ON THE RANGE OF THE SUBDIFFERENTIAL IN NON REFLEXIVE BANACH SPACES

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ABSTRACT. We present the following unbounded version for James's theorem on weak compactness in Banach spaces: let C be a closed, convex but not necessarily bounded subset in the Banach space E, and Λ be a non-void and $\tau(E^*, E)$-open subset of E^*; i.e. Mackey open in the dual space, such that

$$\sup\{z^*(c) : c \in C\} < +\infty \text{ whenever } z^* \in \Lambda.$$

If C is not $\sigma(E^{**}, E^*)$-closed in E^{**} there is a linear form $z^* \in \Lambda$ such that the $\sup\{z^*(c) : c \in C\}$ is not attained.

As a main application we have the following: if $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$ is a proper and bounded below function such that the range of the subdifferential $\partial f(E)$ contains a nonvoid open subset for the Mackey topology on the dual space $(E^*, \tau(E^*, E))$, then for each set $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c]$ is relatively weakly compact. If in addition the function f has a domain with non-empty norm interior, the Banach space E must be reflexive.

Straightforward applications to robust representation of risk measures and weak solutions of variational equations are also derived.

1. INTRODUCTION

One of the most important results about weak compactness in Banach spaces is James's sup theorem. We refer to [3] for a detailled description of the state of the art around it until 2013. James's theorem asserts that a weakly closed subset A of a real Banach space E is weakly compact provided that each continuous and linear functional on E attains its supremum on A. In the last years, a few generalizations of James's sup theorem have appeared, some motivated by its use in mathematical finance, in which the linear optimization condition is replaced by another one of a perturbed nature; that is, for a fixed and adequate extended real–valued function f, $x^* - f$ attains its supremum, where x^* is any continuous and linear functional on E.

The first of these results deals with a specific subset of the space, its closed unit ball. Inspired by the fact that the set of norm attaining functionals in a real Banach space is not more than the range of the duality mapping, which in turn is the range of the subdifferential of a certain coercive, convex and lower semicontinuous function, S. Fitzpatrick and B. Calvert announced in [7] that a real

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Banach space is reflexive whenever its dual space coincides with the range of an extended real–valued coercive, convex and lower semicontinuous function whose effective domain has nonempty norm–interior. However, the erratum [1] makes [7] more difficult to follow, since the main addendum requires correcting non–written proofs of some statements in [1] which are adapted from [9].

Subsequently, and for arbitrary subsets, in [17] it was proved that a closed and convex subset A of a real Banach space E is weakly compact each time there exists a bounded function $f : A \longrightarrow \mathbb{R}$ such that for all continuous linear functional x^* on E, the function $x^*_{|A} - f$ attains its supremum on A. Finally, [10] contains another James's type result, but for a concrete class of Banach spaces and also under a certain boundedness assumption: if E is a separable real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex and lower semicontinuous function whose effective domain is bounded, and such that for all continuous linear functional x^* on E, the function $x^* - f$ attains its supremum, then its sublevel sets are weakly compact. This same statement has been shown by the first named author in [4] for nonseparable spaces $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ of integrable functions.

In [13] a new version of James's theorem is introduced that generalizes all these results: if E is a real Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a coercive function with subdifferential onto, then its sublevel sets are relatively weakly compact. As a consequence, if for a real Banach space E a suitable abstract optimization problem admits a solution, then E must be reflexive. Such abstract optimization problems include lots of nonlinear or nonsmooth variational ones that arise in connection with numerous applied problems, in the theory of partial differential equations and in many other areas of pure and applied mathematics. For Banach spaces with w^*-sequentially compact dual ball, the coercivity assumption for f was avoided, see [14] where new applications for risk measures on Orlicz spaces are also derived. J. Saint Raymond [18], proved a little bit later the same result without the coercivity assumption on the map f and for an arbitrary Banach space. W. Moors has recently derived, [11, 12] another proof of Saint Raymond's theorem together with a variational approach for James's compactness theorem. The main results in the present paper extend these results when the assumption on f is relaxed asking that the subdifferential map ∂f has an image with non-empty interior for the Mackey topology of the dual space $\tau(E^*, E)$.

We shall do it with a new criteria for a closed convex set C to be w^*-closed in the bidual based on the set of linear functionals attaining its supremum on C, indeed we shall present the following abstract main result:

Theorem 1. *Let C be a closed, convex but not necessarily bounded subset in the Banach space E, and Λ be a non-void and $\tau(E^*, E)$-open subset of E^* such that*

$$\sup\{z^*(c) : c \in C\} < +\infty \text{ whenever } z^* \in \Lambda. \tag{1}$$

If C is not $\sigma(E^{**}, E^*)$-closed in E^{**} there is a linear form $z^* \in \Lambda$ such that the $\sup\{z^*(c) : c \in C\}$ is not attained.

The former result corresponds with the unbounded version of the following James's theorem by the same authors, see [5] where a one-sided result is also included:

Theorem 2. *Let A be a nonempty, closed, convex, bounded and non weakly compact subset of a Banach space E. Let us fix a convex and weakly compact subset D of E, a functional $z_0^* \in E^*$ and $\epsilon > 0$. Then there is a continuous linear form $x_0^* \in B_{p_D}(z_0^*, \epsilon)$ i.e. $x_0^* \in E^*$ and*

$$\sup_{d \in D} |x_0^*(d) - z_0^*(d)| < \epsilon,$$

which does not attain its supremum on A.

Using the Krein-Smulian theorem one can easily see that a convex, norm closed set $C \subset E$ is $\sigma(E^{**}, E^*)$-closed in E^{**} if and only if for each n: $C \cap nB_E = C \cap nB_{E^{**}}$ is weakly compact.

As an application of our abstract results we obtain:

Theorem 3. *Let $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$ be a proper and bounded below function such that $\partial f(E)$ has non-empty interior in the Mackey topology $\tau(E^*, E)$, then for each set $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c]$ is relatively weakly compact.*

Now let us show how our results, following the ideas in [13, Corollary 5], has some consequences for set-valued mappings. Let us recall that given a Banach space E and a set-valued function $\Phi : E \longrightarrow 2^{E^*}$, the *domain* of Φ is the subset of E

$$\mathrm{D}(\Phi) := \{x \in E : \Phi(x) \text{ is nonempty}\},$$

and its *range* is the subset of E^*

$$\Phi(E) := \{x^* \in E^* : \text{there exists } x \in E \text{ with } x^* \in \Phi(x)\}.$$

In addition, Φ is said to be *monotone* if

$$\inf_{\substack{x,y \in \mathrm{D}(\Phi) \\ x^* \in \Phi(x),\ y^* \in \Phi(y)}} \langle x^* - y^*, x - y \rangle \geq 0,$$

and *cyclically monotone* when the inequality

$$\sum_{j=1}^{n} \langle x_j^*, x_j - x_{j-1} \rangle \geq 0$$

holds, whenever $n \geq 2$, $x_0, x_1, \ldots, x_n \in \mathrm{D}(\Phi)$ with $x_0 = x_n$ and for $j = 1, \ldots, n$, $x_j^* \in \Phi(x_j)$.

If Φ is a cyclically monotone operator then there exists a proper and convex function $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that for every $x \in E$,

$$\Phi(x) \subset \partial f(x),$$

see [16, Theorem 1], and so Theorem 3 leads to the following James type result for cyclically monotone operators:

Corollary 4. *Let E be a Banach space and let $\Phi : E \longrightarrow 2^{E^*}$ be a cyclically monotone operator such that $\mathrm{D}(\Phi)$ has nonempty norm-interior and the Mackey interior of $\Phi(E)$ is non void, then E is reflexive and the range of Φ coincides with E^* whenever Φ is maximal cyclically monotone.*

Indeed, reflexive spaces are the adequete frame for the solvability of nonlinear variational equations derived from the weak formulation of a wide range of boundary value problems, see Section 10.6.1 in [3], as the following consequence summarizes:

Corollary 5. *Let E be a real Banach space and let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that $\operatorname{dom}(f)$ has nonempty norm–interior and for all $x^* \in U$ there exists $x_0 \in E$ with*

$$f(x_0) - x^*(x_0) = \inf_{x \in E} \left(f(x) - x^*(x) \right), \tag{2}$$

where U is a non void $\tau(E^*, E)$-open set, then E is a reflexive Banach space. Moreover the minimization problem (2) has solution for all $x^* \in E^*$.

The organization of the paper is as follows. Section 3 is concerned with the analysis of Pryce's approach to James's compactness theorem for not necessarily bounded sets, which is our way to reach conclusions. Section 4 deals with the application of our results to the case of a coercive, convex and lower semicontinuous functions. Finally, in Section 5 we prove our main results and some applications are derived in Section 6.

2. Notation and terminology

Most of our notation and terminology is standard and can be found in standard references on Banach spaces [6].

Unless otherwise stated, E will denote a Banach space with norm $\|\cdot\|$. Given a subset S of a vector space, we write $\operatorname{co}(S)$, to denote its convex hull. If $(E, \|\cdot\|)$ is a normed space then E^* denotes its topological dual. If S is a subset of E^*, then $\sigma(E, S)$ denotes the topology of pointwise convergence on S. Dually, if S is a subset of E, then $\sigma(E^*, S)$ is the topology on E^* of pointwise convergence on S. In particular $\sigma(E, E^*)$ and $\sigma(E^*, E)$ are the weak (ω) and weak* (ω^*) topologies respectively. We denote by $\tau(E^*, E)$ the Mackey topology on E^*, i.e. the topology of uniform convergence on weakly compact sets of the Banach space E.

