bar{Y}_t(t,x) = E\left[(\text{Curl } G)(X_T(t,x))\right]. \tag{51}$$

In view of [34, page 117,Proposition 3.8], we have

$$\|\theta_Z\|_{L^\infty(\mathbb{R}^2)} \leq C(1+\ln^+\|\theta_Y\|_3+\ln^+\|\vartheta_Y\|_0)(1+\|\vartheta_Y\|_{L^\infty(\mathbb{R}^2)}), \tag{52}$$

where $\ln^+ y := 0 \vee \ln y$.

On the other hand, since $\nabla\cdot\theta_Y=0$, we have

$$\det\,(\nabla X_s(t,x)) = \exp\,\left(\int_t^s (\nabla\cdot\theta_Y)(r,X_r(t,x))\,dr\right) = 1,$$

which implies

$$\|\vartheta_Y(t)\|_{L^q(\mathbb{R}^2)} \leq \|\text{Curl }G\|_{L^q(\mathbb{R}^2)} \leq C\|G\|_m,\ q\in[2,\infty].$$

Thus, in view of (52), we have

$$\|\theta_Z\|_{L^\infty(\mathbb{R}^2)} \leq C(\|G\|_m)(1+\ln^+\|\theta_Y\|_m). \tag{53}$$

From the identity equation and the estimate (5) of Lemma 2, we have

$$\begin{aligned}\|\theta_Y(s)\|_m^2+\nu\int_s^T\|\theta_Z(r)\|_m^2\,dr &= \|G\|_m^2+2\int_s^T\langle\theta_Z\theta_Y(r),\,\theta_Y(r)\rangle_m\,dr\\ &\leq \|G\|_m^2+C\int_s^T\|\theta_Z(r)\|_{L^\infty(\mathbb{R}^2)}\|\theta_Y(r)\|_m^2\,dr\end{aligned}$$

which by Gronwall inequality implies

$$\|\theta_Y(s)\|_m \leq C\|G\|_m\exp\left(\int_s^T\|\theta_Z(r)\|_{L^\infty(\mathbb{R}^2)}\,dr\right).$$

In view of (53), we have

$$\ln^+\|\theta_Y(s)\|_m \leq C(\|G\|_m,T)\left(1+\int_s^T\ln^+\|\theta_Y(r)\|\,dr\right).$$

Gronwall inequality yields that

$$\sup_{s\in[t,T]}\|\theta_Y(s)\|_m \leq C(\|G\|_m,T), \quad \forall t\in(T_0,T]. \tag{54}$$

using a bootstrap argument, we can extend the local H^m-solution $(X,Y,Z,\widetilde{Y}_0)$ of the FBSDS (3) into a global one. Therefore, we have

Theorem 10 *Let $d=2$, $m\geq 3$ and $G\in H_\sigma^m$. Then our FBSDS (3) with $f=0$ admits a unique H^m-solution $(X,Y,Z,\widetilde{Y}_0)$.*

6 Approximation of the Navier-Stokes equations

In view of the FBSDS (3) and Theorem 5, we can expect to approximate the Navier-Stokes equations in this section by truncating the time interval of the BSDE associated with $\widetilde{Y}$.

First, we introduce without proof a lemma which follows from the interpolation inequalities of Gilbarg and Trudinger [24, Lemma 6.32].

Lemma 11 *For any $\phi\in C^{k+1,\alpha}$, $k\in\mathbb{N}\cup\{0\}$, $\gamma,\alpha\in(0,1)$, $\gamma\leq\alpha$, there is a constant C depending on α such that*

$$\begin{aligned}&\|\phi\|_{C^{k,\gamma}} \leq C\|\phi\|_{C^{k,\alpha}},\\ &\|\phi\|_{C^{k,\alpha}} \leq C\left(\|\phi\|_{C^{k,\gamma}}+\|\nabla\phi\|_{C^k}\right).\end{aligned}$$

To approximate the Navier-Stokes equations, we truncate the time interval of the infinite-time-interval BSDE of the FBSDS (3).

Lemma 12 *For any $\phi, \psi \in C^{k,\alpha}$, $k \in \mathbb{N}$, $\alpha \in (0,1)$, the following BSDE*

$$\begin{cases} -d\widetilde{Y}_s(x) = \dfrac{27}{2s^3}\phi^i\psi^j(x+B_s)\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)B_{\frac{s}{3}}\,ds - dM_s, \ s \in (0,\infty) \\ \widetilde{Y}_\infty(x) = 0 \end{cases} \tag{55}$$

belongs to class $\mathcal{E}$ for each $x \in \mathbb{R}^d$ and there holds

$$\widetilde{Y}_0 = \nabla(-\Delta)^{-1} div\ div(\phi\otimes\psi) \in C^{k-1,\alpha},$$

and

$$\|\widetilde{Y}_0\|_{C^{k-1,\frac{\alpha}{2}}} \le C\|\phi\|_{C^{k,\alpha}}\|\psi\|_{C^{k,\alpha}}, \tag{56}$$

with C a positive constant independent of ϕ and ψ.

Proof (Sketeched proof) For any $\varepsilon \in (0,1)$ and $N \in \mathbb{N}$, in a similar way to the proof of Lemma 4,

$$\begin{aligned} &E\int_\varepsilon^N \frac{27}{2s^3}\phi^i\psi^j(x+B_s)\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)B_{\frac{s}{3}}\,ds \\ =&E\left(\int_\varepsilon^1 + \int_1^N\right)\frac{27}{2s^3}\phi^i\psi^j(x+B_s)\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)B_{\frac{s}{3}}\,ds \\ =&E\int_\varepsilon^1 \frac{9}{2s^2}\left[\nabla(\phi^i\psi^j)(x+B_s) - \nabla(\phi^i\psi^j)(x)\right]\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)ds \\ &+E\int_1^N \frac{27}{2s^3}\phi^i\psi^j(x+B_s)\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)B_{\frac{s}{3}}\,ds \\ \le& C\|\phi\otimes\psi\|_{C^{1,\alpha}}\int_\varepsilon^1 \frac{1}{s^{1-\frac{\alpha}{2}}}\,ds + C\|\phi\otimes\psi\|_{L^\infty}\int_1^N \frac{1}{s^{\frac{3}{2}}}\,ds \\ \le& C\|\phi\otimes\psi\|_{C^{1,\alpha}}\left(2 - \varepsilon^{\frac{\alpha}{2}} - \frac{1}{\sqrt{N}}\right). \end{aligned} \tag{57}$$

Letting $\varepsilon \to 0$ and $N \to \infty$, we conclude that BSDE (55) belongs to class $\mathcal{E}$ for each $x \in \mathbb{R}^d$.

On the other hand, for each $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &E\int_1^N \frac{27}{2s^3}\left(\phi^i\psi^j(x+B_s) - \phi^i\psi^j(y+B_s)\right)\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)B_{\frac{s}{3}}\,ds \\ \le&\ C|x-y|^{\frac{\alpha}{2}}\|\phi\otimes\psi\|^{\frac{1}{2}}_{C^{1,\frac{\alpha}{2}}}\|\phi\otimes\psi\|^{\frac{1}{2}}_{L^\infty}E\int_1^N \frac{1}{s^3}\left|\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)B_{\frac{s}{3}}\right|ds \\ \le&\ C|x-y|^{\frac{\alpha}{2}}\|\phi\otimes\psi\|_{C^{1,\alpha}}\left(1 - \frac{1}{\sqrt{N}}\right) \end{aligned} \tag{58}$$

