REMARK ON THE PAPER "ENTROPIC VALUE-AT-RISK: A NEW COHERENT RISK MEASURE" BY AMIR AHMADI-JAVID, J. OPT. THEORY AND APPL., 155 (2001),1105–1123

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Abstract. The paper mentioned in the title introduces the entropic value at risk. I give some extra comments and using the general theory make a relation with some commonotone risk measures.

1. Introduction

In [1], see also [2] for corrections and precisions, Amir Ahmadi-Javid introduced the Entropic Value at Risk (EVAR). He also relates this risk measure to the CVaR or expected shortfall measure and gives some generalisations. In this note I make some remarks, relating the EVAR to some other commonotone risk measure. In particular I show that there is a best commonotone risk measure that is dominated by EVAR. The presentation uses the concept of monetary utility functions, which has the advantage that it has a better economic interpretation. Up to some sign changes it is the same as in the papers of [1]. The general theory of such coherent utility functions was introduced in [3], [4], [5]. For a more complete presentation I refer to [6] and [7].

We make the standing assumption that all random variables are defined on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a level $\alpha, 0 < \alpha < 1$, the Entropic Value at Risk of a bounded random variable, ξ, is defined as

$$e_\alpha(\xi) = \sup_{0<z} \left(-\frac{1}{z} \log \mathbb{E}\left[\frac{\exp(-z\xi)}{\alpha}\right]\right).$$

The reader will — already said above — notice that this is up to sign changes, the definition of [1]. The entropic value at risk is the supremum of entropic utility functions perturbed by the introduction of the level α. For $\alpha = 1$ we would find the $\mathbb{E}_\mathbb{P}[\xi]$. For $\alpha \to 0$, we get ess.inf ξ. The mapping

$$e_\alpha \colon L^\infty \to \mathbb{R},$$

defines a coherent utility function, see [1]. Using duality arguments, it is shown there that

$$e_\alpha(\xi) = \inf \left\{ \mathbb{E}_\mathbb{Q}[\xi] \mid H(\mathbb{Q}|\mathbb{P}) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \le -\log(\alpha) \right\}.$$

The proof of the equality uses a standard duality argument (see the last line on page 1111 of the paper) but also needs the (not mentioned) property that the sets $\{\mathbb{Q} \mid H(\mathbb{Q} \mid \mathbb{P}) \le \beta\}$ $(\beta \ge 0)$ are weakly compact in L^1. For later use let us denote

$$\mathcal{S} = \{\mathbb{Q} \mid H(\mathbb{Q} \mid \mathbb{P}) \le -\log(\alpha)\}.$$

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Because the proof in [1] seems incomplete we will give an alternative and in fact easier approach. Let us remark that the coherence of e_α is easily proved as seen in [1]. Since it only depends on the law of the random variable the results of [8] show that it has the Fatou property and hence it is given by a scenario set of probability measures, see [5]. This set is described as

$$\{\mathbb{Q} \mid \text{ for all } \xi : \mathbb{E}_{\mathbb{Q}}[\xi] \geq e_\alpha(\xi)\}\,.$$

The required inequality is equivalent to: for all ξ and for all $z > 0$ we have

$$\mathbb{E}_{\mathbb{Q}}[\xi] \geq \left(-\frac{1}{z}\log \mathbb{E}\left[\frac{\exp(-z\xi)}{\alpha}\right]\right),$$

which can be rewritten as:

$$\mathbb{E}_{\mathbb{Q}}[z\xi] \geq \left(-\log \mathbb{E}\left[\frac{\exp(-z\xi)}{\alpha}\right]\right).$$

Because the inequality has to be satisfied for all $\xi \in L^\infty$, it is also equivalent to: for all $\xi \in L^\infty$ we must have

$$\mathbb{E}_{\mathbb{Q}}[\xi] \geq \left(-\log \mathbb{E}\left[\frac{\exp(-\xi)}{\alpha}\right]\right).$$