Given $x^* \in E^*$ and $x \in E$, we write $\langle x^*, x\rangle = \langle x, x^*\rangle = x^*(x)$ for the evaluation of x^* at x. If $x \in E$ and $\delta > 0$ we denote by $B(x, \delta)$ (resp. $B[x, \delta]$) the open (resp. closed) ball centred at x of radius δ. We simply write $B_E := B[0, 1]$ and the unit sphere $\{x \in E\colon \|x\| = 1\}$ will be denoted by S_E. When dealing with the Mackey topology $\tau(E^*, E)$ we denote by $B_{p_D}(x^*, \delta)$ (or $B_{p_D}[x^*, \delta]$) the open (resp. closed) ball centred at x of radius δ for the seminorm

$$p_D(x^*) = \sup\{|x^*(x)| : x \in D\}$$

of uniform convergence on the weakly compact set $D \subset E$. An element $x^* \in E^*$ is *norm-attaining* if there is $x \in B_E$ with $x^*(x) = \|x^*\|$. The set of norm-attaining functionals of E is usually denoted by $NA(E)$. The famous Bishop-Phelps theorem asserts that $NA(E)$ is norm dense in E^*.

If $(x_n)_n$ is a bounded sequence in a sequentially complete, Hausdorff locally convex space G, we denote by

$$\text{co}_\sigma(x_n; n \geq 1) = \left\{ \sum_{n=1}^{\infty} \xi_n x_n : \xi_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \xi_n = 1 \right\},$$

the σ-convex hull of the sequence $(x_n)_n$. The convex hull operator is denoted by co. A *pseudo-subsequence* of $(x_n)_n$ is a sequence $(y_n)_n$ such that $y_n \in \text{co}_\sigma(x_p; p \geq n)$ for every $n \in \mathbb{N}$.

For an absolutely convex subset W of the vector space E we shall denote by E_W the linear span of W: i.e. $E_W = \cup_{n=1}^{\infty} nW$.

For a function $a \in \mathbb{R}^X$ we denote by:

$$S_X(a) := \sup\{a(x) : x \in X\} \in (-\infty, +\infty].$$

S_X is positively homogeneous and subadditive. It should not be confused with the norm $\|a\|_\infty = \sup\{|a(x)| : x \in X\}$.

We will also need some elementary facts from convex function theory. If $f : E \to \mathbb{R} \cup \{+\infty\}$ is a function defined on a Banach space E, we say that f is proper if f is not identically $+\infty$. Its domain is the set $D(f) = \{x \in E \mid f(x) < \infty\}$. We remark that f cannot take the value $-\infty$. The epigraph of f is the set

$$\text{epi}(f) = \{(x, t) \mid x \in E \text{ and } +\infty > t \geq f(x)\}.$$

A function f is convex if $\text{epi}(f)$ is a convex set of $E \times \mathbb{R}$. The function f is lower semi continuous if $\text{epi}(f)$ is closed. If $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper function then its subdifferential at a point $x_0 \in D(f)$ is defined as

$$\partial f(x_0) = \{x^* \in E^* \mid \text{ for all } x \in E : f(x) \geq f(x_0) + x^*(x - x_0)\}.$$

The subdifferential can be the empty set! For a proper function f, the closed convex hull $\overline{\text{co}}(\text{epi}(f))$ of $\text{epi}(f)$ defines a lower semi continuous convex function as follows:

$$\hat{f}(x) = \inf\{t \mid (x, t) \in \overline{\text{co}}(\text{epi}(f))\}.$$

The function $\hat{f}$ is the biggest lower semi continuous convex function that is smaller than f. It is called the convex envelope. In case the subdifferential of f at x_0 is non-empty we have (as easily seen) $\hat{f}(x_0) = f(x_0)$ and $\partial \hat{f}(x_0) = \partial f(x_0)$. If $\partial f(x_0) \neq \emptyset$ for some $x_0 \in E$, then f is bounded below on bounded sets of E. The range of the subdifferential is

$$\partial f(E) = \{x^* \mid \text{there is } x \in E \text{ with } x^* \in \partial f(x)\}.$$

For proper functions f the above shows that $\partial \hat{f}(E) \supseteq \partial f(E)$.

3. The undetermined function technique for the unbounded case

We will extend the technique of the undetermined function of James and Pryce as further developed by Galán and Simons, see [8]. We shall complete the study we did in sections 2 of [14] and [3], where we were dealing with Simons' inequality and the so called sup-limsup theorem. We shall deal here with pointwise bounded sequences $\{a_j\}_{j\geq 1}$ in $\mathbb{R}^X$ with

$$\sup\{S_X(a_j) : j = 0, 1, 2, \cdots\} < +\infty$$

instead of requiring boundedness in $l^\infty(X)$, i.e. to have

$$\sup\{S_X(|a_j|) : j = 0, 1, 2, \cdots\} < +\infty.$$

We will first observe that the Galán-Simons approach in [8] remains valid in this one-sided boundedness case. Let us assume in what follows that $\sum_{i=1}^{0} \cdots$ is always interpreted to be zero.

We shall use the following elementary result:

Lemma 6. *Let $\{a_j\}_{j\geq 1}$ be a sequence in $\mathbb{R}^X$ with for some $x_0 \in X$*

$$\inf\{a_j(x_0) : j = 1, 2, \cdots\} = \alpha > -\infty.$$

Then we have

$$\inf\{S_X(b) : b \in co\{a_j : j \geq 1\}\} \geq \alpha > -\infty.$$

In case the sequence $(a_j)_j$ is pointwise bounded the same inequality holds for $co_\sigma\{a_j : j \geq 1\}$.

Proof. Obviously for $b \in co\{a_j : j \geq 1\}$ we have $b(x_0) \geq \alpha$ and in case $(a_j)_j$ is pointwise bounded, the same holds for $b \in co_\sigma\{a_j : j \geq 1\}$. Therefore $S_X(b) \geq \alpha$ for all b. □

For a complete treatment of all inequalities needed to prove our main result we need to adapt versions of the ones given in[8]. The proofs are almost the same and we shall strictly follow [8] to do it.

The first one is a direct consequence of Lemma 2 in [8]:

Lemma 7. *Let us assume that $\{a_j : j = 1, 2\cdots\}$ is a pointwise bounded sequence in $\mathbb{R}^X$, i.e. a bounded subset for the product topology in $\mathbb{R}^X$. If $b_i \in co_\sigma\{a_j : j \geq 1\}$ for all $i \geq 1$ then*

$$co_\sigma\{b_j : j \geq 1\} \subset co_\sigma\{a_j : j \geq 1\}.$$

Proof. The pointwise topology on $\mathbb{R}^X$ is sequentially complete, Hausdorff and locally convex and Lemma 2 in [8] gives us the proof. □

The second one follows Lemma 4 in [8], of course our sequence now is not assumed to be uniformly bounded from below:

Proposition 8. *Let $\{a_j\}_{j\geq 1}$ be a pointwise bounded sequence in $\mathbb{R}^X$ which is assumed to be uniformly upper bounded, i.e.*

$$\sup\{S_X(a_j) : j = 0, 1, 2, \cdots\} < +\infty.$$

We have that:

(i) *For every* $\rho, \eta \in (0,1)$ *there is a pseudo-subsequence* $\{b_i\}_{i\geq 1}$ *of* $\{a_j\}_{j\geq 1}$ *such that:*

$$k \geq 0 \Rightarrow S_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right) \geq S_X\left(\sum_{i=1}^{k} \rho^i b_i\right) + \rho^k \left[S_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right) - \eta\right] \tag{3}$$

(ii) *Let* $B < \inf S_X(co_\sigma\{a_j : j \geq 1\})$. *Then there is a pseudo-subsequence* $\{b_i\}_{i\geq 1}$ *of* $\{a_j\}_{j\geq 1}$ *such that:*

$$k \geq 0 \Rightarrow S_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right) \geq S_X\left(\sum_{i=1}^{k} \rho^i b_i\right) + B \sum_{i=k+1}^{\infty} \rho^i \tag{4}$$

Proof. For all $m \geq 1$ let us denote by $C_m := \mathrm{co}_\sigma\{a_j : j \geq m\}$. Once $\rho, \eta \in (0,1)$ has been fixed we proceed by induction. Since the reasoning for $m = 1$ is the same as the induction step we only give the step from $m-1$ to m. Suppose we have found $b_i \in C_i$ for $i = 1, 2, \cdots, m-1$. We then and choose $b_m \in C_m$ so that:

$$S_X\left(\sum_{i=1}^{m-1} \rho^i b_i + \rho^m b_m\right) \leq \inf_{b\in C_m} S_X\left(\sum_{i=1}^{m-1} \rho^i b_i + \rho^m b\right) + \eta(\rho/2)^m. \tag{5}$$

The induction process can be carried out since we have

$$\inf S_X(\mathrm{co}_\sigma\{a_j : j \geq 1\}) > -\infty$$

by Lemma 6. Let us define $c := \sum_{i=1}^{\infty} \rho^i b_i$, and for all $m \geq 1$, $c_m := \sum_{i=1}^{m} \rho^i b_i$. Then (5) gives for all $m \geq 1$,

$$S_X(c_m) \leq \inf_{b\in C_m} S_X(c_{m-1} + \rho^m b) + \eta(\rho/2)^m. \tag{6}$$

Let us now prove (i): For $k = 0$ the inequality is trivial. Now let $k \geq 1$ and $1 \leq m \leq k$. Then, since

$$(1-\rho)(c - c_{m-1})/\rho^m = \sum_{i=0}^{\infty}(1-\rho)\rho^i b_{i+m} \in C_m,$$

inequality (6) implies

$$\begin{aligned} S_X(c_m) &\leq S_X(c_{m-1} + (1-\rho)(c - c_{m-1})) + \eta(\rho/2)^m \\ &= S_X((1-\rho)c + \rho c_{m-1}) + \eta(\rho/2)^m \\ &\leq (1-\rho)S_X(c) + \rho S_X(c_{m-1}) + \eta(\rho/2)^m. \end{aligned}$$

Dividing both expresions by ρ^m, we obtain

$$(1/\rho^m - 1/\rho^{m-1})S_X(c) \geq S_X(c_m)/\rho^m - S_X(c_{m-1})/\rho^{m-1} - \eta/2^m.$$

Adding up these inequalities for $m = 1, 2, \cdots, k$ (and noting that $c_0 = 0$) yields

$$(1/\rho^k - 1)S_X(c) \geq S_X(c_k)/\rho^k - \eta,$$

which after rearrangement gives (i).