and

$$\begin{aligned} &E\int_\varepsilon^1 \frac{9}{2s^2}\left[\nabla(\phi^i\psi^j)(x+B_s) - \nabla(\phi^i\psi^j)(y+B_s)\right]\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)ds \\ =&\ E\int_\varepsilon^1 \frac{9}{2s^2}\nabla\left[\phi^i(x+B_s)\left(\psi^j(x+B_s) - \psi^j(y+B_s)\right) + \left(\phi^i(x+B_s) - \phi^i(y+B_s)\right)\psi^j(y+B_s)\right] \\ &\qquad\left(B^j_s - B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}} - B^i_{\frac{s}{3}}\right)ds \\ =&\ E\int_\varepsilon^1 \frac{9}{2s^2}\nabla\Big[\left(\phi^i(x+B_s) - \phi^i(x)\right)\left(\psi^j(x+B_s) - \psi^j(y+B_s)\right) \\ &\quad + \phi^i(x)\left(\psi^j(x+B_s) - \psi^j(x) - \psi^j(y+B_s) + \psi^j(y)\right) \\ &\quad + \left(\phi^i(x+B_s) - \phi^i(y+B_s)\right)\left(\psi^j(y+B_s) - \psi^j(y)\right) \end{aligned}$$

$$
\begin{aligned}
&+\psi^j(y)\left(\phi^i(x+B_s)-\phi^i(x)-\phi^i(y+B_s)+\phi^i(y)\right)\Big]\left(B^j_s-B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}}-B^i_{\frac{s}{3}}\right)ds\\
\le\;& C|x-y|^{\frac{\alpha}{2}}\|\phi\|_{C^{1,\alpha}}\|\psi\|_{C^{1,\alpha}}E\int_\varepsilon^1\frac{1}{s^2}\left||B_s|^{\frac{\alpha}{2}}\left(B^j_s-B^j_{\frac{2s}{3}}\right)\left(B^i_{\frac{2s}{3}}-B^i_{\frac{s}{3}}\right)\right|ds\\
\le\;& C|x-y|^{\frac{\alpha}{2}}\|\phi\|_{C^{1,\alpha}}\|\psi\|_{C^{1,\alpha}}\left(1-\varepsilon^{\frac{\alpha}{4}}\right),
\end{aligned}
\tag{59}
$$

where for $h=\phi^i$, $\nabla\phi^i$, ψ^j or $\nabla\psi^j$, we note that

$$
\begin{aligned}
&|h(x+B_s)-h(x)-h(y+B_s)+h(y)|\\
\le&|h(x+B_s)-h(x)-h(y+B_s)+h(y)|^{\frac12}\left(|h(x+B_s)-h(x)|^{\frac12}+|h(y+B_s)+h(y)|^{\frac12}\right)\\
\le&4\|h\|_{C^{1,\alpha}}|B_s|^{\frac{\alpha}{2}}|x-y|^{\frac{\alpha}{2}}.
\end{aligned}
$$

Hence, combining (57), (58) and (59), we obtain

$$\|\widetilde{Y}_0\|_{C^{0,\frac{\alpha}{2}}}\le C\|\phi\|_{C^{1,\alpha}}\|\psi\|_{C^{1,\alpha}}.$$

Taking $k-1$-th derivatives in the above arguments, we prove (56).

Remark 12 *In view of* (58) *and* (58) *of the above proof, we can deduce easily that for any* $\varepsilon\in(0,1)$ *and* $N\in\mathbb{N}$,

$$\left\|E\left[\widetilde{Y}_\varepsilon-\widetilde{Y}_N\right]\right\|_{C^{k,\frac{\alpha}{2}}}\le C\|\phi\|_{C^{k,\alpha}}\|\psi\|_{C^{k,\alpha}}\left(2-\varepsilon^{\frac{\alpha}{4}}-\frac{1}{\sqrt{N}}\right)$$

with the constant C *independent of* ϕ, ψ, ε *and* N. *Moreover, in a similar way to the above proof, we obtain*

$$\|E\left[\widetilde{Y}_\varepsilon-\widetilde{Y}_N-\widetilde{Y}_0\right]\|_{C^{k-1,\frac{\alpha}{2}}}\le C\|\phi\|_{C^{k,\alpha}}\|\psi\|_{C^{k,\alpha}},$$

with the constant C *independent of* ϕ, ψ, ε *and* N.

Denote the heat kernel

$$\mathcal{H}^\nu(t,x)=\frac{1}{(2\pi\nu t)^{\frac{d}{2}}}\exp\Big(-\frac{|x|^2}{2\nu t}\Big),$$

and the convolution

$$\mathcal{H}^\nu(t)*g(x)=\int_{\mathbb{R}^d}\mathcal{H}^\nu(t,x-y)g(y)\,dy,\quad\forall\,g\in C(\mathbb{R}^d).$$

Lemma 13 *There exists a constant* C *such that for any* $\phi\in C^{k,\gamma}$ *with* $k\in\mathbb{N}\cup\{0\}$ *and* $\gamma\in(0,1)$,

$$\|\mathcal{H}^\nu(t)*\phi\|_{C^{k+1,\gamma}}\le C\left(1+\frac{1}{\sqrt{t}}\right)\|\phi\|_{C^{k,\gamma}};\tag{60}$$

$$\sum_{i,j=1}^d\|\partial_{x^i}\partial_{x^j}\mathcal{H}^\nu(t)*\phi\|_{C^k}\le\frac{C}{t^{1-\frac{\gamma}{4}}}\|\phi\|_{C^{k,\frac{\gamma}{2}}};\tag{61}$$

$$\|\mathcal{H}^\nu(t)*\phi\|_{C^{k+1,\gamma}}\le C\left(1+\frac{1}{t^{1-\frac{\gamma}{4}}}\right)\|\phi\|_{C^{k,\frac{\gamma}{2}}}.\tag{62}$$

Proof (Sketched proof) The estimate (60) follows from

$$|\mathcal{H}^\nu(t)*\phi(x)|=\left|\int_{\mathbb{R}^d}\mathcal{H}^\nu(t,x-y)\phi(y)\,dy\right|\le\|\phi\|_{C(\mathbb{R}^d)},\quad\forall\,x\in\mathbb{R}^d$$

and for any $x,z\in\mathbb{R}^d$,

$$|\nabla\mathcal{H}^\nu(t)*\phi(x)-\nabla\mathcal{H}^\nu(t)*\phi(z)|=\left|\int_{\mathbb{R}^d}\frac{y}{(2\pi)^{\frac{d}{2}}(\nu t)^{\frac{d}{2}+1}}\exp\Big(-\frac{|y|^2}{2\nu t}\Big)\left(\phi(x-y)-\phi(z-y)\right)dy\right|$$

$$
\begin{aligned}
&\le |x-z|^{\gamma}\|\phi\|_{C^{0,\gamma}}\int_{\mathbb{R}^d}\frac{|y|}{(2\pi)^{\frac{d}{2}}(\nu t)^{\frac{d}{2}+1}}\exp\Big(-\frac{|y|^2}{2\nu t}\Big)\,dy\\
&\le \frac{C}{\sqrt{t}}\,|x-z|^{\gamma}\|\phi\|_{C^{0,\gamma}}.
\end{aligned}
$$

For any $x\in\mathbb{R}^d$ and $i,j=1,\cdots,d$,

$$
\begin{aligned}
|\partial_{x^i}\partial_{x^j}\mathcal{H}^{\nu}(t)*\phi(x)| &= \left|\int_{\mathbb{R}^d}\frac{1}{(2\pi\nu t)^{\frac{d}{2}}}\left(\frac{|y|^2}{(t\nu)^2}-\frac{1}{\nu t}\right)\exp\Big(-\frac{|y|^2}{2\nu t}\Big)\left(\phi(x-y)-\phi(x)\right)\,dy\right|\\
&\le \|\phi\|_{C^{0,\frac{\gamma}{2}}}\int_{\mathbb{R}^d}\frac{|y|^{\frac{\gamma}{2}}}{(2\pi\nu t)^{\frac{d}{2}}}\left(\frac{|y|^2}{(t\nu)^2}+\frac{1}{\nu t}\right)\exp\Big(-\frac{|y|^2}{2\nu t}\Big)\,dy\\
&\le \frac{C}{t^{1-\frac{\gamma}{4}}}\|\phi\|_{C^{0,\frac{\gamma}{2}}},
\end{aligned}
$$

which implies estimate (61). From Lemma 12 and estimates (60) and (61), we conclude estimate (62). The proof is completed.

