RISK MEASURES FOR NON-INTEGRABLE RANDOM VARIABLES

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Abstract. We show that when a real-valued risk-measure is defined on a solid, rearrangement invariant space of random variables, then necessarily it satisfies a weak compactness, also called continuity from below, property and the space necessarily consists of integrable random variables. As a result we see that a risk-measure defined for, say Cauchy distributed random variables, must take infinite values for some of the random variables.

We use standard notation. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ denotes an atomless probability space. In practice this is not a restriction since the property of being atomless is equivalent to the fact that on $(\Omega, \mathcal{F}, \mathbb{P})$, we can define a random variable that has a continuous distribution function. By L^p we denote the standard spaces, i.e. for $0 < p < \infty$, $X \in L^p$ if and only if $\mathbb{E}[|X|^p] < \infty$. L^0 stands for the space of all random variables and L^∞ is the space of bounded random variables. The topological dual of L^∞ is denoted by $\mathbf{ba}$, the space of bounded finitely additive measures μ defined on $\mathcal{F}$ with the property that $\mathbb{P}[A] = 0$ implies $\mu(A) = 0$. The subset of normalised non-negative finitely additive measures – the so called finitely additive probability measures – is denoted by $\mathcal{P}^{\mathbf{ba}}$. By E we will denote a vector space of random variables defined on Ω. We will assume that E contains the space L^∞ of bounded random variables. We do not assume that E carries a norm neither do we assume completeness properties. However we will assume that E satisfies the following properties.

Definition. *We say that the vector space E is rearrangement invariant if for random variables Y and X having the same distribution, $X \in E$ implies $Y \in E$.*

Definition. *We say that the vector space E is solid if for random variables Y and X, $|Y| \le |X|$, $X \in E$ implies $Y \in E$.*

Remark. If E contains the constants and is solid then automatically E contains the space of bounded random variables.

We assume that u is a utility function defined on E. The function u satisfies the following properties:

(1) $u\colon E \to \mathbb{R}$, $u(0) = 0$,
(2) if $X \in E$ and $X \ge 0$ a.s. then $u(X) \ge 0$,
(3) u is monetary, i.e. for $X \in E$ and $a \in \mathbb{R}$ we have: $u(X + a) = u(X) + a$,
(4) u is concave.

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Remark. We do not assume any continuity property except that u is non-negative for non-negative random variables. In particular we do not assume that u has the Fatou property. We need the property that u is monetary in order to apply the representation theorem. However we have the impression that this assumption can be relaxed. We also do not require that u is law invariant (rearrangement invariant). We only require that u is defined on a rearrangement invariant space. The mapping $\rho(X) = -u(X)$ is a risk-measure but to keep the paper in line with decision theory we prefer to use utility functions. Other terminolgy for such mappings is "risk adjusted value".

Remark. The paper was motivated after discussion with P. Embrechts and J. Neslehova. They were interested in defining risk-measures for spaces containing Pareto random variables with "bad exponent $\alpha \leq 1$" having e.g. densities of the form $c/(c' + x^{\alpha+1})\mathbf{1}_{\{x>0\}}$. These random variables showed up in operational risk studies, see [Neslehova, Embrechts and Chavez-Demoulin (2006)] for more information. In order to be able to do risk capital allocation there is a need for concave (even coherent) risk measures, see [Delbaen (2000)]. The results below show that there is no immediate solution for this problem.

Lemma [1]. *The function u is monotone in the following sense. If $X \leq Y$ are elements of E, if moreover $Y \in L^\infty$, then $u(X) \leq u(Y)$.*