Let us now prove (ii): Let

$$0 < \eta < \sum_{i=1}^{\infty} \rho^i \left(\inf S_X(\text{co}_\sigma\{a_j : j \geq 1\}) - B\right],$$

and $\{b_i\}$ be chosen as in the first part of our lemma for this value of η. Then we have:

$$S_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right) - \eta \geq \left(\sum_{i=1}^{\infty} \rho^i\right) \inf S_X\left(\text{co}_\sigma\{a_j : j \geq 1\}\right) - \eta \geq \sum_{i=1}^{\infty} \rho^i B,$$

and the conclusion follows by substituting this into (3). □

The next consequence will be crucial to find non attaining functions later and it corresponds to Lemma 5 in [8]:

Corollary 9. *Under the former conditions, if*

$$\sup\{S_X(a_j) : j = 0, 1, 2, \cdots\} = M < +\infty,$$

any pseudo-subsequence $\{b_i\}_{i\geq 1}$ that verifies (4) above is such that the following inequality holds:

$$\inf_{k\geq 1} b_k(x_0) \geq \frac{B - \rho M}{1 - \rho} \tag{7}$$

whenever $x_0 \in X$ can be found so that the supremum is attained, i.e. :

$$\sum_{i=1}^{\infty} \rho^i b_i(x_0) = S_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right),$$

Proof. Our hypothesis implies for each $k \geq 0$:

$$S_X\left(\sum_{i=1}^{\infty} \rho^i b_i\right) \geq S_X\left(\sum_{i=1}^{k} \rho^i b_i\right) + B \sum_{i=k+1}^{\infty} \rho^i$$

So we have:

$$\sum_{i=1}^{\infty} \rho^i b_i(x_0) \geq \sum_{i=1}^{k} \rho^i b_i(x_0) + B \sum_{i=k+1}^{\infty} \rho^i$$

from which for $k \geq 1$

$$\sum_{i=k}^{\infty} \rho^{i-k} b_i(x_0) \geq B \sum_{i=0}^{\infty} \rho^i,$$

and so for $k \geq 1$

$$b_k(x_0) \geq B \sum_{i=0}^{\infty} \rho^i - \sum_{i=1}^{\infty} \rho^i M = \frac{B - \rho M}{1 - \rho}.$$

□

We can now state the main result needed for the construction of non attaining linear functionals, it corresponds to Corollary 8 in [8]:

Theorem 10. *Let φ be a nonnegative function in $\mathbb{R}^X$ and $\{h_j\}_{j\geq 1}$ a pointwise bounded sequence in $\mathbb{R}^X$. Let $0 < A < K$ be positive real numbers such that for all $h_0 \in \text{co}_\sigma\{h_j : j \geq 1\}$*

(8) $0 < A \leq S_X(h_0 - \limsup_j h_j - \varphi) = S_X(h_0 - \liminf_j h_j - \varphi) \leq K < \infty.$

Then there is a pseudo-subsequence $\{g_i\}_{i\geq 1}$ of $\{h_j\}_{j\geq 1}$, and $g_0 \in \text{co}_\sigma\{g_i : i \geq 1\}$, such that for every $\hat{g}$ satisfying

$$\text{for every } x \in X : \liminf g_i(x) \leq \hat{g}(x) \leq \limsup g_i(x),$$

the function $g_0 - \hat{g} - \varphi$ does not attain its supremum on X.

Proof. Let us denote by

$$\underline{h} := \liminf_j h_j \text{ and } \overline{h} := \limsup h_j,$$

and for all $j \geq 1$, $a_j := h_j - \underline{h} - \varphi$, so our hypothesis implies:

$$0 < A \leq \inf\{S_X(a_j) : j = 0, 1, 2, \cdots\} \leq \sup\{S_X(a_j) : j = 0, 1, 2, \cdots\} \leq K$$

We now apply Proposition 8 with $B := \rho(K + 1)$ small enough to have

$$B < \inf\{S_X(a_j) : j = 0, 1, 2, \cdots\}.$$

It follows that there exists a pseudo-subsequence $\{g_i\}$ of $\{h_j\}$ such that for each $k \geq 0$:

$$S_X\left(\sum_{i=1}^{\infty} \rho^i(g_i - \underline{h} - \varphi)\right) \geq S_X\left(\sum_{i=1}^{k} \rho^i(g_i - \underline{h} - \varphi)\right) + B\sum_{i=k+1}^{\infty} \rho^i$$

Let us take any pointwise cluster point $\hat{g}$ of the pointwise bounded sequence of functions $\{g_j\}$ in $\mathbb{R}^X$, we have

$$\underline{h}(x) \leq \liminf_j g_j(x) \leq \hat{g}(x) \leq \limsup_j g_j(x) \leq \overline{h}(x)$$

for every $x \in X$. We can apply Corollary 9, from where it follows that if we have

$$S_X\left(\sum_{i=1}^{\infty} \rho^i(g_i - \hat{g} - \varphi)\right) = \sum_{i=1}^{\infty} \rho^i(g_i - \hat{g} - \varphi)(x_0)$$

for some $x_0 \in X$, then we have:

$$\inf_{k\geq 1}(g_k - \hat{g} - \varphi)(x_0) \geq \frac{(K+1)\rho - K\rho}{1-\rho} = \frac{\rho}{1-\rho} > 0$$

which is impossible, since $\inf_{k\geq 1}(g_k - \hat{g}) \leq 0$ and $\varphi \geq 0$ on X . Consequently, the function $\sum_{i=1}^{\infty} \rho^i(g_i - \hat{g} - \varphi)$ cannot attain its supremum on X. The required result follows with

$$g_0 := \frac{1}{\sum_{i=1}^{\infty} \rho^i} \sum_{i=1}^{\infty} \rho^i g_i$$

□

In order to find the former conditions in our next application we need the following result which is an extension of Lemma 9 (b) in [8]:

Lemma 11. *Let* $\{a_j\}_{j\geq 1}$ *be a pointwise bounded sequence in* $\mathbb{R}^X$*. Then there exists a subsequence* $\{b_j\}_{j\geq 1}$ *of* $\{a_j\}_{j\geq 1}$ *such that*

$$S_X(\liminf_j b_j) = S_X(\limsup_j b_j)$$

Proof. Let us first assume that $S_X(\limsup_j a_j)$ is finite. We inductively choose subsequences and then use Cantor's diagonal argument. We start with $b_j^0 = a_j$ for all $j \geq 1$. The first step in the construction is similar to the induction reasoning. Suppose we already found successive subsequences $(b_j^0)_j, \dots, (b_j^{m-1})_j$. Take $x \in X$ with

$$\limsup_j b_j^{m-1}(x) \geq S_X\left(\limsup_j b_j^{m-1}\right) - \frac{1}{m}.$$

Take now a subsequence $(b_j^m)_j$ of $(b_j^{m-1})_j$ with $\lim_j b_j^m(x) = \limsup_j b_j^{m-1}(x)$. Clearly

$$S_X\left(\liminf_j b_j^m\right) \geq S_X\left(\limsup_j b_j^{m-1}\right) - 1/m.$$

Using Cantor's diagonal argument we find a pointwise bounded subsequence $\{b_n^n\}_{n\geq 1}$ such that

$$S_X\left(\liminf_j b_j^j\right) = S_X\left(\limsup b_j^j\right).$$

Indeed, we have for all $m \geq 1$:

$$\begin{aligned} S_X\left(\liminf_j b_j^j\right) &\geq S_X\left(\liminf_j b_j^m\right) \\ &\geq S_X\left(\limsup_j b_j^{m-1}\right) - 1/m \\ &\geq S_X\left(\limsup_j b_j^j\right) - 1/m. \end{aligned}$$

from where the conclusion now follows.

In case we have $S_X(\limsup_j a_j) = +\infty$, we select the subsequences in the same way as above but using the sequences $\big(\arctan(b_j^m)\big)_j$. □

Let us finally write here the main construction of the undefined function technique in our context, which finishes the extension of Lemma 9 in [8]:

Proposition 12. *Let* φ *a nonnegative function in* $\mathbb{R}^X$ *let* $\{f_j\}_{j\geq 1}$ *be a pointwise bounded sequence in* $\mathbb{R}^X$*. Then there exists a subsequence* $\{h_j\}_{j\geq 1}$ *of* $\{f_j\}_{j\geq 1}$ *such that for every* $h_0 \in \mathrm{co}_\sigma\{f_j : j \geq 1\}$*:*

$$S_X(h_0 - \limsup_j h_j - \varphi) = S_X(h_0 - \liminf_j h_j - \varphi).$$

Proof. For a positive integer N let us consider the set

$$X_N := \{x \in X : |\varphi(x) \leq N \text{ and for all } j : |f_j(x)| \leq N\}$$

and the norm $\|g\|_N := \sup\{|g(x)| : x \in X_N\}$ for $g \in l^\infty(X_N)$. This norm is a well defined seminorm on $\mathrm{co}_\sigma\{f_j : j \geq 1\}$. Because the sequence $(f_j)_j$ is pointwise bounded we have $X = \cup_N X_N$. Since the Banach space l^1 is separable, the set of functions $\mathrm{co}_\sigma\{f_j : j \geq 1\}$ is going to be $\|\cdot\|_N$-separable for every $N \in \mathbb{N}$, so we can fix a dense subset $\{d_m^N : m \geq 1\}$ for it. We fix $h_j^{0,1} := f_j$ for all $j \geq 1$. Let us inductively use the former lemma with $a_j^1 := d_m^1 - h_j^{m-1,1} - \varphi$ for all $j \geq 1$, to get a subsequence $\{h_j^{m,1}\}_{j\geq 1}$ of $\{h_j^{m-1,1}\}_{j\geq 1}$ so that:

$$S_{X_1}(d_m^1 - \limsup_j h_j^{m,1} - \varphi) = S_{X_1}(d_m^1 - \liminf_j h_j^{m,1} - \varphi)$$

Using Cantor's diagonal argument we obtain a pointwise bounded subsequence $\{h_n^{n,1}\}_{n\geq 1}$ that verifies:

$$S_{X_1}(d_m^1 - \limsup_n h_n^{n,1} - \varphi) = S_{X_1}(d_m^1 - \liminf_n h_n^{n,1} - \varphi)$$

for all $m \geq 1$. Therefore we have

$$S_{X_1}(h_0 - \limsup_n h_n^{n,1} - \varphi) = S_{X_1}(h_0 - \liminf_n h_n^{n,1} - \varphi)$$

for all $h_0 \in \mathrm{co}_\sigma\{f_j : j \geq 1\}$.