For each $N\in\mathbb{N}$, define

$$
\mathbf{P}_N(\phi\otimes\psi)=E\left[\widetilde{Y}_{\frac{1}{N}}-\widetilde{Y}_N\right],\quad \phi,\psi\in H^m,\ m>\frac{d}{2}+1,
$$

where $\widetilde{Y}_{\cdot}$ satisfies BSDE (55). In view of Remark 6, we have

$$
\|\mathbf{P}_N(\phi\otimes\psi)\|_k\le C\bigg(\frac{1}{\sqrt{N}}+\sqrt{N}\bigg)\|\phi\otimes\psi\|_k,\quad 0\le k\le m. \tag{63}
$$

In a similar way to Theorem 5, we have

Theorem 14 *Let $\nu>0, G\in H^m$, and $f\in L^2(0,T;H^{m-1})$ with $m>d/2$. Then our FBSDS*

$$
\begin{cases}
\quad dX_s(t,x)=Y_s(t,x)\,ds+\sqrt{\nu}\,dW_s,\ s\in[t,T];\quad X_t(t,x)=x;\\
-dY_s(t,x)=\left[f(s,X_s(t,x))+\widetilde{Y}_0(s,X_s(t,x))\right]\,ds-\sqrt{\nu}Z_s(t,x)\,dW_s;\\
\quad Y_T(t,x)=G(X_T(t,x));\\
-d\widetilde{Y}_s(t,x)=\displaystyle\sum_{i,j=1}^{d}\frac{27}{2s^3}Y_t^iY_t^j(t,x+B_s)\left(B^i_{\frac{2s}{3}}-B^i_{\frac{s}{3}}\right)\left(B^j_s-B^j_{\frac{2s}{3}}\right)B_{\frac{s}{3}}\mathbb{I}_{\left[\frac{1}{N},N\right]}(s)\,ds\\
\qquad\qquad -dM_s,\quad s\in(0,\infty);\\
\quad \widetilde{Y}_\infty(t,x)=0.
\end{cases} \tag{64}
$$

admits one and only one local H^m-solution $(X,Y,Z,\widetilde{Y}_0)$ on some time interval $(T_0,T]$ with $\theta_Y\in C_{loc}((T_0,T];H^m)\cap L^2_{loc}(T_0,T;H^{m+1})$ and $\theta_Z\in C_{loc}((T_0,T];H^{m-1})\cap L^2_{loc}(T_0,T;H^m)$, where T_0 depends on $\|f\|_{L^2(0,T;H^{m-1})}$, ν, T, N and $\|G\|_m$. Moreover, there hold the following representations

$$
\theta_Z(t,\cdot)=\nabla\theta_Y(t,\cdot), Y_s(t,\cdot)=\theta_Y(s,X_s(t,\cdot))\ \text{and}\ Z_s(t,\cdot):=\theta_Z(s,X_s(t,\cdot)), \tag{65}
$$

for $T_0<t\le s\le T$, and $(\theta_Y,\theta_Z,\widetilde{Y}_0)$ satisfies

$$
\begin{aligned}
\theta_Y(r,X_r(t,x))=&\,G(X_T(t,x))+\int_r^T\left[f(s,X_s(t,x))+\widetilde{Y}_0(s,X_s(t,x))\right]\,ds\\
&-\sqrt{\nu}\int_r^T\theta_Z(s,X_s(t,x))\,dW_s,\ T_0<t\le r\le T,\ a.e.x\in\mathbb{R}^d,\ a.s..
\end{aligned} \tag{66}
$$

In addition, θ_Y coincides with the unique strong solution $\widetilde{u}^N$ of the following PDE:

$$
\partial_t\widetilde{u}^N+\frac{\nu}{2}\Delta\widetilde{u}^N+(\widetilde{u}^N\cdot\nabla)\widetilde{u}^N+\boldsymbol{P}_N(\widetilde{u}^N\otimes\widetilde{u}^N)+f=0,\ \ T_0<t\le T;\quad \widetilde{u}^N(T)=G. \tag{67}
$$

As the proof is similar to that of Theorem 5, we omit it.

Letting $r = t$, making Girsanov thransformation in a similar way to (4.2) of the proof for Theorem 5 and then taking expectations on both sides of (66), we have

$$\widetilde{u}^N(t,x) = \mathcal{H}^\nu(T-t) * G(x) + \int_t^T \mathcal{H}^\nu(s-t) * \left(f + (\widetilde{u}^N \cdot \nabla)\widetilde{u}^N + \mathbf{P}_N(\widetilde{u}^N \otimes \widetilde{u}^N)\right)(s,x)\, ds. \quad (68)$$

Assume

$$G \in C^{k,\alpha},\ f \in C([0,T]; C^{k-1,\alpha}), \quad k \in \mathbb{N}, \alpha \in (0,1). \quad (69)$$

By Lemmas 12 and 13 and in view of Remark 12, we get

$$\begin{aligned}
\|\widetilde{u}^N(t)\|_{C^{k,\alpha}} \leq & C\|G\|_{C^{k,\alpha}} + C\int_t^T \Bigg[\Big(1 + \frac{1}{\sqrt{s-t}}\Big)\big(\|f(s) + (\widetilde{u}^N \cdot \nabla)\widetilde{u}^N(s)\|_{C^{k-1,\alpha}} \\
& + \Big(1 + \frac{1}{(s-t)^{1-\frac{\alpha}{4}}}\Big)\|\mathbf{P}_N(\widetilde{u}^N \otimes \widetilde{u}^N)(s)\|_{C^{k-1,\frac{\alpha}{2}}}\Bigg] ds \\
\leq & C\|G\|_{C^{k,\alpha}} + C\int_t^T \Bigg[\Big(1 + \frac{1}{\sqrt{s-t}}\Big)\big(\|f(s)\|_{C^{k-1,\alpha}} + \|\widetilde{u}^N(s)\|^2_{C^{k,\alpha}}\big) \\
& + \Big(1 + \frac{1}{(s-t)^{1-\frac{\alpha}{4}}}\Big)\|\widetilde{u}^N(s)\|^2_{C^{k,\alpha}}\Bigg] ds \\
\leq & C\|G\|_{C^{k,\alpha}} + C\|f\|_{C([0,T];C^{k-1,\alpha})} + C\int_t^T \Big(1 + \frac{1}{(s-t)^{1-\frac{\alpha}{4}}}\Big)\|\widetilde{u}^N(s)\|^2_{C^{k,\alpha}}\, ds, \quad (70)
\end{aligned}$$

which by Gronwall inequality implies that

$$\sup_{s\in[t,T]} \|\widetilde{u}^N(s)\|_{C^{k,\alpha}} \leq \frac{C\big(\|G\|_{C^{k,\alpha}} + \|f\|_{C([0,T];C^{k-1,\alpha})}\big)}{1 - C^2(\|G\|_{C^{k,\alpha}} + \|f\|_{C([0,T];C^{k-1,\alpha})})\big[(T-t) + (T-t)^{\frac{\alpha}{4}}\big]}, \quad (71)$$

where $T - t$ is small enough and the constant C is independent of t and N. In view of (61) of Lemma 13, we further have $\widetilde{u}^N(t) \in C^{k+1}$ when t is away from T.

Basing on estimate (71) and Theorem 14, in a similar way to Theorem 5 we obtain the following corollary.

Corollary 15 *Let $\nu > 0$. Under assumption (69), our FBSDS (64) admits one and only one local solution $(X, Y, Z, \widetilde{Y}_0)$ on some time interval $(T_0, T]$ with $\theta_Y \in C_{loc}((T_0, T]; C^{k,\alpha})$ and $\theta_Z \in C_{loc}((T_0, T]; C^{k-1,\alpha})$, where T_0 depends on $\|f\|_{C([0,T];C^{k-1,\alpha})}$, ν, T and $\|G\|_{C^{k,\alpha}}$. Moreover, there hold the representations in (65), and $(\theta_Y, \theta_Z, \widetilde{Y}_0)$ satisfies BSDE (66).*

In addition, θ_Y coincides with the unique solution $\widetilde{u}^N$ of the PDE (67).