In particular, if we write $f = \frac{d\mathbb{Q}}{d\mathbb{P}}$ then for all $n \in \mathbb{N}$ and $\epsilon > 0$ it has to be satisfied by the choice $\xi = -\log(\min(n, f+\epsilon)) \in L^\infty$. This yields:

$$-\mathbb{E}[f\log(\min(n, f+\epsilon))] \geq -\log\left(\mathbb{E}[\min(n, f+\epsilon)]\right) + \log\alpha.$$

Now we let $n \to +\infty$. On the left we use Beppo Levi's theorem and on the right we can use the dominated convergence theorem. This gives:

$$\mathbb{E}\left[f\log(f+\epsilon)\right] \leq \log(1+\epsilon) - \log\alpha.$$

This inequality already shows that $\mathbb{E}[f\log(f+1)] < \infty$ and an application of the dominated convergence theorem (or just Fatou's lemma) yields the result $\mathbb{E}[f\log(f)] \leq -\log\alpha$. Conversely if $\mathbb{E}[f\log(f)] \leq -\log\alpha$ then the calculation of the entropic risk measure yields:

$$-\log\mathbb{E}\left[e^{-\xi}\right] \leq \mathbb{E}[f\xi] + \mathbb{E}[f\log(f)] \leq \mathbb{E}_{\mathbb{Q}}[\xi] - \log\alpha.$$

This shows that

$$\mathcal{S} = \{\mathbb{Q} \mid H(\mathbb{Q} \mid \mathbb{P}) \leq -\log(\alpha)\}.$$

It is now clear that $\mathcal{S}$ is weakly compact (by de la Vallée-Poussin's theorem) and hence e_α has the Lebesgue property. This means that if ξ_n is a uniformly bounded sequence of random variables converging in probability to ξ then $e_\alpha(\xi_n) \to e_\alpha(\xi)$.

In the next sections we will calculate the values for indicators and using these we will prove that e_α is not commonotone. We will show that e_α is between two commonotone utility functions.

2. The value for indicators

In case $\xi = \mathbf{1}_A$ is the indicator of a set $A \in \mathcal{F}$, we can more or less calculate the value $e_\alpha(\mathbf{1}_A)$. Because the function $x\log(x)$ is convex, the entropy decreases if we replace $\mathbb{Q}$ by the probability given by the density $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \mid A, A^c\right]$. Because the space is atomless, it is easily seen that the value

$$e_\alpha(\mathbf{1}_A) = \inf\left\{\mathbb{E}_\mathbb{Q}[\mathbf{1}_A] \mid H(\mathbb{Q}|\mathbb{P}) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] \le -\log(\alpha)\right\},$$

is given by a minimum attained by a function of the form ($a = \mathbb{P}[A]$):

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \lambda\frac{\mathbf{1}_A}{a} + (1-\lambda)\frac{\mathbf{1}_{A^c}}{1-a}.$$

In this case the value $\mathbb{E}_\mathbb{Q}[\mathbf{1}_A]$ is simply λ. The entropy of such a measure is given by

$$F(\lambda, a) = \lambda\log(\lambda) + (1-\lambda)\log(1-\lambda) - \lambda\log(a) - (1-\lambda)\log(1-a).$$

The function F takes a unique minimum equal to zero at $\lambda = a$ and for $\lambda = 0$ it gives the value $-\log(1-a)$, whereas for $\lambda = 1$ we get $-\log(a)$. The derivatives of F satisfy:

$$\frac{\partial F}{\partial\lambda} = \log\left(\frac{\lambda}{1-\lambda}\right) - \log\left(\frac{a}{1-a}\right) < 0 \qquad \frac{\partial F}{\partial a} = \frac{a-\lambda}{a(1-a)} > 0$$

$$\frac{\partial^2 F}{\partial\lambda^2} = \frac{1}{\lambda} + \frac{1}{1-\lambda}; \quad \frac{\partial^2 F}{\partial\lambda\partial a} = -\frac{1}{a} - \frac{1}{1-a}; \quad \frac{\partial^2 F}{\partial a^2} = \frac{\lambda}{a^2} + \frac{1-\lambda}{(1-a)^2}$$