Inductively, for every $p \in \mathbb{N}$ we construct a subsequence $\{h_n^{n,p}\}_{n\geq 1}$ of $\{h_n^{n,p-1}\}_{n\geq 1}$ that verifies:

$$S_{X_p}(d_m^p - \limsup_n h_n^{n,p} - \varphi) = S_{X_p}(d_m^p - \liminf_n h_n^{n,p} - \varphi)$$

for all $m \geq 1$, and therefore:

$$S_{X_p}(h_0 - \limsup_n h_n^{n,p} - \varphi) = S_{X_p}(h_0 - \liminf_n h_n^{n,p} - \varphi)$$

for all $h_0 \in \mathrm{co}_\sigma\{f_j : j \geq 1\}$.

Using Cantor's diagonal argument once again we have a pointwise bounded subsequence $\{h_n^{n,n} : n = 1, 2, \cdots\}$ such that

$$S_{X_p}(h_0 - \limsup_n h_n^{n,n} - \varphi) = S_{X_p}(h_0 - \liminf_n h_n^{n,n} - \varphi)$$

for all $h_0 \in \mathrm{co}_\sigma\{f_j : j \geq 1\}$ and every $p = 1, 2, \cdots$. Since $X = \cup_p X_p$ it follows that $S_X(g) = \sup_p S_{X_p}(g)$ for all $g \in \mathbb{R}^X$ and so we finally have:

$$S_X(h_0 - \limsup_n h_n^{n,n} - \varphi) = S_X(h_0 - \liminf_n h_n^{n,n} - \varphi)$$

for all $h_0 \in \mathrm{co}_\sigma\{f_j : j \geq 1\}$. □

4. On the range of the subdifferential map

In this section we shall present an extension of the main result of [13]. We decided to include it for readers interested only in this result and in its applications. In the next section we shall obtain an abstract version for it together with another proof that does not use the coercivity assumption.

Theorem 13. *Let $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$ be a proper and coercive function, i.e.*

$$\lim_{\|x\| \longrightarrow \infty} \frac{f(x)}{\|x\|} = +\infty,$$

such that $\partial f(E)$ contains a non-empty open subset for the $\tau(E^*, E)$-topology. Then for each set $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c])$ is a relatively weakly compact subset of E.

Proof. Let us first assume that our function f is convex. Since $\partial f(E) \neq \emptyset$, the function f is bounded below on bounded sets. Because f is also coercive, f must then be bounded below. So we may without loss of generality, assume that $f \geq 0$. Our hypothesis means there is $z_0^* \in \partial f(E)$, together with a $\sigma(E, E^*)$-compact $W \subset E$ and $\epsilon > 0$, such that

$$B_{P_W}(z_0^*, \epsilon) \subset \partial f(E) \tag{9}$$

We are going to work in the Banach space $F = E \times \mathbb{R}$ with the norm $\|(x, t)\| = \max(\|x\|, |t|)$ and where we denote by $C = \text{epi}(f)$ the epigraph of f.

We claim that the norm closure $\hat{C} := \overline{C}^{\|\cdot\|}$ is $\sigma(F^{**}, F^*)$-closed from where the conclusion with the sublevel sets easily follows.

By the Krein-Smulian theorem it is enough to show that $\hat{C} \cap nB_{F^{**}}$ is $\sigma(F^{**}, F^*)$-closed for every positive integer n. Let us assume, arguing by contradiction, that $\hat{A} := \hat{C} \cap B_{F^{**}} = \hat{C} \cap B_F$ is not $\sigma(F^{**}, F^*)$-closed, and thus not weakly compact. Hence $A := C \cap B_F$ is not relatively weakly compact and Eberlein's theorem allows us to find a sequence $\{(x_n, \lambda_n) \mid n \geq 1\} \subset A$ which has no weak cluster point in F. Then we find $x_0^{**} \in \overline{\{x_n : n \geq 1\}}^{\sigma(E^{**}, E^*)}$ and $\lambda \in \mathbb{R}^+$ such that

$$(x_0^{**}, \lambda) \in \overline{\{(x_n, \lambda_n) \mid n \geq 1\}}^{\sigma(F^{**}, F^*)} \setminus F. \tag{10}$$

Let us assume that $\lambda_n = \lambda = 1$ for every n. This is not a restriction since we always have $0 \leq \lambda_n \leq 1$, thus $(x_n, 1) \in C$ for every n and (x_n, λ_n) will not have a cluster point if and only if the sequence x_n has no cluster point.

In the duality $\langle F^*, F^{**} \rangle$, if B denotes the $\sigma(F^{**}, F^*)$-closed convex hull of the line segment $[-(x_0^{**}, 1)(x_0^{**}, 1)] = \{(\lambda x_0^{**}, 1) : -1 \leq \lambda \leq 1\}$ and C, we can fix any $v_0 \in F \setminus B$ and use the separation theorem to find a functional $(x^*, \xi) \in F^*$, $\|(x^*, \xi)\| = 1$ so that

$$(x^*, \xi)(B) \subset (\infty, +\mu],$$

$$(x^*, \xi)(v_0) = a$$

where

$$0 \leq \mu < \beta < \alpha < a \text{ and } (\alpha - \beta) > -\langle (x_0^{**}, 1), (z_0^*, -1) \rangle. \tag{11}$$

Indeed, it is sufficient to use $\zeta v_0 \in F$, with $\zeta > 1$ big enough to get (11), instead of v_0 if it should be necessary. Let us observe we have:

$$\langle (x^*, \xi), (x_n, 1) \rangle < \beta \text{ for every } n \in \mathbb{N}. \tag{12}$$

We now use a standard result for non reflexive spaces. This result follows from the Hahn–Banach theorem. Since $x_0^{**} \notin E$ and since E is a norm closed subspace of E^{**} there is a linear form $z^{***} \in B_{E^{***}}$ such that $z^{***}(x_0^{**}) > 0$ but $z^{***}(E) = \{0\}$. Let us choose $\lambda > 0$ such that $\lambda z^{***}(x_0^{**}) + \langle (x^*, \xi), (x_0^{**}, 1) \rangle) > \alpha$ and consider the linear functional

$$(x^{***}, \xi) := (x^*, \xi) + \lambda(z^{***}, 0).$$

We will have

$$(x^{***}, \xi)(v_0) = (x^*, \xi)(v_0) = a > \alpha > 0, \tag{13}$$

$$\langle (x^{***}, \xi), (x_0^{**}, 1) \rangle = \lambda z^{***}(x_0^{**}) + \langle (x^*, \xi), (x_0^{**}, 1) \rangle > \alpha > 0.$$

Let us remind the reader that by (12) we also have for every $n \in \mathbb{N}$:

$$\langle (x^{***}, \xi), (x_n, 1) \rangle = \langle (x^*, \xi), (x_n, 1) \rangle < \beta. \tag{14}$$

Without loss of generality we may and do assume $\|x^{***}\| = 1$. Otherwise all former inequalities are adapted to the new situation by using $\|x^{***}\|^{-1} x^{***}$ instead of x^{***} any time. The only inequality which may require something more is again (11) which we get with another multiplication with a big enough positive scalar as above, if it should be necessary. The accounting is analogous to [5].

By Goldstine's theorem B_{E^*} is weak*, i.e. $\sigma(E^{***}, E^{**})$, dense in $B_{E^{***}}$. Therefore by the Mackey-Arens theorem it is also Mackey, i.e $\tau(E^{***}, E^{**})$, dense. We can find a sequence $x_n^* \in B_{E^*}$ such that

$$\text{for all } p \geq 1, \qquad \lim_{n \geq 1} x_n^*(x_p) = x^{***}(x_p) = x^*(x_p),$$

$$\lim_{n \geq 1} x_0^{**}(x_n^*) = x^{***}(x_0^{**}),$$

$$\lim_{n \geq 1} \langle (x_n^*, \xi), v_0 \rangle = (x^{***}, \xi)(v_0) = \langle (x^*, \xi), (v_0) \rangle$$

and

$$\lim_{n \geq 1} \langle (x_n^*, \xi), (w^{**}, 0) \rangle = \langle (x^{***}, \xi)(w^{**}, 0) \rangle = x^{***}(w^{**}),$$

uniformly on $w^{**} \in W$. Indeed, for every positive integer n we deal with the $\tau(E^{***}, E^{**})$-continuous seminorm p_n of uniform convergence on $\{x_1, x_2, \cdots, x_n\} \cup \{x_0^{**}, \} \cup W$, and we can require to find linear forms x_n^* such that:

$$p_n(x_n^* - x^{***}) < 1/2^n,$$

what means, for every $n \in \mathbb{N}$ the following inequalities:

$$|x_n^*(x_p) - x^{***}(x_p)| < 1/2^n \tag{15}$$

for all $p \leq n$

$$|(x_n^* - x^{***})(x_0^{**})| < 1/2^n \tag{16}$$

and

$$|(x_n^* - x^{***})(w^{**})| < 1/2^n \tag{17}$$

for every $w^{**} \in W$. Moreover, without loss of generality we do assume that:

$$|\langle (x_n^*, \xi) - (x^{***}, \xi), (v_0)\rangle| < 1/2^n \tag{18}$$

Then we have:

$$\lim_{n\to+\infty} x_n^*(x_p) = \langle x^*, x_p\rangle \tag{19}$$

for every $p \in \mathbb{N}$,

$$\lim_{n\to+\infty} x_n^*(x_0^{**}) = \langle x^{***}, x_0^{**}\rangle \tag{20}$$

$$\lim_{n\to+\infty} (x_n^*, \xi)(v_0) = (x^*, \xi)(v_0) \tag{21}$$

and

$$\lim_{n\to+\infty} x_n^*(w^{**}) = \langle x^{***}, w^{**}\rangle \tag{22}$$

uniformly on $w^{**} \in W$

The inequalities above have some consequences which follow from (19) and (14):

(23) for every $p \geq 1,$ there is n_p such that $\beta > \langle (x_n^*, \xi), (x_p, 1)\rangle$ for $n \geq n_p$,

and without loss of generality we may assume that:

$$\langle (x_0^{**}, 1), (x_n^*, \xi)\rangle) > \alpha, \tag{24}$$

by (20) and(13).