Since $C^{l,\alpha} \cap H^m$ is dense in $C^{l,\alpha}$ for any $m > \frac{d}{2}$ and $l \in \mathbb{N}$, by Theorem 14 we can prove the existence of the local solution $(X, Y, Z, \widetilde{Y}_0)$ through standard density arguments. In view of representation (68), we can prove the uniqueness of the solution through a priori estimates in a similar way to (70). Basing on the estimate (71), we extend the uniqueness and existence of local solution to the maximal time interval $(T_0, T]$. The proof of Corollary 15 is omitted. For more general results and the connections between Navier-Stokes equations and forward-backward stochastic differential systems in Hölder spaces, we refer to [16] for details. It is worth noting that in Corollary 15, the T_0 is independent of N, while in Theorem 14, the T_0 depends on N and in fact it converge to T if we use the similar method of Theorem 5.

Now we shall use the solution $\widetilde{u}^N$ of PDE (67) to approximate the velocity field $\widetilde{u}$ of Navier-Stokes equations (2).

Theorem 16 *Let $\nu > 0, G \in H^m_\sigma$, and $f \in C([0,T]; H^{m-1}_\sigma)$ with $m > \frac{d}{2}+1$. Let $\widetilde{u} \in C_{loc}((T_0, T]; H^m) \cap L^2_{loc}(T_0, T; H^{m+1})$ be the strong solution of Navier-Stokes equation (23) in Theorem 5. Since*

$$H^m \hookrightarrow C^{k-1,\alpha}, \quad \textit{with } k = \lceil m - \frac{d}{2} \rceil \textit{ and } 0 < \alpha < m + 1 - \frac{d}{2} - k,$$

we are allowed to assume that $\widetilde{u}^N \in C_{loc}((T_1,T];C^{k,\alpha})$ *be the solution of PDE* (67) *in Corollary 15. Then, for any* $t \in (T_0 \wedge T_1, T]$*, there exists a constant* C *independent of* N *such that*

$$\left\|\widetilde{u} - \widetilde{u}^N\right\|_{C(t,T;C^{k,\alpha})} \le \frac{C}{N^{\frac{\alpha}{4}}}. \tag{72}$$

Proof In a similar way to (68), we get for any $\tau \in [t,T]$

$$\widetilde{u}(\tau,x) = \mathcal{H}^\nu(T-\tau) * G(x) + \int_\tau^T \mathcal{H}^\nu(s-\tau) * \Big(f + (\widetilde{u}\cdot\nabla)\widetilde{u} - \mathbf{P}^\perp \operatorname{div}(\widetilde{u}\otimes\widetilde{u})\Big)(s,x)\,ds.$$

Putting $\delta u = \widetilde{u}^N - \widetilde{u}$, we have for $\tau \in [t,T]$

$$\begin{aligned}
&\delta u(\tau,x)\\
=&\int_\tau^T \mathcal{H}^\nu(s-\tau) * \Big((\widetilde{u}^N\cdot\nabla)\widetilde{u}^N - (\widetilde{u}\cdot\nabla)\widetilde{u} + \mathbf{P}^N(\widetilde{u}^N\otimes\widetilde{u}^N) + \mathbf{P}^\perp \operatorname{div}(\widetilde{u}\otimes\widetilde{u})\Big)(s,x)\,ds\\
=&\int_\tau^T \mathcal{H}^\nu(s-\tau) * \Big((\delta\widetilde{u}\cdot\nabla)\widetilde{u}^N + (\widetilde{u}\cdot\nabla)\delta\widetilde{u} + (\mathbf{P}^N + \mathbf{P}^\perp \operatorname{div})(\widetilde{u}\otimes\widetilde{u}) + \mathbf{P}^N(\delta\widetilde{u}\otimes\widetilde{u}^N + \widetilde{u}\otimes\delta\widetilde{u})\Big)(s,x)\,ds.
\end{aligned}$$

From Lemmas 12 and 13 and Remark 12, it follows that

$$\begin{aligned}
&\|\delta u(\tau)\|_{C^{k,\alpha}}\\
\le& C\int_\tau^T\Big(1+\frac{1}{(s-\tau)^{1-\frac{\alpha}{4}}}\Big)\Big(\big\|(\delta\widetilde{u}\cdot\nabla)\widetilde{u}^N(s) + (\widetilde{u}\cdot\nabla)\delta\widetilde{u}(s) + (\mathbf{P}^N + \mathbf{P}^\perp \operatorname{div})(\widetilde{u}\otimes\widetilde{u})(s)\\
&\qquad + \mathbf{P}^N(\delta\widetilde{u}\otimes\widetilde{u}^N + \widetilde{u}\otimes\delta\widetilde{u})(s)\big\|_{C^{k-1,\frac{\alpha}{2}}}\Big)\,ds\\
\le& C\int_\tau^T\Big(1+\frac{1}{(s-\tau)^{1-\frac{\alpha}{4}}}\Big)\Big(\big(\|\widetilde{u}(s)\|_{C^{k,\alpha}} + \big\|\widetilde{u}^N(s)\big\|_{C^{k,\alpha}}\big)\|\delta u(s)\|_{C^{k,\alpha}} + \frac{1}{N^{\frac{\alpha}{4}}}\|\widetilde{u}(s)\|^2_{C^{k,\alpha}}\Big)\,ds\\
\le& C\int_\tau^T\Big(1+\frac{1}{(s-\tau)^{1-\frac{\alpha}{4}}}\Big)\Big(\|\delta u(s)\|_{C^{k,\alpha}} + \frac{1}{N^{\frac{\alpha}{4}}}\Big)\,ds,
\end{aligned}$$

which implies the estimate (72) by Gronwall inequality. We complete the proof.

Remark 13 *In view of Theorem 16, we can approximate numerically the strong solution of Navier-Stokes equation* (23)*, by approximating the PDE* (67)*. By Theorem 14 and Corollary 15, we rewrite the FBSDS* (64) *into the following form*

$$\begin{cases}
dX_s(t,x) = \theta_Y(s,X_s(t,x))\,ds + \sqrt{\nu}\,dW_s,\ \ s\in[t,T]; \quad X_t(t,x)=x;\\
-d\theta_Y(s,X_s(t,x)) = \Big[f(s,X_s(t,x)) + \boldsymbol{P}^N(\theta_Y\otimes\theta_Y)(s,X_s(t,x))\Big]\,ds - \sqrt{\nu}Z_s(t,x)\,dW_s;\\
\theta_Y(T,x) = G(x);\\
\boldsymbol{P}^N(\theta_Y\otimes\theta_Y)(s,x) = \displaystyle\sum_{i,j=1}^d E\int_{\frac{1}{N}}^N \frac{27}{2r^3}\theta_Y^i\theta_Y^j(s,x+B_r)\Big(B^i_{\frac{2r}{3}} - B^i_{\frac{r}{3}}\Big)\Big(B^j_r - B^j_{\frac{2r}{3}}\Big)B_{\frac{r}{3}}\,dr;\\
\qquad\qquad = \displaystyle\sum_{i,j=1}^d E\int_{\frac{1}{3N}}^{\frac{N}{3}} \frac{3}{2r^3}\theta_Y^i\theta_Y^j(s,x+\bar{B}_r+\tilde{B}_r+\hat{B}_r)\bar{B}^i_r\tilde{B}^j_r\hat{B}_r\,dr,
\end{cases} \tag{73}$$

where $\bar{B}$*,* $\tilde{B}$ *and* $\hat{B}$ *are three independent d-dimensional Brownian motions. The numerical approximation theory of FBSDEs (see [4, 14, 15] and references therein) allows us to approximate numerically the FBSDS* (73) *and the PDE* (67)*. Indeed, in the spirit of Delarue and Menozzi [14, 15], we can define roughly the following algorithm:*