for $0 < \lambda < a$, $1-\alpha < a < 1$. It follows easily that for $a \le (1-\alpha)$, the minimum in the expression for $e_\alpha(\mathbf{1}_A)$ is attained for $\lambda = 0$ and consequently $e_\alpha(\mathbf{1}_A) = 0$. For $1 > a > 1-\alpha$ there is one solution of the equation

$$F(\lambda, a) = \lambda\log(\lambda) + (1-\lambda)\log(1-\lambda) - \lambda\log(a) - (1-\lambda)\log(1-a) = -\log(\alpha)$$

that is smaller than a. The function $\Lambda(a) = e_\alpha(\mathbf{1}_A) < a$ is well defined as an implicit function. From the definition of e_α, it is already seen that $\Lambda(a)$ must be increasing but the implicit function theorem gives that $\frac{d\Lambda}{da} > 0$ on $1-\alpha < a < 1$. For $a \to 1$, the derivative tends to $+\infty$. The Hessian of F is positive definite and hence on $1-\alpha < a < 1, 0 < \lambda < a$, the function F is strictly convex. If we take the derivative of $\frac{\partial F}{\partial a} + \frac{\partial F}{\partial\lambda}\frac{d\Lambda}{da} = 0$ with respect to a we find at the points $(a, \Lambda(a))$ that

$$\frac{\partial^2 F}{\partial a^2} + 2\frac{\partial^2 F}{\partial\lambda\partial a}\frac{d\Lambda}{da} + \frac{\partial^2 F}{\partial\lambda^2}\left(\frac{d\Lambda}{da}\right)^2 = -\frac{\partial F}{\partial\lambda}\frac{d^2\Lambda}{da^2}.$$

Since the Hessian of F is positive definite the left hand side is always positive. Hence we must have $\frac{d^2\Lambda}{da^2} > 0$ on $1-\alpha < a < 1, 0 < \lambda < a$. The function Λ is therefore strictly convex on $1-\alpha \le a \le 1$. Summarising $\Lambda\colon [0,1] \to [0,1]$ is convex, $\Lambda(0) = 0, \Lambda(1) = 1$ and $\Lambda(a) = 0$ for $0 \le a \le 1-\alpha$. The latter shows that the utility function e_α is not strictly monotone.

3. Relation with commonotone utilities

For a convex function $f\colon [0,1] \to [0,1]$ with $f(0)=0, f(1)=1$ we can define a commonotone utility. The relation with convex games and non-additive expectations is well known, see [11], [7], [6]. The basic ingredient is the scenario set defined as

$$\mathcal{S}_f = \{\mathbb{Q} \mid \text{ for all } A \in \mathcal{F} : \mathbb{Q}[A] \geq f(\mathbb{P}[A])\}.$$

With this we associate the utility

$$u_f(\xi) = \inf\{\mathbb{E}_{\mathbb{Q}}[\xi] \mid \mathbb{Q} \in \mathcal{S}_f\}.$$

We immediately get $\mathcal{S} \subset \mathcal{S}_\Lambda$ and $\mathcal{S}_c \subset \mathcal{S}$, where c denotes the convex function defined as $c(x) = 0$ for $x \leq 1-\alpha$ and $c(x) = \frac{x-1+\alpha}{\alpha}$ for $1-\alpha \leq x \leq 1$. The latter defines the utility function known as CVAR at level α, see [6] or [7]. We get that

$$CVAR_\alpha \geq e_\alpha \geq u_\Lambda.$$

The calculation of u_Λ can be done using Ryff's theorem (see [10], [7]). We get:

$$u_\Lambda(\xi) = \int_0^1 q_{1-a}\frac{d\Lambda}{da}da = \int_{1-\alpha}^1 q_{1-a}\frac{d\Lambda}{da}da,$$

where q is a quantile function of ξ.