Note that given any pointwise–cluster point x_0^* on E^* for the $\sigma(E^*, E)$-topology of the bounded sequence $\{x_n^*\}_{n\geq 1}$, we have that

$$\langle (x_0^*, \xi), (x_0^{**}, 1)\rangle \leq \beta, \tag{25}$$

because $x_0^{**} \in \overline{\{x_p : p \geq 1\}}^{\sigma(E^{**}, E^*)}$ and for all $p \geq 1, \langle (x_0^*, \xi), (x_p, 1)\rangle) \leq \beta$ by (23).

We now have all ingredients to find the non-attaining linear functionals following the *unbounded undetermined function procedure*, which goes back to Pryce and James,[9],[15], as well as Galan and Simons in [8], as we have seen in the former section, see Theorem 10. Indeed, we are going to work on $\mathbb{R}^C$ with the pointwise bounded sequence $\{(x_n^*, \xi) : n = 1, 2, \cdots\}$.

First let us remark that for $(x, \lambda) \in C$ we have:

$$\langle (-z_0^*, 1), (x, \lambda)\rangle = -z_0^*(x) + \lambda \geq -z_0^*(x) + f(x) \geq f(x_0) - z_0^*(x_0)$$

where we are setting $z_0^* \in \partial f(x_0)$ by hypothesis. Thus it is not a restriction to assume the requirement $(-z_0^*, 1)$ nonnegative on C from the very beginning, just a translation of C should be enough in case it should be necessary. Proposition 12 provides us a subsequence

$$\{(x_{n_k}^*, \xi) : k = 1, 2, \cdots\}$$

such that, for all $h_0 \in \mathrm{co}_\sigma \{(x_n^*, \xi) : n \geq 1\}$, we have:

(26)
$$\sup_C \left(h_0 - \limsup_{k \geq 1} (x_{n_k}^*, \xi) + (z_0^*, -1) \right) = \sup_C \left(h_0 - \liminf_{k \geq 1} (x_{n_k}^*, \xi) + (z_0^*, -1) \right).$$

Let us now observe that for x_0^{**} we have by (24) that

$$\forall h_0 \in \mathrm{co}_\sigma \left\{ (x_{n_k}^*, \xi) : k \geq 1 \right\}, \qquad \langle (x_0^{**}, 1), h_0 \rangle > \alpha. \tag{27}$$

Let us fix a $\sigma(E^*, E)$-cluster point x_0^* for the bounded sequence $\left\{ x_{n_k}^* : k \geq 1 \right\}$; then it follows that for all $c \in C$,

$$\limsup_{k \geq 1} (x_{n_k}^*, \xi)(c) \geq (x_0^*, \xi)(c) \geq \liminf_{k \geq 1} (x_{n_k}^*, \xi)(c)$$

and thus, for all $c \in C$,

$$\begin{aligned} &\left(h_0(c) - \liminf_{k \geq 1} (x_{n_k}^*, \xi)(c) + (z_0^*, -1)(c) \right) \\ &\geq (h_0 - (x_0^*, \xi) + (z_0^*, -1))(c) \\ &\geq \left(h_0(c) - \limsup_{k \geq 1} (x_{n_k}^*, \xi)(c) + (z_0^*, -1)(c) \right). \end{aligned}$$

Therefore, in view of (26) we deduce that for all $h_0 \in \mathrm{co}_\sigma \left\{ (x_{n_k}^*, \xi) : k \geq 1 \right\}$

$$\begin{aligned} &\sup_C \left(h_0 - \limsup_{k \geq 1} (x_{n_k}^*, \xi) + (z_0^*, -1) \right) \\ &= \sup_C \left(h_0 - \liminf_{k \geq 1} (x_{n_k}^*, \xi) + (z_0^*, -1) \right) \\ &= \sup_C (h_0 - (x_0^*, \xi) + (z_0^*, -1)). \end{aligned}$$

Let us observe now that for $h_0 \in \mathrm{co}_\sigma \left\{ (x_{n_k}^*, \xi) : k \geq 1 \right\}$ we have:

$$\begin{aligned} &\sup_C (h_0 - (x_0^*, \xi) + (z_0^*, -1)) \\ &= \sup_{\overline{C}^{w^*}} (h_0 - (x_0^*, \xi) + (z_0^*, -1)) \\ &\geq \langle (x_0^{**}, 1), h_0 - (x_0^*, \xi) + (z_0^*, -1) \rangle \end{aligned}$$

and

$$\langle (x_0^{**}, 1), h_0 - (x_0^*, \xi) + (z_0^*, -1) \rangle > \alpha - \beta + \langle (x_0^{**}, 1), (z_0^*, -1) \rangle > 0,$$

by (25), (27) and (11), moreover we also have

$$\begin{aligned}
&\sup_{h_0\in \mathrm{co}_\sigma\left\{(x^*_{n_k},\xi):k\geq 1\right\}} \sup_{c\in C}\langle c, h_0 - \lim_k\sup(x^*_{n_k},\xi) + (z^*_0,-1)\rangle \\
&= \sup_{h_0\in \mathrm{co}_\sigma\left\{(x^*_{n_k},\xi):k\geq 1\right\}} \sup_{c\in C}\langle\ c, h_0 - \liminf_k(x^*_{n_k},\xi) + (z^*_0,-1)\rangle \\
&= \sup_{h_0\in \mathrm{co}_\sigma\left\{(x^*_{n_k},\xi):k\geq 1\right\}} \sup_{c\in C}\langle c, h_0 - (x^*_0,\xi) + (z^*_0,-1)\rangle < +\infty
\end{aligned}$$

because of the coercivity assumption on the function f. Indeed, the first coordinate of the linear functionals

$$h_0 - (x^*_0,\xi) + (z^*_0,-1)$$

corresponds with a linear functional $h^1_0 - x^*_0 + z^*_0 \in \partial f(E)$ since

$$|h^1_0(w) - x^*_0(w)| < \epsilon \text{ for every } w \in W \tag{28}$$

as we are going to see now, and just apply our hypothesis (9). Indeed, by uniform convergence of the sequence (x^*_n) to x^{***} on W, we may assume that the cluster point x^*_0 does coincide with x^{***} on W, where we have:

$$x^{***}(w) + \epsilon > x^*_n(w) > x^{***}(w) - \epsilon$$

for all $w \in W$ and every $n \in \mathbb{N}$, and finally

$$x^{***}(w) + \epsilon > h^1_0(w) > x^{***}(w) - \epsilon$$

for all $w \in W$ and every $i \in \mathbb{N}$, therefore

$$+\epsilon > (h^1_0 - x^*_0)(w)) > -\epsilon$$

for every $w \in W$ and (28) follows. Consequently we have

$$h^1_0 - x^*_0 + z^*_0 \in \partial f(E),$$

thus $h^1_0 - x^*_0 + z^*_0 = \partial f(x(h^1_0))$ for some $x(h^1_0) \in E$ and every $h^1_0 \in \mathrm{co}_\sigma\left\{x^*_{n_k} : k \geq 1\right\}$. As we mention above, the coercivity of f ensures the boundedness of

$$H := \sup\{\left\{\|x(h^1_0)\| : h^1_0 \in \mathrm{co}_\sigma\left\{x^*_{n_k} : k \geq 1\right\}\right\}\} < +\infty,$$

see the proof of Lemma 2 in [13]. Finally we see that

$$\sup_{h_0\in \mathrm{co}_\sigma\left\{(x^*_{n_k},\xi):k\geq 1\right\}} \sup_{y\in C}\langle y, h_0 - (x^*_0,\xi) + z^*_0,-1)\rangle < +\infty$$

because

$$\begin{aligned}
\sup_{c\in C}\langle c, h_0 - (x^*_0,\xi) + (z^*_0,-1)\rangle &= \sup_{c\in C}\langle (h^1_0 - x^*_0 + z^*_0,-1)\rangle \\
&= \langle h^1_0 - x^*_0 + z^*_0, x(h^1_0)\rangle - f(x(h^1_0))
\end{aligned}$$

and therefore

$$\sup_{h_0\in \mathrm{co}_\sigma\left\{(x^*_{n_k},\xi):k\geq 1\right\}} \sup_{c\in C}\langle c, h_0 - (x^*_0,\xi) + (z^*_0,-1)\rangle \leq (2 + \|z^*_0\|)H$$

since we are assuming form the beginning that $f \geq 0$.

All ingredients needed to apply Theorem 10 are verified, thus we get a sequence $(g_i^*, \xi)_{i\geq 1}$ with $(g_i^*, \xi) \in \mathrm{co}_\sigma\{(x_{n_k}^*, \xi) : k \geq i\}$ and $(g_0^*, \xi) \in \mathrm{co}_\sigma \{(g_i^*, \xi) : i \geq 1\}$ such that for all $(\tilde{g}, \xi) \in \ell_\infty(C)$ with

$$\liminf_{i\geq 1}(g_i^*, \xi) \leq (\tilde{g}, \xi) \leq \limsup_{i\geq 1}(g_i^*, \xi) \text{ on } C$$

we have that

$$((g_0^*, \xi) - (\tilde{g}, \xi) + (z_0^*, -1)) \text{ does not attain its supremum on } C.$$

In particular $(g_0^* - \tilde{g} + z_0^*) \notin \partial f(E)$ if $\tilde{g} \in E^*$, so we have

$$(g_0^* - \tilde{x_0^*} + z_0^*) \notin \partial f(E),$$

whenever we take a cluster point $\tilde{x_0^*}$ of the bounded sequence $(g_i^*)_{i=1}^\infty$ in $E^*[\sigma(E^*, E)]$, which is a contradiction with the fact that $B_{p_W}(z_0^*, \epsilon) \subset \partial f(E)$.