$$\begin{aligned}
&\forall\, x\in\mathbb{R}^d, \quad \bar{u}^N(T,x) = G(x),\\
&\forall\, k\in[0,\tilde{N}-1], \quad \forall\, x\in\varXi,\\
&\quad \mathcal{J}(t_k,x) = \bar{u}^N(t_{k+1},x)h + \sqrt{\nu}\varDelta W_{t_k},
\end{aligned}$$

$$
\begin{aligned}
\mathcal{P}^N(t_k,x) &= \sum_{i,j=1}^{d} E\int_{\frac{1}{3N}}^{\frac{N}{3}} \frac{3}{2r^3}(\bar{u}^N)^i(\bar{u}^N)^j(t_{k+1},x+\bar{B}_r+\tilde{B}_r+\hat{B}_r)\bar{B}^i_r\tilde{B}^j_r\hat{B}_r\,dr,\\
\bar{u}^N(t_k,x) &= E\bar{u}^N(t_{k+1},x+\mathcal{J}(t_k,x)) + h\left(f(t_k,x)+\mathcal{P}^N(t_k,x)\right),
\end{aligned}
$$

where $\Xi = \delta\mathbb{Z}^d$ is the infinite Cartesian grid of step $\delta > 0$ and compared with Delarue and Menozzi [14, 15], we omit the projection mapping on the grid, quantized algorithm for the Brownian motions and the approximations for the diffusion coefficient of the BSDEs in (73). *The error estimates, algorithm complexity and convergence rate can be analyzed in a similar way to Delarue and Menozzi [14, 15], where quadratic growth examples are given as well. In this way, we derive a time-space discretization scheme for the Navier-Stokes equations.*

7 Connections with the Lagrangian approach

With the Lagrangian approach, Constantin and Iyer [10,11] and Iyer [26–28] derived a stochastic representation for the incompressible Navier-Stokes equations based on stochastic Lagrangian paths and gave a self-contained proof of the existence. Later, Zhang [48] considered a backward analogue and provided short elegant proofs for the classical existence results. In this section, we shall derive from our representation (see Theorem (5)) an analogous Lagrangian formula, through which we show the connections with the Lagrangian approach.

Let $\nu > 0, G \in H^m_\sigma$, and $f \in L^2(0,T;H^{m-1}_\sigma)$ with $m > d/2$. By Theorem 5, the following FBSDS

$$
\begin{cases}
dX_s(t,x) = Y_s(t,x)\,ds + \sqrt{\nu}\,dW_s,\ \ s\in[t,T];\quad X_t(t,x)=x;\\
-dY_s(t,x) = \left[f(s,X_s(t,x)) + \widetilde{Y}_0(s,X_s(t,x))\right]ds - \sqrt{\nu}Z_s(t,x)\,dW_s;\\
Y_T(t,x) = G(X_T(t,x));\\
-d\widetilde{Y}_s(t,x) = \displaystyle\sum_{i,j=1}^{d}\frac{27}{2s^3}Y^i_tY^j_t(t,x+B_s)\big(B^i_{\frac{2s}{3}}-B^i_{\frac{s}{3}}\big)\big(B^j_s-B^j_{\frac{2s}{3}}\big)B_{\frac{s}{3}}\,ds - dM_s,\ s\in(0,\infty);\\
\widetilde{Y}_\infty(t,x) = 0.
\end{cases}
\tag{74}
$$

admits a unique local H^m-solution $(X,Y,Z,\widetilde{Y}_0)$ on some time interval $(T_0,T]$, with

$$
\theta_Z(t,\cdot) = \nabla\theta_Y(t,\cdot), Y_s(t,\cdot) = \theta_Y(s,X_s(t,\cdot)) \text{ and } Z_s(t,\cdot) := \theta_Z(s,X_s(t,\cdot)),
\tag{75}
$$

and there exists $p \in L^2(T_0,T;H^m)$ such that $\nabla p := \widetilde{Y}_0$ and (u,p) coincides with the unique strong solution to Navier-Stokes equation:

$$
\partial_t u + \frac{\nu}{2}\Delta u + (u\cdot\nabla)u + \nabla p + f = 0,\ \ T_0 < t \le T;\quad \nabla\cdot u = 0,\quad u(T) = G.
\tag{76}
$$

For each $t \in (T_0,T]$ a.e. $x \in \mathbb{R}^d$, define the following equivalent probability $Q^{t,x}$:

$$
d\mathbb{Q}^{t,x} := \exp\left(-\frac{1}{\sqrt{\nu}}\int_t^T \theta_Y(s,X_s(t,x))\,dW_s - \frac{1}{2\nu}\int_t^T |\theta_Y(s,X_s(t,x))|^2\,ds\right)d\mathbb{P}.
\tag{77}
$$

Then we have

$$
\begin{cases}
dX_s(t,x) = \sqrt{\nu}\,dW'_s,\ \ s\in[t,T];\quad X_t(t,x) = x;\\
-dY_s(t,x) = \left[(\theta_Y\cdot\nabla)\theta_Y + f + \nabla p\right](s,X_s(t,x))\,ds - \sqrt{\nu}\nabla\theta_Y(s,X_s(t,x))\,dW'_s;\\
Y_T(t,x) = G(X_T(t,x)),
\end{cases}
$$

where $(W',\mathbb{Q}^{t,x})$ is a standard Brownian motion.

Consider the following BSDE

$$
\begin{cases}
-d\overline{Y}_s(t,x) = [f(s,X_s(t,x)) + Z^T_s(t,x)\overline{Y}_s(t,x)]\,ds - \sqrt{\nu}\,\overline{Z}_s(t,x)dW_s,\ \ T_0 < t \le s \le T;\\
\overline{Y}_T(t,x) = G(X_T(t,x)).
\end{cases}
\tag{78}
$$

Putting

$$(\delta Y, \delta Z)_s(t,x) = (Y - \overline{Y}, Y - \overline{Y})_s(t,x),$$

we have $\delta Y_T(t,x) = 0$ and

$$\begin{aligned}
&-d\delta Y_s(t,x)\\
=&\big[\widetilde{Y}_0(s, X_s^(t,x)) - Z^{\mathcal{T}}\big(Y - \delta Y\big)_s(t,x)\big]\, ds - \sqrt{\nu}\,\delta Z_s(t,x) dW_s\\
=&\big[(\nabla p - \nabla^{\mathcal{T}}\theta_Y\theta_Y)(s, X_s(t,x)) + \nabla^{\mathcal{T}}\theta_Y(s, X_s(t,x))\delta Y_s(t,x)\big] dt - \sqrt{\nu}\,\delta Z_s(t,x) dW_s\\
=&\Big[\nabla\Big(p - \frac{1}{2}|\theta_Y|^2\Big)(s, X_s(t,x)) + \nabla^{\mathcal{T}}\theta_Y(s, X_s(t,x))\delta Y_s(t,x)\Big] dt - \sqrt{\nu}\,\delta Z_s(t,x) dW_s\\
=&\Big[\nabla\Big(p - \frac{1}{2}|\theta_Y|^2\Big)(s, X_s(t,x)) + \nabla^{\mathcal{T}}\theta_Y(s, X_s(t,x))\delta Y_s(t,x) + \delta Z_s(t,x)\theta_Y(s, X_s(t,x))\Big] dt\\
&- \sqrt{\nu}\,\delta Z_s(t,x) dW'_s,\\
=&\Big[\nabla\Big(p - \frac{1}{2}|\theta_Y|^2\Big)(s, X_s(t,x)) + \nabla^{\mathcal{T}}\theta_Y\theta_{\delta Y}(s, X_s(t,x)) + \theta_{\delta Z}\theta_Y(s, X_s(t,x))\Big] dt\\
&- \sqrt{\nu}\,\delta Z_s(t,x) dW'_s,
\end{aligned}$$

where by Proposition 7, we have

$$\theta_{\delta Z}(t,\cdot) = \nabla\theta_{\delta Y}(t,\cdot), \delta Y_s(t,\cdot) = \theta_{\delta Y}(s, X_s(t,\cdot)) \text{ and } \delta Z_s(t,\cdot) := \theta_{\delta Z}(s, X_s(t,\cdot)). \tag{79}$$