Proof. We may and do suppose that $u(X) = 0$. It is then sufficient to show that $u(Y) \geq 0$. This will follow from the concavity of u. Let $1 \geq \varepsilon > 0$ and let $\alpha \geq \max(2, 2\|Y\|_\infty)/\varepsilon$. We claim that $\alpha(Y - X + \varepsilon) + X \geq 0$. Indeed on the set $\{X \geq -2\|Y\|_\infty\}$ we have $\alpha(Y - X + \varepsilon) + X \geq 0$ since $\alpha\varepsilon \geq 2\|Y\|_\infty$. On the set $\{X \leq -2\|Y\|_\infty\}$ we have $\alpha(Y - X) \geq \alpha(-\|Y\|_\infty - X) \geq -X$ since $\alpha \geq 2$. Since u is non-negative for non-negative random variables we find that $u(\alpha(Y - X + \varepsilon) + X) \geq 0$. Since u is concave we then get for $0 \leq \lambda = \frac{1}{\alpha} \leq 1$: $u(Y + \varepsilon) = u(\lambda(\alpha(Y - X + \varepsilon) + X) + (1 - \lambda)X) \geq \lambda u(\alpha(Y - X + \varepsilon) + X) + (1 - \lambda)u(X) \geq 0$. Since ε was arbitrary we proved $u(Y) \geq 0$.

Remark. The proof is a little bit curious. The fact that $Y \in L^\infty$ seems to be needed. However if u is superadditive (coherent), then $u(Y) \geq u(Y - X) + u(X)$ would give a trivial proof.

Representation Theorem [2]. *The function $c: \mathcal{P}^{\mathbf{ba}} \to \mathbb{R}_+ \cup \{+\infty\}$ defined as $c(\mu) = \sup\{\mu(-X) \mid X \in L^\infty, u(X) \geq 0\}$ satisfies $\inf_{\mu \in \mathcal{P}^{\mathbf{ba}}} c(\mu) = 0$ and for $X \in L^\infty$:*

$$u(X) = \inf\{\mu[X] + c(\mu) \mid \mu \in \mathcal{P}^{\mathbf{ba}}\}.$$

Furthermore the function c is convex and lower semi continuous, meaning that for $k \in \mathbb{R}$ the set $\{\mu \in \mathcal{P}^{\mathbf{ba}} \mid c(\mu) \leq k\}$ is convex and weak compact, i.e. $\sigma(\mathbf{ba}, L^\infty)$ compact.*

Remark. The proof of this representation theorem is based on duality and can be found in [Föllmer and Schied (2002)]. We remark that the representation theorem is only stated for bounded random variables. We do not claim any representation for other elements of E. A counter-example is given at the end of the paper.

Theorem [3]. *Suppose that $E \setminus L^\infty \neq \emptyset$,then u satisfies*

(1) *For $0 \leq k < \infty$ the set $\{\mu \in \mathcal{P}^{\mathbf{ba}} \mid c(\mu) \leq k\}$ is weakly compact in L^1. Hence $c(\mu) < \infty$ implies that μ is sigma-additive and is absolutely continuous with respect to $\mathbb{P}$, i.e. $\mu \in L^1$. Furthermore $0 = \min_{\mu \in \mathcal{P}^{\mathbf{ba}}} c(\mu)$.*

(2) *u is continuous from below: for non-decreasing sequences $X_n \uparrow X$, uniformly bounded $\sup \|X_n\|_\infty < \infty$, we have $\lim u(X_n) = u(X)$. This implies the weaker property that u is continuous from above, i.e. has the Fatou property.*