In the general case, when we have no assumption of convexity for the function f, it is enough to consider the convex envelope $\hat{f}$ of f. The function $\hat{f}$ is coercive when f is. Since $\hat{f}$ is a proper, convex and lower semicontinuous function such that

$$\partial(\hat{f}) \supseteq \partial(f)$$

our former reasoning says that the sublevel sets of $\hat{f}$ are weakly compact, and hence those of $f \geq \hat{f}$ are relatively weakly compact as well. □

Now we have the following consequence that we already have proved in [5]:

Corollary 14. *A closed, convex and bounded subset A of the Banach space E is weakly compact if, and only if, there is a nonvoid $\tau(E^*, E)$-open subset V of E^* such that every $x^* \in V$ attains its supremum on A*

Proof. It is enough to consider the function $f(x) = 0$ if $x \in A$ and $f(x) = +\infty$ when $x \notin A$. For every $x^* \in E^*$, if there is $x_0 \in A$ such that

$$x^*(x_0) = \sup\{x^*(a) : a \in A\},$$

then we have:

$$f(x_0) - x^*(x_0) = \inf_{x\in E}(f(x) - x^*(x)) \Longleftrightarrow x^* \in \partial f(x_0)$$

from where a straightforward application of Theorem 13 gives us the conclusion. □

Remark 15. Let us observe that as a consequence of the fact that the sublevel sets of a function g are relatively weakly compact it follows that $\partial g(E) = E^*$. If this should be our hypothesis from the beginning we have another proof of the main result in [13] which answered a question of J. Jouini, W. Schachermayer and N. Touzi, see[10].

5. A NEW JAMES' TYPE RESULT FOR UNBOUNDED SETS

We shall describe here unbounded and hence one-sided versions of James's theorem. They go in the same line as we have done with the epigraph of a given function in the former section, but in an abstract context here. Next result provides strong extensions of the main one in ([2], Theorem 9) for arbitrary Banach spaces.

Theorem 16. *Let C be a closed, convex but not necessarily bounded subset in the Banach space E and D be a weakly compact subset of E. Let us assume that*

$$\sup\{z^*(c) : c \in C\} < +\infty. \tag{29}$$

whenever $\sup\{z^*(d) : d \in D\} < 0$. *Let us fix an open subset $U \subset E^*$ for the Mackey topology $\tau(E^*, E)$, together with a functional $z_0^* \in U$ such that*

$$\sup\{z_0^*(d) : d \in D\} < 0.$$

If C is not $\sigma(E^{**}, E^*)$-closed in E^{**}, there is a linear form $z^* \in U$ such that

$$\sup\{z^*(d) : d \in D\} < 0,$$

and so

$$\sup\{z^*(c) : c \in C\} < +\infty,$$

but z^* does not attain its supremum on C.

Proof. The set $L_D := \{z^* \in E^* : z^*(D) < 0\}$ is a Mackey open subset of E^*. Let us denote by:

$$J^k := \{z^* \in E^* : \sup z^*(C) \leq k\}$$

which is a closed subset of E^*. Since $z_0^* \in L_D \cap U$, and U is $\tau(E^*, E)$-open, we can choose

$$\Omega := B_{p_W}(z_0^*, \epsilon/2) \subset B_{p_W}(z_0^*, \epsilon) \subset U$$

for some W absolutely convex and weakly compact subset of E and $\epsilon > 0$, and we have $z_0^* \in L_D \cap \Omega$. This set is norm open and can be written as

$$L_D \cap \Omega = \cup_{k=1}^{\infty} J^k \cap L_D \cap \Omega.$$

The Baire category theorem tells us that there is an integer n_0 such that $J^{n_0} \cap L_D \cap \Omega$ has non void relative interior in $L_D \cap \Omega$, i.e. for some $y_0^* \in J^{n_0} \cap L_D \cap \Omega$ and $\rho_0 > 0$ we have that

$$B_{\|\cdot\|}(y_0^*, \rho_0) \cap L_D \cap \Omega \subset J^{n_0} \cap L_D \cap \Omega.$$

and without loss of generality we may and do assume that:

$$B_{\|\cdot\|}(y_0^*, \rho_0) \subset J^{n_0} \cap L_D \cap \Omega \tag{30}$$

since y_0^* belongs to the norm open set $\in L_D \cap \Omega$

Let us assume that $A := C \cap B_{E^{**}} = C \cap B_E$ is not $\sigma(E^{**}, E^*)$-closed, and thus not weakly compact. Eberlein's theorem allows us to find a sequence $\{x_n \mid n \geq 1\} \subset A$ which has no weak cluster point in E. We then find $x_0^{**} \in \overline{\{x_n : n \geq 1\}}^{\sigma(E^{**}, E^*)}$ such that

$$x_0^{**} \in \overline{A}^{\sigma(E^{**}, E^*)} \setminus E. \tag{31}$$

Dealing in the duality $\langle E^*, E^{**}\rangle$, if B denotes the $\sigma(E^{**}, E^*)$-closed convex hull of the line segment $[-x_0^{**}, x_0^{**}] = \{\lambda x_0^{**} : -1 \leq \lambda \leq 1\}$ and C, we can fix any $v_0 \in E \setminus B$ and use the separation theorem to find a functional $x^* \in E^*, \|x^*\| = 1$ so that

$$x^*(B) \subset (\infty, +\mu],$$
$$x^*(v_0) = a$$

where

$$0 \leq \mu < \beta < \alpha < a; \text{ and } (\alpha - \beta) > -x_0^{**}(y_0^*). \tag{32}$$

Indeed, it is sufficient to use $\xi v_0 \in E$ with $\xi > 1$ big enough for (32), instead of v_0, if it should be necessary. Let us observe we have:

$$x^*(x_n) < \beta \text{ for every } n \in \mathbb{N}. \tag{33}$$

We now use a standard result for non reflexive spaces. This result follows from the Hahn–Banach theorem. Since $x_0^{**} \notin E$ and since E is a norm closed subspace of E^{**} there is a linear form $z^{***} \in B_{E^{***}}$ such that $z^{***}(x_0^{**}) > 0$ but $z^{***}(E) = \{0\}$. Let us choose $\lambda > 0$ such that $\lambda z^{***}(x_0^{**}) + x^*(x_0^{**}) > \alpha$ and consider the linear functional

$$x^{***} := x^* + \lambda z^{***}.$$

We will have

$$x^{***}(v_0) = x^*(v_0) = a > \alpha > 0, \tag{34}$$
$$x^{***}(x_0^{**}) = \lambda z^{***}(x_0^{**}) + x^*(x_0^{**}) > \alpha > 0.$$

Let us remind the reader that we also have by (33)

$$x^{***}(x_n) = x^*(x_n) < \beta \tag{35}$$

for every $n \in \mathbb{N}$.

Without loss of generality we may and do assume $\|x^{***}\| = 1$. Otherwise all former inequalities are adapted to the new situation by using $\|x^{***}\|^{-1}x^{***}$ instead of x^{***} any time. The only inequality which may require something more is again (32) which we get with another multiplication with a big enough positive scalar as above, if it should be necessary. Even more, once we have fixed the positive numbers n_0 and ρ_0 such that (30) is verified, we can fix a positive real number r with

$$0 < \sqrt{r} < \frac{1}{2n_0},$$
$$2\sqrt{r}\sup\{\|d\| : d \in D\} < \eta$$

where $\eta > 0$ is fixed so that $y_0^*(D) < -\eta < 0$, and

$$2\sqrt{r} < \rho_0,$$

then we can and do assume:

$$(r\alpha - r\beta) > \sqrt{r}|\langle x_0^{**}, y_0^*\rangle| \tag{36}$$

since this inequality is always possible, again with some $\xi' v_0$, with $\xi' > 1$ in case it should be necessary only. Summarising, after all possible points selected in the line $L := \{\lambda v_0 : \lambda \geq 1\}$ we finally set $u_0 \in L$ and replace v_0 by u_0 to be sure that all former conditions are satisfied with u_0 instead of v_0.

By Goldstine's theorem B_{E^*} is weak*, i.e. $\sigma(E^{***}, E^{**})$, dense in $B_{E^{***}}$. Therefore by the Mackey-Arens theorem it is also Mackey, i.e $\tau(E^{***}, E^{**})$, dense. We can find a sequence $x_n^* \in B_{E^*}$ such that

$$\text{for all } p \geq 1, \qquad \lim_{n\geq 1} x_n^*(x_p) = x^{***}(x_p) = x^*(x_p),$$

$$\lim_{n\geq 1} x_0^{**}(x_n^*) = x^{***}(x_0^{**}),$$

$$\lim_{n\geq 1} x_n^*(u_0) = x^{***}(u_0) = x^*(u_0),$$

$$\lim_{n\geq 1} x_n^*(w) = x^{***}(w) = x^*(w),$$

uniformly on $w \in W$, and

$$\lim_{n\geq 1}(x_n^*(d)) = x^{***}(d) = x^*(d)$$

uniformly on $d \in D$.