On the other hand, through basic calculations it is easy to check that

$$v(r,x) := E\int_r^T \Big(p - \frac{1}{2}|\theta_Y|^2\Big)(s, X_s(r,x))\, ds, \quad \forall\, r \in (T_0, T],$$

satisfies BSDE

$$\begin{cases}
-dv(s, X_s(t,x)) =\Big[p - \frac{1}{2}|\theta_Y|^2 + \nabla v\theta_Y(s, X_s(t,x))\Big]\, ds - \sqrt{\nu}\,\nabla v(s, X_s(t,x)) dW'_s\\
\qquad\qquad\qquad\quad =\Big[p - \frac{1}{2}|\theta_Y|^2\Big]\, ds - \sqrt{\nu}\,\nabla v(s, X_s(t,x)) dW_s;\\
\qquad\quad v(T,x) =0.
\end{cases} \tag{80}$$

In view of the following relation

$$\nabla(\nabla v)\theta_Y + \nabla^{\mathcal{T}}\theta_Y\nabla v = \nabla\left((\theta_Y\cdot\nabla)v\right),$$

we further check that $\nabla v = \theta_{\delta Y}$. Therefore, we have

Proposition 17 *Let $\nu > 0, G \in H_\sigma^m$, and $f \in L^2(0,T; H_\sigma^{m-1})$ with $m > d/2$. Let $(X, Y, Z, \widetilde{Y}_0)$ be the unique local H^m-solution of the FBSDS* (74) *on some time interval $(T_0, T]$ and $(\overline{Y}, \overline{Z})$ satisfy BSDE* (82)*. Then, the strong solution of Navier-Stokes equation* (76) *admits a probabilistic representation:*

$$u(t,x) = Y_t(t,x) = \overline{Y}_t(t,x) + \nabla v(t,x) = \boldsymbol{P}\overline{Y}_t(t,x), \quad (t,x) \in (T_0, T] \times \mathbb{R}^d, \tag{81}$$

with $\overline{Y}$ and v satisfies BSDEs (78) *and* (80) *respectively.*

Remark 14 *In view of the relation* (75)*, we rewrite* (78) *into*

$$\begin{cases}
-d\overline{Y}_s(t,x) = [f(s, X_s(t,x)) + \nabla^{\mathcal{T}}u(s, X_s(t,x))\overline{Y}_s(t,x)]\, ds - \sqrt{\nu}\,\overline{Z}_s(t,x) dW_s;\\
\overline{Y}_T(t,x) = G(X_T(t,x)).
\end{cases} \tag{82}$$

It follows that

$$\overline{Y}_t(t,x) = E\Big[\nabla^{\mathcal{T}}X_T(t,x)G(X_T(t,x) + \int_t^T \nabla^{\mathcal{T}}X_s(t,x)f(s, X_s(t,x))\, ds\Big], \quad T_0 < t \le T,$$

and thus,

$$u(t,x) = \boldsymbol{P}E\left[\nabla^{\mathcal{T}} X_T(t,x) G(X_T(t,x) + \int_t^T \nabla^{\mathcal{T}} X_s(t,x) f(s, X_s(t,x))\, ds\right], \quad T_0 < t \le T, \tag{83}$$

with

$$dX_s(t,x) = u(s, X_s(t,x))\, ds + \sqrt{\nu}\, dW_s, \quad s \in [t,T]; \quad X_t(t,x) = x; \tag{84}$$

and

$$d\,\nabla^{\mathcal{T}} X_s(t,x) = \nabla^{\mathcal{T}} X_s(t,x) \nabla^{\mathcal{T}} u(s, X_s(t,x))\, ds, \quad s \in [t,T]; \quad \nabla_t^{\mathcal{T}}(t,x) = I_{d\times d}.$$

Hence, by the relation (81) *or* (83)*, we derive a probabilistic representation along the stochastic particle systems for the strong solutions of Navier-Stokes equations. In particular, the representation formula* (83) *is analogous to those of [10,48] with the Lagrangian approach.*

8 Appendix

8.1 Proof of Lemma 3

It is sufficient for us to prove (8) with $l = 1$, from which (9) follows by Fubini Theorem.

First, taking a nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$, we consider the following trivial FBSDE:

$$\begin{cases} dX_r(t,x) = b(r, X_r(t,x))\, dr + \sqrt{\nu}\, dW_r, \quad T-\varepsilon \le t \le r \le s; \quad X_t(t,x) = x; \\ dY_r(t,x) = \sqrt{\nu} Z_r(t,x)\, dW_r, \quad r \in [t,s]; \quad Y_T(t,x) = \varphi(X_s(t,x)). \end{cases} \tag{85}$$

In view of Lemma 6 and the proof therein, the FBSDE (85) is a particular case with $\phi = 0$ therein, and moreover, the assertions of Lemma 6 still hold for (85), as Lemma 3 will never be involved in the proof of Lemma 6 if $\phi = 0$. Therefore, for almost all $x \in \mathbb{R}^d$ our FBSDE (85) admits a unique solution

$$\big(X_\cdot(t,x), Y_\cdot(t,x), Z_\cdot(t,x)\big) \in S^2(t,s;\mathbb{R}^d) \times S^2(t,s;\mathbb{R}^d) \times L^2_{\mathscr{F}}(t,s;\mathbb{R}^d),$$

and for this solution (X,Y,Z), there hold

$$\begin{aligned} \theta_Y &\in C([t,s];H^m) \cap L^2(t,s;H^{m+1}) \\ \theta_Y(r, X_r(t,x)) &= \varphi(X_s(t,x)) - \sqrt{\nu} \int_r^s \theta_Z(\tau, X_\tau(t,x))\, dW_\tau, \ a.s. \end{aligned}$$

and

$$\theta_Z(t,x) = \nabla\theta_Y(t,x),\ (Y_r(t,x), Z_r(t,x)) = (\theta_Y, \theta_Z)(r, X_r(t,x)),\ a.s.. \tag{86}$$

In an obvious way, we have

$$Y_r(t,x) = E\left[\varphi(X_s(t,x)) \middle| \mathscr{F}_r\right] \ge 0, \quad a.s.\, r \in [t,s].$$

Define the following equivalent probability measure

$$d\mathbb{Q}^{t,x} =: \exp\left(-\nu^{-\frac{1}{2}} \int_t^s b(r, X_r(t,x))\, dW_s - \frac{1}{2}\nu^{-1} \int_t^T |b(r, X_r(t,x))|^2\, ds\right) d\mathbb{P}.$$

In view of (86), the FBSDE (85) reads

$$\begin{cases} dX_r(t,x) = \sqrt{\nu}\, dW'_r, \quad t \le r \le s; \quad X_t(t,x) = x; \\ -dY_r(t,x) = Z_r(t,x) b(r, X_r(t,x))\, dr - \sqrt{\nu} Z_r(t,x)\, dW'_r \\ \qquad\qquad = (b\cdot\nabla)\theta_Y(r, X_r(t,x))\, dr - \sqrt{\nu} Z_r(t,x)\, dW'_r, \quad r \in [t,s]; \\ Y_s(t,x) = \varphi(X_s(t,x)), \end{cases} \tag{87}$$

where $(W', \mathbb{Q}^{t,x})$ is a standard Brownian motion. Therefore,

$$\int_{\mathbb{R}^d} \theta_Y(r,x)\, dx = \int_{\mathbb{R}^d} E_{\mathbb{Q}^{t,x}}\left[\theta_Y(r, X_r(t,x))\right] dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} E_{\mathbb{Q}^{t,x}}\left[\varphi(X_s(t,x))\right]\,dx + \int_{\mathbb{R}^d}\int_r^s E_{\mathbb{Q}^{t,x}}\left[(b\cdot\nabla)\theta_Y(\tau, X_\tau(t,x))\right]\,d\tau dx \\
&= \int_{\mathbb{R}^d} \varphi(x)\,dx + \int_{\mathbb{R}^d}\int_r^s (b\cdot\nabla)\theta_Y(\tau,x)\,d\tau dx \\
&= \int_{\mathbb{R}^d} \varphi(x)\,dx - \int_{\mathbb{R}^d}\int_r^s (\operatorname{div} b)\theta_Y(\tau,x)\,d\tau dx \\
&\leq \int_{\mathbb{R}^d} \varphi(x)\,dx + \int_r^s \|\operatorname{div} b(\tau)\|_{L^\infty}\int_{\mathbb{R}^d}\theta_Y(\tau,x)\,dxd\tau, \\
&\text{or} \\
&\geq \int_{\mathbb{R}^d} \varphi(x)\,dx - \int_r^s \|\operatorname{div} b(\tau)\|_{L^\infty}\int_{\mathbb{R}^d}\theta_Y(\tau,x)\,dxd\tau.
\end{aligned}
$$