Proof. Take k a real number $0 \le k < \infty$ and suppose that the set $\{\mu \in \mathcal{P}^{\mathbf{ba}} \mid c(\mu) \le k\}$ contains a measure that is not sigma-additive. The Yosida-Hewitt theorem allows us to write $\mu = \mu^a + \mu^s$ where $\mu^a \in L^1$ and μ^s is purely finitely additive. Moreover if $\mu^s \neq 0$, there is a decreasing sequence of sets, say $(A_n)_n$ such that $\mu^s(A_n) \ge \varepsilon > 0$ and $\mathbb{P}[A_n] \downarrow 0$. Let us now take $X \in E \setminus L^\infty$. We may suppose that $X \le 0$ (since E is a solid vector space). Let $\beta_n = \inf\{x \mid \mathbb{P}[X \le x] \ge \mathbb{P}[A_n]\}$. Because X is unbounded we have that $\beta_n \to -\infty$. Now by rearrangement – see the lemma at the end of the paper – we may suppose that $X \le \beta_n$ on the set A_n. Monotonicity as in the lemma above, implies $u(X) \le u(\beta_n \mathbf{1}_{A_n})$. The representation theorem then implies that the latter term is bounded by $u(\beta_n \mathbf{1}_{A_n}) \le \mu(\beta_n \mathbf{1}_{A_n}) + c(\mu) \le \beta_n \mu^s(A_n) + k \le \beta_n \varepsilon + k$. Since β_n tends to $-\infty$ this would imply that $u(X) \le -\infty$, a contradiction to the hypothesis that u is real-valued. So we proved that $\mu^s = 0$ and consequently the set $\{\mu \in \mathcal{P}^{\mathbf{ba}} \mid c(\mu) \le k\}$ is a weakly compact subset of L^1. An easy compactness argument shows that the infimum in $0 = \inf_{\mu \in \mathcal{P}^{\mathbf{ba}}} c(\mu)$ is a minimum. The continuity from below is now a consequence of weak compactness as can be seen from [Jouini, Schachermayer and Touzi (2006)].

Theorem [4]. *With the above notation we have $E \subset L^1$.*

Proof. We may suppose that there is $X \in E \setminus L^\infty$ since otherwise $E = L^\infty$ and the statement becomes trivial. The previous theorem implies a weak compactness property. Take $\mathbb{Q} \in L^1$ a probability measure such that $c(\mathbb{Q}) < \infty$, we can even take $c(\mathbb{Q}) = 0$ but it does not simplify the proof. The existence of such a sigma additive probability measure is guaranteed by the previous theorem and $\inf\{c(\mu) \mid \mu \in \mathcal{P}^{\mathbf{ba}}\} = 0$. Of course we have $u(Y) \le \mathbb{E}_\mathbb{Q}[Y] + c(\mathbb{Q})$ for any $Y \in L^\infty$. By the monotonicity proved above and the Beppo Levi theorem this inequality extends to non-positive elements of E. For given $X \in E$, X unbounded, we have that $|X| \in E$ and we may by rearrangement, suppose that there is $\beta > 0$ so that $\{|X| \ge \beta\} \subset \{\frac{d\mathbb{Q}}{d\mathbb{P}} \ge 1/2\}$. The change of X to a rearrangement does not change the problem since rearrangements have the same integral under $\mathbb{P}$! We then find

$$-\infty < u(-|X|) \le \mathbb{E}_\mathbb{Q}[-|X|] + c(\mathbb{Q}).$$

In particular we find that $\mathbb{E}_\mathbb{Q}[|X|] < \infty$. This in turn implies that $\mathbb{E}_\mathbb{P}[|X|\mathbf{1}_{\{|X| \ge \beta\}}] < \infty$. This in turn implies that $\mathbb{E}_\mathbb{P}[|X|] < \infty$, as required.

Remark. The above theorem is related to the automatic continuity theorem for positive linear functionals defined on ordered spaces. From the continuity of such functionals it is easily derived that the space E cannot be too big, see [Biagini and Frittelli (2006)] and [Cheridito and Li (2006)] for a discussion. The difference with our result and the Namioka-Klee theorem is that we replaced the completeness assumption by the hypothesis that the space is rearrangement invariant. Together with the assumption that the space is solid, this is a convenient substitute to construct elements of E.

We now give a sketch of the proof of the rearrangement result used during the proofs.

Lemma [5]. *Let* $(A_t)_{t\in J\subset]0,1[}$ *be an increasing family of sets, i.e.* $A_t \subset A_s$ *if* $t \le s$. *Suppose that* $\mathbb{P}[A_t] = t$ *for* $t \in J$. *Then there exists an increasing family* $(A_t)_{t\in]0,1[}$ *such that for all* t: $\mathbb{P}[A_t] = t$. *There is also a random variable* U, *uniformly distributed on* $]0,1[$ *such that* $\{U \le t\} = A_t$ *for all* $0 < t < 1$.