Indeed, for every positive integer n we deal with the $\tau(E^{***}, E^{**})$-continuous seminorm p_n of uniform convergence on $\{x_1, x_2, \cdots, x_n\} \cup \{x_0^{**}, u_0\} \cup D \cup W$, and we can require to find linear forms x_n^* such that:

$$p_n(x_n^* - x^{***}) < 1/2^n,$$

what means, for every $n \in \mathbb{N}$ the following inequalities:

$$|x_n^*(x_p) - x^{***}(x_p)| < 1/2^n \tag{37}$$

for all $p \leq n$

$$|(x_n^* - x^{***})(x_0^{**})| < 1/2^n \tag{38}$$

$$|(x_n^* - x^{***})(u_0)| < 1/2^n \tag{39}$$

$$|(x_n^* - x^{***})(w)| < 1/2^n \tag{40}$$

for every $w \in W$, and

$$|x_n^*(d) - x^{***}(d)| < 1/2^n \tag{41}$$

for every $d \in D$. Then we have:

$$\lim_{n\to+\infty} x_n^*(x_p) = \langle x^*, x_p\rangle \tag{42}$$

for every $p \in \mathbb{N}$,

$$\lim_{n\to+\infty} x_n^*(x_0^{**}) = \langle x^{***}, x_0^{**}\rangle \tag{43}$$

$$\lim_{n\to+\infty} x_n^*(u_0) = \langle x^*, u_0\rangle \tag{44}$$

$$\lim_{n\to+\infty} x_n^*(w) = \langle x^{***}, w\rangle \tag{45}$$

uniformly on $w \in W$ and

$$\lim_{n\to+\infty} x_n^*(d) = \langle x^*, d\rangle \tag{46}$$

uniformly on $d \in D$.

The inequalities above have some consequences which follow from (42) and (35):

$$\text{for every } p \geq 1, \text{ there is } n_p \text{ such that } \beta > x_n^*(x_p) \text{ for } n \geq n_p, \tag{47}$$

and without loss of generality we may assume that:

$$\langle x_0^{**}, x_n^*\rangle \geq \alpha, \tag{48}$$

by (43) Note that given any pointwise–cluster point x_0^* on E^* for the $\sigma(E^*, E)$-topology of the bounded sequence $\{x_n^*\}_{n\geq 1}$, we have that

$$\langle x_0^{**}, x_0^*\rangle \leq \beta, \tag{49}$$

because $x_0^{**} \in \overline{\{x_p : p \geq 1\}}^{\sigma(E^{**},E^*)}$ and for all $p \geq 1, x_0^*(x_p) \leq \beta$ by (47).

We now have all ingredients to find the non-attaining linear functionals following the *unbounded undetermined function procedure*, which come baks to Pryce and James,[9],[15], as well as Galan and Simons in [8], as we have seen in the former section, see Theorem 10 above. Indeed, we are going to work on $\mathbb{R}^{\sqrt{r}C}$ with the pointwise bounded sequence $\{\sqrt{r}x_n^* : n = 1, 2, \cdots\}$.

Proposition 12 provides us a subsequence

$$\{\sqrt{r}x_{n_k}^* : k = 1, 2, \cdots\}$$

such that, for all $h_0 \in \mathrm{co}_\sigma \{\sqrt{r}x_n^* : n \geq 1\}$, we have:

$$\sup_{\sqrt{r}C}\left(h_0 - \limsup_{k\geq 1}\sqrt{r}x_{n_k}^* + y_0^*\right) = \sup_{\sqrt{r}(C)}\left(h_0 - \liminf_{k\geq 1}\sqrt{r}x_{n_k}^* + y_0^*\right). \tag{50}$$

Let us now observe that for x_0^{**} we have by (48) that

$$\forall h_0 \in \mathrm{co}_\sigma\left\{\sqrt{r}x_{n_k}^* : k \geq 1\right\}, \qquad \sqrt{r} < x_0^{**}, h_0 > r\alpha. \tag{51}$$

Let us fix a $\sigma(E^*, E)$-cluster point x_0^* of the bounded sequence $\left\{x_{n_k}^* : k \geq 1\right\}$; then it follows that for all $c \in C$,

$$\limsup_{k\geq 1}\sqrt{r}x_{n_k}^*(c) \geq \sqrt{r}x_0^*(c) \geq \liminf_{k\geq 1}\sqrt{r}x_{n_k}^*(c)$$

and thus, for all $c \in C$,

$$\begin{aligned}
&\left(h_0(\sqrt{r}c) - \liminf_{k\geq 1}\sqrt{r}x_{n_k}^*(\sqrt{r}c) + y_0^*(\sqrt{r}c)\right)\\
&\geq \left(h_0 - \sqrt{r}x_0^* + y_0^*\right)(\sqrt{r}c)\\
&\geq \left(h_0(\sqrt{r}c) - \limsup_{k\geq 1}\sqrt{r}x_{n_k}^*(\sqrt{r}c) + y_0^*(\sqrt{r}c)\right).
\end{aligned}$$

Therefore, in view of (50) we deduce that for all $h_0 \in \mathrm{co}_\sigma \left\{ \sqrt{r} x^*_{n_k} : k \geq 1 \right\}$

$$\begin{aligned}
&\sup_{\sqrt{r}C} \left(h_0 - \limsup_{k\geq 1} \sqrt{r} x^*_{n_k} + y^*_0 \right) \\
&= \sup_{\sqrt{r}C} \left(h_0 - \liminf_{k\geq 1} \sqrt{r} x^*_{n_k} + y^*_0 \right) \\
&= \sup_{\sqrt{r}C} \left(h_0 - \sqrt{r} x^*_0 + y^*_0 \right).
\end{aligned}$$

Let us observe that for $h_0 \in \mathrm{co}_\sigma \left\{ \sqrt{r} x^*_{n_k} : k \geq 1 \right\}$ we have:

$$\begin{aligned}
&\sup_{\sqrt{r}C} \left(h_0 - \sqrt{r} x^*_0 + y^*_0 \right) \\
&= \sup_{\overline{\sqrt{r}C}^{w^*}} \left(h_0 - \sqrt{r} x^*_0 + y^*_0 \right) \\
&\geq \langle \sqrt{r} x^{**}_0, h_0 - \sqrt{r} x^*_0 + y^*_0 \rangle
\end{aligned}$$

and

$$\langle \sqrt{r} x^{**}_0, h_0 - \sqrt{r} x^*_0 + y^*_0 \rangle > r\alpha - r\beta + \sqrt{r} \langle x^{**}_0, y^*_0 \rangle > 0,$$

by (49), (51) and (36). Moreover we also have

$$\begin{aligned}
&\sup_{h_0 \in \mathrm{co}_\sigma \left\{ \sqrt{r} x^*_{n_k} : k \geq 1 \right\}} \sup_{c\in C} \langle \sqrt{r} c, h_0 - \limsup_{k} \sqrt{r} x^*_{n_k} + y^*_0 \rangle \\
= &\sup_{h_0 \in \mathrm{co}_\sigma \left\{ \sqrt{r} x^*_{n_k} : k \geq 1 \right\}} \sup_{c\in C} \langle \sqrt{r} c, h_0 - \liminf_{k} \sqrt{r} x^*_{n_k} + y^*_0 \rangle \\
= &\sup_{h_0 \in \mathrm{co}_\sigma \left\{ \sqrt{r} x^*_{n_k} : k \geq 1 \right\}} \sup_{c\in C} \langle \sqrt{r} c, h_0 - \sqrt{r} x^*_0 + y^*_0 \rangle < +\infty
\end{aligned}$$

because of our construction, indeed every one of the linear functionals

$$h_0 - \sqrt{r} x^*_0 + y^*_0 \in J^{n_0}$$

since $|h_0(x) - \sqrt{r} x^*_0(x)| \leq 2\sqrt{r}\|x\| < \rho_0$ for all $x \in B_E$ and we fixed ρ_0 so that $B_{\|\cdot\|}(y^*_0, \rho_0) \subset J^{n_0}$.

All needed to apply Theorem 10 is verified, with the only possible exception of the condition $\sup\{y^*_0(c) : c \in C\} \leq 0$ which we easily adjust too after a suitable translation of our set C fixed in the very beginning of our reasoning. Thus we get a sequence $\{g^*_i\}_{i\geq 1}$ with $g^*_i \in \mathrm{co}_\sigma\{\sqrt{r} x^*_{n_k} : k \geq i\}$ and $g^*_0 \in \mathrm{co}_\sigma \{g^*_i : i \geq 1\}$ such that for all $\tilde{g} \in \ell_\infty(C)$ with

$$\liminf_{i\geq 1} g^*_i \leq \tilde{g} \leq \limsup_{i\geq 1} g^*_i \text{ on } \sqrt{r}C$$

we have that

$$(g^*_0 - \tilde{g} + y^*_0) \text{ does not attain its supremum on } \sqrt{r}C.$$

Thus it does not attain its supremum on C whenever $\tilde{g} \in E^*$.

We know take care of the fact that

$$|(g^*_0 - \tilde{g})(d)| \leq 2\|d\|\sqrt{r} < \eta$$

for every $d \in D$, and thus

$$(y_0^* + (g_0^* - \tilde{g}))(D) < 0.$$

Let us observe that $\tilde{g}$ does coincide with $\sqrt{r}x^{***}$ on $W \subset E$ and that the elements g_i are σ-convex combinations of linear forms of the sequence $\{\sqrt{r}x_n^*\}$. By uniform convergence on W we may assume that

$$x^{***}(w) + \frac{\epsilon}{2\sqrt{r}} > x_n^*(w) > x^{***}(w) - \frac{\epsilon}{2\sqrt{r}}$$

for all $w \in W$ and every $n \in \mathbb{N}$, and finally

$$\sqrt{r}x^{***}(w) + \frac{\epsilon}{2} > g_i(w) > \sqrt{r}x^{***}(w) - \frac{\epsilon}{2}$$

for all $w \in W$ and every $i \in \mathbb{N}$, therefore

$$+\frac{\epsilon}{2} > (g_0^* - \tilde{g})(w) > -\frac{\epsilon}{2}$$

for every $w \in W$ and the proof is complete. In particular, for every $\sigma(E^*, E)$–cluster point of the sequence $\{g_i^*\}_{i\geq 1}$, let us say $\tilde{g}^*$, we have that $g_0^* - \tilde{g}^* + y_0^* \in E^*$, it does not attain its supremum on C and $g_0^* - \tilde{g}^* + y_0^* \in B_{pW}(y_0^*, \epsilon/2)$, thus $(y_0^* - \tilde{g}^* + g_0^*) \in B_{pW}(z_0^*, \epsilon) \subset U$ and it does not attains its finite supremum on C although

$$(y_0^* + \tilde{g}^* - g_0^*)(D) < 0$$

□

As an application we can get rid of the coercivity assumption we have in the former section for our function f of Theorem 13, see previous versions in [3, 11, 13, 14, 18]. Indeed, next result extends Saint Raymond's theorem[18] dealing with the case of $\partial f(E) = E^*$, firstly obtained in [14] for Banach spaces with weak*-sequentially compact dual unit ball:

Theorem 17. *Let $f : \mathbb{E} \to \mathbb{R} \cup \{\infty\}$ be a proper and bounded below function such that $\partial f(E)$ has non-empty interior in the Mackey topology $\tau(E^*, E)$, then for each set $c \in \mathbb{R}$ the sublevel set $f^{-1}((-\infty, c]$ is relatively weakly compact.*

Proof. First let us assume that our function f is convex. By translation of the graph we may and do assume that $f \geq 0$. Let us consider the epigraph of the given function f, i.e:

$$\text{epi}(f) := \{(x, t) \in E \times \mathbb{R} : f(x) \leq t\}$$

epi(f) is a subset of $E \times \mathbb{R}$ such that for every $y_0^* \in \partial f(E)$ there is $x_0 \in E$ such that $f(x_0) - f(y) \leq y_0^*(x_0 - y)$ for every $y \in E$ and thus

$$\langle (y_0^*, -1), (x_0, f(x_0)) \rangle \geq \langle (y_0^*, -1), (y, f(y)) \rangle \geq \langle (y_0^*, -1), (y, t) \rangle$$

whenever $(y, t) \in \text{epi}(f)$. Let us take C as the epigraph $Epi(f)$ in the Banach space $E \times \mathbb{R}$. We shall see that C is a subset satisfying the conditions of the former theorem. Let us consider the singleton $D := \{(0, -1)\}$, and we have that every linear form (z^*, λ) attains its supremum on C whenever $\langle (z^*, \lambda), (0, 1) \rangle = \lambda < 0$, thus we can apply Theorem 16 to finally see that C must be $weak^*$-closed in the bidual space $E^{**}\times\mathbb{R}$, from where the conclusion with the level sets clearly follows once we see that level sets are bounded, but this fact follows from the Uniform Bounded Principle once we have $\partial f(E)$ with non-empty interior in the Mackey topology $\tau(E^*, E)$. Indeed every $z^* \in E^*$ can be written as $z^* = z_0^* + \mu(v^* - z_0^*)$ where z_0^* and v^* are in $\partial f(E)$ and $\mu > 0$, so any level set of f is bounded above for both, from where the conclusion follows.

For the general case it is enough to consider the convex envelope $\hat{f}$:

$$\hat{f}(x) := \inf\left\{t \in \mathbb{R} : (x,t) \in \overline{\mathrm{co}\{(y,s) \in E \times \mathbb{R} : f(y) \leq s\}}^{\sigma(E\times\mathbb{R}, E^*\times\mathbb{R})}\right\}.$$

Since $\hat{f}$ is a proper, convex and lower semicontinuous function such that

$$\partial(\hat{f}) \supseteq \partial(f)$$

our former reasonning says that the level sets of $\hat{f}$, and a fortiori those of $f \geq \hat{f}$ are relatively weakly compact.

□

Remark 18. *To finish this section let us remark that Theorem 1 in the Introduction is also valid with a completely similar proof. Indeed, it is a reformulation of Theorem 16. We have followed this different formulation thinking in the direct application of their hypothesis to the epigraph of a function as described in Theorem 17 above.*

6. Some Applications

For a monetary concave utility function $u : \mathbb{L}^\infty \to \mathbb{R}$, its Fenchel-Legendre transform (or penalty function) is defined as

$$c : (\mathbb{L}^\infty)^* \to \mathbb{R}_+ \cup \{\infty\} \tag{52}$$

by

$$c(\mu) = \sup\{-\mu(\xi) + u(\xi) | \xi \in \mathbb{L}^\infty\} \tag{53}$$

As a straightforward application of our former results we can involve the Mackey topology in $\tau(\mathbb{L}^\infty, \mathbb{L}^1)$ in Jouini-Schachermayer-Touzi result characterizing the Lebesgue property of risk measures, see Theorem 24 in [4] and Theorem 5.2 in [10]:

Theorem 19. *For a concave monetary utility function $u : \mathbb{L}^\infty \to \mathbb{R}$ with the Fatou property, the following are equivalent:*

(i) *u satisfies the property $\lim_n u(\xi_n) = u(\xi)$ for uniformly bounded sequences $((\xi_n)_{n\in\mathbb{N}}$, converging in probability to a random variable ξ.*
(ii) *The convex function c satisfies: for each $0 \leq \alpha < \infty$, $\{\mathbb{Q} | c(\mathbb{Q}) \leq \alpha\}$ is weakly compact in $\mathbb{L}^1$.*
(iii) *For each $\xi \in \mathbb{L}^\infty$ there is a probability $\mathbb{Q}$ so that $u(\xi) = \mathbb{E}_\mathbb{Q}[\xi] + c(\mathbb{Q})$.*
(iv) *The set*

$$\{\xi \in \mathbb{L}^\infty | \text{ there is a probability } \mathbb{Q} \text{ so that } u(\xi) = \mathbb{E}_\mathbb{Q}[\xi] + c(\mathbb{Q})\}$$

contains a non-void open set for the Mackey topology $\tau(\mathbb{L}^\infty, \mathbb{L}^1)$.

Remark 20. All applications considered in [13] can be extended using the hypothesis of non void interior of the subdifferential range in the Mackey topology, instead of the onto of subdifferential required in [13].

Among them we select the following one:

Theorem 21. *Let E be a real Banach space and let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function such that* $\mathrm{dom}(f)$ *has nonempty norm–interior and for all $x^* \in U$ there exists $x_0 \in E$ with*

$$f(x_0) - x^*(x_0) = \inf_{x \in E} \left(f(x) - x^*(x) \right),$$

where U is a non void $\tau(E^*, E)$-open set. Then E is reflexive.

Proof. Let B be a nonempty open ball contained in $\mathrm{dom}(f)$. Then we have that

$$B = \bigcup_{p=1}^{+\infty} B \cap \overline{f^{-1}((-\infty, p])}^{\sigma(E,E^*)}.$$

We can apply the Baire Category theorem to the open set B to get an integer $p \geq 1$ such that $B \cap \overline{f^{-1}((-\infty, p])}^{\sigma(E,E^*)}$ has an interior point relative to B, so that there is an open set G in E such that $\emptyset \neq B \cap G \subset B \cap \overline{f^{-1}((-\infty, p])}^{\sigma(E,E^*)}$ and thus $\emptyset \neq G \cap B \subset \overline{f^{-1}((-\infty, p])}^{\sigma(E,E^*)}$. But $\overline{f^{-1}((-\infty, p])}^{\sigma(E,E^*)}$ is weakly compact by Theorem 13, therefore, we have a closed ball of positive radius which is weakly compact and the space must be reflexive.

□

The same happens with the application for risk measures on Orlicz spaces, see[14]:

Let us recall that a *Young function* Ψ is an even, convex function $\Psi : E \to [0, +\infty]$ with the properties:

(i) $\Psi(0) = 0$.
(ii) $\lim_{x \to \infty} \Psi(x) = +\infty$.
(iii) $\Psi < +\infty$ in a neighborhood of 0.

The Orlicz space $\mathbb{L}^\Psi$ is defined as:

$\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}) := \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$: there exixts $\alpha > 0$ with $\mathbb{E}_\mathbb{P}[\Psi(\alpha X)] < +\infty\}$,

and we consider the Luxemburg norm on it:

$$N_\Psi(X) := \inf\{c > 0 : \mathbb{E}_\mathbb{P}[\Psi(\frac{1}{c}X)] \leq 1\}, \qquad (X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})).$$

With the usual pointwise lattice operations, $\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$ is a Banach lattice and we have the inclusions:

$$\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}) \subset \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}).$$

Moreover, $(\mathbb{L}^\Psi)^* = \mathbb{L}^{\Psi^*} \oplus G$ where G is the singular band and $\mathbb{L}^{\Psi^*}$ is the order continuous band identified with the Orlicz space $\mathbb{L}^{\Psi^*}$, where

$$\Psi^*(y) := \sup_{x \in \mathbb{R}} \{yx - \Psi(x)\}$$

is the Young function conjugate to Ψ, The first named author showed that a risk measure defined on $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ finitely extends to an Orlicz space if, and only if, it is a Lebesgue measure, [4, Section 4.16].

The Orlicz heart $\mathbb{M}^\Psi$ is the Morse subspace of all $X \in L^\Psi$ such that for every $\beta > 0$

$$\mathbb{E}_\mathbb{P}[\Psi(\beta X)] < +\infty.$$

We arrive at the following, see Theorem 1 in [14]:

Theorem 22. *Let Ψ be a Young function with finite conjugate Ψ^* and*

$$\alpha : (\mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}))^* \to \mathbb{R} \cup \{+\infty\}$$

be a $\sigma((\mathbb{L}^\Psi)^*, \mathbb{L}^\Psi)$-lower semicontinuous penalty function representing a finite monetary risk measure ρ as

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}.$$

The following statements are equivalent:

(i) *For each $c \in \mathbb{R}$, $\alpha^{-1}((-\infty, c])$ is a weakly compact subset of $\mathbb{M}^{\Psi^*}(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For every $X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P})$, the supremum in the equality*

$$\rho(X) = \sup_{Y \in \mathbb{M}^{\Psi^*}} \{-\mathbb{E}[XY] - \alpha(Y)\}$$

is attained.

(iii) *ρ is order sequentially continuous.*

(iv) *The set*

$$\{X \in \mathbb{L}^\Psi(\Omega, \mathcal{F}, \mathbb{P}) : \text{ there is } Y \in \mathbb{M}^{\Psi^*} : \rho(X) = \{-\mathbb{E}[XY] - \alpha(Y)\}\}$$

has non-empty interior in the Mackey topology $\tau(\mathbb{L}^\Psi, \mathbb{M}^{\Psi^*})$

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