Using Gronwall inequality, we have

$$
\kappa\int_{\mathbb{R}^d}\varphi(x)\,dx \leq \int_{\mathbb{R}^d}\theta_Y(r,x)\,dx \leq \kappa^{-1}\int_{\mathbb{R}^d}\varphi(x)\,dx, \quad \forall r\in[t,s]
$$

with

$$
\kappa := e^{-\|\operatorname{div} b\|_{L^1(0,T;L^\infty)}}.
$$

Taking $r=t$, we have (8) for $K:=\kappa^{-1}$.

For the general function $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ without the nonnegative assumption, we choose a positive Schwartz function h and a nonnegative function $\widetilde{\varphi}\in C_c^\infty(\mathbb{R}^d;\mathbb{R})$ such that

$$
\operatorname{supp}\varphi \subset \left\{x\in\mathbb{R}^d : \widetilde{\varphi}(x)=1\right\}.
$$

Set

$$
\varphi_\varepsilon := \sqrt{\varphi^2+\varepsilon h}\widetilde{\varphi}, \text{ for } \varepsilon\in(0,1).
$$

Then in view of the above arguments, we have

$$
\kappa\|\varphi\|_{L^1(\mathbb{R}^d)} \leq \kappa\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} E\left[|\varphi_\varepsilon(X_s(t,x))|\right]dx \leq K\|\varphi_\varepsilon\|_{L^1(\mathbb{R}^d)}.
$$

Letting $\varepsilon\to 0$, we conclude from Lebesgue dominant convergence theorem that (8) holds for all $\varphi\in C_c^\infty(\mathbb{R}^d;\mathbb{R})$.

Finally, for any $\varphi\in L^1(\mathbb{R})$, we choose a sequence $\{\varphi^n, n\in\mathbb{Z}^+\}\subset C_c^\infty(\mathbb{R}^d;\mathbb{R})$ such that $\lim_{n\to\infty}\|\varphi-\varphi^n\|_{L^1(\mathbb{R})}=0$. Then, by (8), $\{\varphi^n(X_s(t,x))\}$ is a Cauchy sequence in $L^1(\Omega\times\mathbb{R}^d;\mathbb{R})$. It remains to show that $\varphi(X_s(t,\cdot))$ is the limit.

Through the above approximation, we can check that (8) holds for any continuous function of a compact support. Therefore, if $A\subset\mathbb{R}^d$ is a measurable, bounded subset of zero Lebesgue measure, then the $d\mathbb{P}\times dx$-measure of the set $\{(\omega,x)\in\Omega\times\mathbb{R}^d : X_s(t,x)\in A\}$ is zero. Thus, the almost everywhere convergence of φ^n to φ in $\mathbb{R}^d$ implies that of $\varphi^n(X_s(t,\cdot))$ to $\varphi(X_s(t,\cdot))$.

Hence, $\varphi^n(X_s(t,x))$ converges to $\varphi(X_s(t,x))$ in $L^1(\Omega\times\mathbb{R}^d; d\mathbb{P}\times dx)$. Since (8) holds for each φ^n, passing to the limit, (8) holds for any $\varphi\in L^1(\mathbb{R})$. We complete the proof.

8.2 Proof of Lemma 6

For $m>d/2$, $H^m\hookrightarrow C^{0,\delta}$, $H^{m+1}\hookrightarrow C^{1,\delta}$. By Theorems 3.4.1 and 4.5.1 of [31], the *forward* SDE is well posed for each $(t,x)\in[T_0,T]\times\mathbb{R}^d$ and defines a stochastic flow of homeomorphisms. Moreover, from Lemma 3 and Remark 2, the backward SDE is also well posed for every $x\in\mathbb{R}^d/F_t$ with Lebesgue's measure of F_t being zero. Therefore, for each $(t,x)\in[T_0,T]\times(\mathbb{R}^d/F_t)$, the FBSDE (26) has unique solution

$$
\left(X_\cdot(t,x), Y_\cdot(t,x), Z_\cdot(t,x)\right)\in S^2(T_0,T;\mathbb{R}^d)\times S^2(T_0,T;\mathbb{R}^d)\times L^2_{\mathscr{F}}(T_0,T;\mathbb{R}^d).
$$

For each $(t,x) \in [T_0, T) \times (\mathbb{R}^d/F_t)$, define the following equivalent probability measure:

$$d\mathbb{Q}^{t,x} := \exp\left(-\frac{1}{\sqrt{\nu}}\int_t^T b(s, X_s(t,x))\,dW_s - \frac{1}{2\nu}\int_t^T |b(s, X_s(t,x))|^2\,ds\right) d\mathbb{P}.$$

Then there is a standard brownian motion $(W', \mathbb{Q}^{t,x})$ such that the FBSDE (26) is written into the following form:

$$\begin{cases} dX_s(t,x) = \sqrt{\nu}\,dW'_s, \quad T_0 \le t \le s \le T; \quad X_t(t,x) = x; \\ -dY_s(t,x) = \phi(s, X_s(t,x)) + Z_s(t,x)b(s, X_s(t,x))\,ds - \sqrt{\nu}Z_s(t,x)\,dW'_s; \\ Y_T(t,x) = \psi(X_T(t,x)). \end{cases} \tag{88}$$

Choose a sequence $\{(b^n, \phi^n, \psi^n), n \in \mathbb{Z}^1\} \subset C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^{d+1}) \times C_c^\infty(\mathbb{R}^d)$ satisfying

$$\lim_{n\to\infty}\left(\|b^n - b\|_{C([t,T];H^m)} + \|b^n - b\|_{L^2([t,T];H^{m+1})} + \|\phi^n - \phi\|_{L^2(t,T;H^{m-1})} + \|\psi^n - \psi\|_{H^m}\right) = 0.$$

Let (X, Y^n, Z^n) be the unique solution of the FBSDE (88) with (b, ϕ, ψ) being replaced by (b^n, ϕ^n, ψ^n). Then for each n, we have by the standard relationship between Markovian BSDEs and PDEs (for instance, see [1,25,33,39–41]),

$$\theta_{Y^n} \in C([t,T]; H^m) \cap L^2(t,T; H^{m+1}),$$

and for each $x \in \mathbb{R}^d$ and all $t \le r \le T$,

$$\theta_{Y^n}(r, X_r(t,x)) = \psi^n(X_T(t,x)) + \int_r^T (\phi^n + \theta_{Z^n}b^n)(s, X_s(t,x))\,ds - \sqrt{\nu}\int_r^T \theta_{Z^n}(s, X_s(t,x))\,dW'_s,$$
$$\theta_{Z^n}(t,x) = \nabla\theta_{Y^n}(t,x), Y_r^n(t,x) = \theta_{Y^n}(r, X_r(t,x)), Z_r^n(t,x) = \theta_{Z^n}(r, X_r(t,x)).$$