This is a well known exercise in the study of atomless spaces. We skip the lengthy but straightforward proof.

Lemma [6]. *Let* $(A_t)_{t\in J\subset]0,1[}$ *be an increasing family of sets such that* $\mathbb{P}[A_t] = t$. *For a random variable* X, *let* $\mathbb{F}(x) = \mathbb{P}[X \le x]$ *and* $\mathbb{F}^{-1}(t) = \inf\{x \mid \mathbb{F}(x) \ge t\}$. *There exists a random variable* Y *having the same distribution as* X *and such that on* A_t *we have* $Y \le \mathbb{F}^{-1}(t)$.

Proof. We complete the system as in the previous lemma. We get a uniformly $]0,1[$ distributed random variable U such that $\{U \le t\} = A_t$. We then define $Y = \mathbb{F}^{-1}(U)$. The reader can check that Y satisfies all the desired properties.

Remark. If E is a solid, rearrangement invariant vector space containing non-integrable random variables, and if we want to define a risk measure on E, we need to consider a utility function u that takes infinite values. Of course we would like to have functions u such that $+\infty$ is avoided as this does not make economic sense, see [Delbaen (2002)] and [Delbaen(2000)] for a discussion and a condition that guarantees that $u(X) < +\infty$ for all $X \in E$.

Remark and Example. Let u be coherent and defined on a space E (rearrangement invariant and solid) containing an unbounded random variable. The first theorem then says that there is a weakly compact convex set $\mathcal{S}$ of probability measures $\mathbb{Q} \in L^1$ such that for $X \in L^\infty$: $u(X) = \min_{\mathbb{Q}\in\mathcal{S}} \mathbb{E}_{\mathbb{Q}}[X]$. There is no reason to believe that the same representation holds for all elements $X \in E$. As the following example shows this is related to (the failure of) a density property of L^∞ in the space E. Indeed let us introduce the following conjugate Young functions: $\Phi(x) = (x+1)\log(x+1) - x$, $\Psi(y) = \exp(y) - y - 1$. This leads to the following Orlicz spaces, see [Krasnoselskii and Ruticki(1961)] for more details on Orlicz spaces: $L^\Phi = \{X \in L^0 \mid \mathbb{E}_{\mathbb{P}}[\Phi(|X|)] < \infty\}$, $L^\Psi = \{X \mid \text{ there is } \alpha > 0, \mathbb{E}_{\mathbb{P}}[\Psi(\alpha|X|)] < \infty\}$, $L^{(\Psi)} = \{X \mid \text{ for all } \alpha > 0, \mathbb{E}_{\mathbb{P}}[\Psi(\alpha|X|)] < \infty\}$. The latter space is the closure of L^∞ in L^Ψ. It is clear that $L^{(\Psi)} \ne L^\Psi$, e.g. look at a random variable X that is exponentially distributed with density $\exp(-x)\mathbf{1}_{\{x>0\}}$. Furthermore L^Φ is the dual of $L^{(\Psi)}$ and L^Ψ is the dual of L^Φ. Take now X exponentially distributed and let $\mu \in (L^\Psi)^*$ so that $\mu \ge 0$, μ is zero on Ł$^{(\Psi)}$ and $\mu(X) \ne 0$. Since $X + L^\Psi_+$ is at a strictly positive distance from $L^{(\Psi)}$, the Hahn-Banach theorem gives the existence of such an element. Now define $u(Y) = \mathbb{E}_{\mathbb{P}}[Y] + \mu(Y)$. The functional u defined on $E = L^\Psi$ is linear, positive and monetary (!). When restricted to $L^{(\Psi)}$ and hence to L^∞, it coincides with the expectation operator. But on $E = L^\Psi$ it is different, since $\mu(X) \ne 0$. This shows that the representation theorem does not hold for all elements of E. Of course the reason is that L^∞ is not dense in the space E and therefore approximation by bounded random variables is not possible.

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