For indicators $\mathbf{1}_A$ we have $u_\Lambda(\mathbf{1}_A) = e_\alpha(\mathbf{1}_A)$. Therefore (see the lemma below) we get that u_Λ is the greatest commonotone utility that is dominated by e_α. The utility e_α is not commonotone. Indeed if it were, then for $1-\alpha < a < b < 1$, $A \subset B$, $a = \mathbb{P}[A], b = \mathbb{P}[B]$ we would find an element of $\mathcal{S}$, say $\mathbb{Q}$ such that $\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A] = \Lambda(a), \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_B] = \Lambda(b)$. However the minimisers for indicators have a special form and this would imply that $\frac{\Lambda(a)}{a} = \frac{\Lambda(b)}{b}$ as well as $\frac{\Lambda(b)}{b} = \frac{1-\Lambda(a)}{1-a}$. The only solution would then be $\Lambda(a) = a$, a contradiction. The reader can check that for $\xi = \mathbf{1}_A + \mathbf{1}_B$ we have that $e_\alpha(\xi) > u_\Lambda(\xi)$. With some extra effort the reader can also see that for random variables ξ such that the distribution has a support of more than two points, we have the same strict inequality.

Lemma 1. *Let u be a coherent utility function. Suppose that $v(A) = u(\mathbf{1}_A)$ defines a convex game. Let w be the coherent utility function defined by v. Then $w \leq u$ and w is the largest commonotone utility function that is smaller than u.*

Proof For $\xi \geq 0$ the function w takes the value given by the Choquet integral

$$w(\xi) = \int_0^\infty v(\xi \geq x)\, dx.$$

The scenario set of w is given as the core of the game v:

$$\mathcal{S}_w = \{\mathbb{Q} \mid \text{ for all } A \in \mathcal{F} : \mathbb{Q}[A] \geq v(A) = u(\mathbf{1}_A)\}.$$

Clearly the scenario set, $\mathcal{S}_u$ of u is smaller since it is defined as:

$$\mathcal{S}_u = \{\mathbb{Q} \mid \text{ for all } \xi \in L^\infty : \mathbb{E}_{\mathbb{Q}}[\xi] \geq u(\xi)\} \subset \mathcal{S}_w.$$

This shows that $w \leq u$. Suppose now that w' is a commonotone utility that is smaller than u. We then have for $\xi \geq 0$ (and hence for all $\xi \in L^\infty$):

$$w'(\xi) = \int_0^\infty w'(\mathbf{1}_{\{\xi \geq x\}})\, dx \leq \int_0^\infty u(\mathbf{1}_{\{\xi \geq x\}})\, dx = w(\xi).$$

Remark 1. One could ask whether the same conclusion holds when u is only a concave monetary utility function. This is wrong as one can see by looking at the entropic risk measure. We leave it as an exercise.

4. The Kusuoka Representation

In [9], Kusuoka proved that coherent utilities that only depend on the law of the random variables are given by a convex set K of probabilities on $[0,1]$. They are in fact averages of CVAR utilities:

$$u(\xi) = \inf\left\{\int_{[0,1]} CVAR_x(\xi)\,\nu(dx) \mid \nu\in K\right\}.$$

To find the set K needed to represent e_α, we proceed as in [7]. Unfortunately besides a transformation of the problem we cannot give a more explicit analytic form for the set K. The first step is to find decreasing probability densities η on $[0,1]$ such that $\int_0^1 \eta(x)\log(\eta(x))\,dx \le -\log(\alpha)$. Then we write each η in the form $\eta(x)=\int_{(x,1]}\frac{1}{a}\,\nu(da)$ where ν is a probability on $[0,1]$. The set K is then the set of the measures ν so obtained. As shown in [7] the set K is a weak* compact convex set of probability measures on $(0,1]$. For more information the reader could also check Remark 41 on page 95 and exercise 18 in the same book. Somewhat more elegant descriptions can be given but essentially no better characterisation was found. I do not pursue this direction here.

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