If we consider further

$$\begin{cases} -d\bar{Y}^n(s,x) = I_m(\phi^n + \theta_{Z^n}b^n)(s, X_s(t,x))\,ds - \sqrt{\nu}\bar{Z}_s^n(t,x), \; t \le s \le T; \\ \bar{Y}_T(t,x) = I_m\psi^n(X_T(t,x)), \end{cases}$$

it is not difficult to show that for almost every $x \in \mathbb{R}^d$ and all $s \in [t,T]$,

$$\tilde{Y}_s^n(t,x) = I_m\theta_{Y^n}(s, X_s(t,x)) \quad \text{and} \quad \bar{Z}_s^n(t,x) = I_m\theta_{Z^n}(s, X_s(t,x)), \quad a.s..$$

Applying Itô's formula, we have

$$\begin{aligned} &E_{\mathbb{Q}^{t,x}}\left[|\bar{Y}_t^n(t,x)|^2 + \nu\int_t^T |\bar{Z}_s^n(t,x)|^2\,ds\right] \\ =&2\int_t^T E_{\mathbb{Q}^{t,x}}\left[\langle \bar{Y}_s^n(t,x),\, I_m\phi^n(s, X_s(t,x)) + I_m\{\theta_{Z^n}b^n(s, X_s(t,x))\}\rangle\right] ds \\ &+ E_{\mathbb{Q}^{t,x}}\left[|I_m\theta_{Y^n}(T, X_T(t,x))|^2\right], \; a.e. x \in \mathbb{R}^d. \end{aligned}$$

Finally, integrating with respect to x on both sides of the last equality, we obtain the energy equality:

$$\|\theta_{Y^n}(t)\|_m^2 + \nu\int_t^T \|\theta_{Z^n}(s)\|_m^2\,ds = \|\theta_{Y^n}(T)\|_m^2 + 2\int_t^T \langle \phi^n(s) + \theta_{Z^n}b^n(s),\, \theta_{Y^n}(s)\rangle_{m-1,m+1}\,ds. \tag{89}$$

Itô's formula yields that for each $(s,x) \in [t,T] \times \mathbb{R}^d/F_t$

$$\begin{aligned} &|Y_s^n(t,x) - Y_s(t,x)|^2 + \nu\int_s^T \left|Z_r^n(t,x) - Z_r(t,x)\right|^2 dr \\ =& -\int_s^T 2\nu\langle Y_r^n(t,x) - Y_r(t,x),\, Z_r^n(t,x) - Z_r(t,x)\rangle\,dW'_r \end{aligned}$$

$$+\int_s^T 2\langle Y_r^n(t,x)-Y_r(t,x),\, \big(\phi^n-\phi\big)(r,X_r(t,x))+Z_r^n(t,x)b^n(r,X_r(t,x))$$
$$-Z_r(t,x)b(r,X_r(t,x))\rangle\,dr+|\psi(X_T(t,x))-\psi^n(X_T(t,x))|^2. \tag{90}$$

Using BDG and Hölder inequalities, we get

$$E_{\mathbb{Q}^{t,x}}\Big[\sup_{\tau\in[s,T]}|Y_\tau^n(t,x)-Y_\tau(t,x)|^2+\nu\int_s^T\big|Z_r^n(t,x)-Z_r(t,x)\big|^2\,dr\Big]$$
$$\le E_{\mathbb{Q}^{t,x}}\Big[|\psi(X_T(t,x))-\psi^n(X_T(t,x))|^2+\frac{1}{2}\sup_{\tau\in[s,T]}|Y_\tau^n(t,x)-Y_\tau(t,x)|^2$$
$$+C\int_s^T|Y_r^n(t,x)-Y_r(t,x)|\big(|\phi_n-\phi|(r,X_r(t,x))+|Z_r^n(t,x)-Z_r(t,x)|$$
$$+\|b^n-b\|_{C([t,T];H^m)}|Z_r(t,x)|\big)\,dr+C\int_s^T\big|Z_r^n(t,x)-Z_r(t,x)\big|^2\,dr\Big], \tag{91}$$

with the constants Cs being independent of n. Combining (90) and (91), we have

$$E_{\mathbb{Q}^{t,x}}\Big[\sup_{\tau\in[t,T]}|Y_\tau^n(t,x)-Y_\tau(t,x)|^2+\nu\int_t^T\big|Z_r^n(t,x)-Z_r(t,x)\big|^2\,dr\Big]$$
$$\le CE_{\mathbb{Q}^{t,x}}\Big[\int_t^T\big(\big|(\phi_n-\phi)(r,X_r(t,x))\big|^2+\|b^n-b\|_{C([t,T];H^m)}^2|Z_r(t,x)|^2\big)\,dr\Big]+C\|\psi-\psi^n\|_{H^m}^2$$
$$\longrightarrow 0,\ \text{ as } n\to\infty,\ \ x\in\mathbb{R}^d/F_t. \tag{92}$$

On the other hand, put

$$(Y^{nk},Z^{nk},b_{nk},\phi_{nk},\psi_{nk}):=(Y^n-Y^k,Z^n-Z^k,b_n-b_k,\phi^n-\phi^k,\psi^n-\psi^k).$$

For $n,k\in\mathbb{Z}^+$, we have by Eq. (89) and Remark 1 that

$$\|\theta_{Y^{nk}}(s)\|_m^2+\nu\int_s^T\|\theta_{Z^{nk}}(r)\|_m^2\,dr$$
$$=\|\psi_{nk}\|_m^2+\int_s^T 2\langle\phi_{nk}(r)+\theta_{Z^n}b_{nk}(r)+\theta_{Z^{nk}}b_k(r),\,\theta_{Y^{nk}}(r)\rangle_{m-1,m+1}\,dr$$
$$\le\|\psi_{nk}\|_m^2+C(\nu)\bigg(\int_s^T\big(\|b_{nk}(r)\|_m^2\|\theta_{Y^n}(r)\|_m^2+\|\phi_{nk}(r)\|_{m-1}^2+\|b_k(r)\|_m^2\|\theta_{Y^{nk}}(r)\|_m^2\big)\,dr$$
$$+\int_s^T\|b_k(r)\|_m^2\|\theta_{Y^{nk}}(r)\|_m^2\,dr\bigg)+\frac{\nu}{2}\int_s^T(\|\theta_{Z^{nk}}(r)\|_m^2+\|\theta_{Y^{nk}}(r)\|_m^2)\,dr$$
$$\le C\bigg\{\|\psi_{nk}\|_m^2+\frac{\nu}{2}\int_s^T(\|\theta_{Z^{nk}}(r)\|_m^2+\|\theta_{Y^{nk}}(r)\|_m^2)\,dr$$
$$+\int_0^T\big(\|b_{nk}(r)\|_m^2+\|\phi_{nk}(r)\|_{m-1}^2+\|\theta_{Y^{nk}}(r)\|_m^2\big)\,dr\bigg\},$$

where we have used the the following priori estimate by taking $(b^k,\phi^k,\psi^k)=0$ in the above,

$$\|\theta_{Y^n}\|_{C([T_0,T];H^m)}+\|\theta_{Y^n}\|_{L^2(T_0,T;H^{m+1})}\le C,$$

with the constant C being independent of n. Thus,

$$\sup_{s\in[T_0,T)}\|\theta_{Y^{nk}}(s)\|_m^2+\nu\int_{T_0}^T\|\theta_{Z^{nk}}(r)\|_m^2\,dr\le C\bigg(\|\psi_{nk}\|_m^2+\int_{T_0}^T\Big(\|b_{nk}(r)\|_m^2+\|\phi_{nk}(r)\|_{m-1}^2\Big)dr\bigg)$$
$$\longrightarrow 0 \text{ as } n,k\to\infty. \tag{93}$$

Combining (92) and (93), we have

$$\lim_{k\to\infty}\Big(\|\theta_{Y^k}-\theta_Y\|_{C([T_0,T];H^m)}+\|\theta_{Z^k}-\theta_Z\|_{L^2_{\mathscr{F}}(T_0,T;H^m)}\Big)=0,$$

with $\theta_Z = \nabla\theta_Y$, and furthermore, by taking limits, we prove (27), (28), (29) and (30). The proof is complete.

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