THE $L^{(u)}$ SPACE FOR A MONETARY UTILITY FUNCTION u

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Abstract. In their paper, [?], Cheridito and Li discussed when a monetary utility function u can be extended to an Orlicz space. We introduce a more abstract way and slightly expend these results.

1. Introduction and notation

Monetary utility functions are defined on an L^∞ space and the extension to larger spaces is related to their continuity for norms that are weaker than the L^∞ norm. The question arises when we can define a natural norm for which the monetary utility function is continuous. The extension problem was first studied by Cheridito and Li in their paper [?]. Here we follow a more abstract version.

The notation we use is standard. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is supposed to be atomless although most of the results do not require this property. The assumption will be used without further mentioning. All random variables will be defined on Ω. The space L^∞ is the space of bounded random variables, modulo equality a.s. . The norm is denoted by $\|\xi\|_\infty = \text{ess.inf}\,\xi = \min\{a \mid |\xi| \le a \text{ a.s. }\}$. The space L^1 is the space of integrable random variables. The space L^0 is the space of all random variables, usually equipped with the topology of convergence in probability. A set, D, of random variables is called rearrangement invariant if $\xi \in D$, η and ξ have the same distribution or law, imply that also $\eta \in D$. A Young function is a function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ that is convex, $\Psi(0) = 0$ and increasing (we use the word increasing also in the meaning of nondecreasing) and satisfies $\lim_{x\to+\infty} \frac{\Psi(x)}{x} = \infty$. The conjugate function Φ is another Young function, defined on $\mathbb{R}_+$as $\Phi(y) = \sup_{x\in\mathbb{R}_+} (xy - \Psi(x))$. With a Young function Φ we associate the Orlicz space:

$$L^\Phi = \{\xi \mid \text{ there exists } \lambda > 0 \text{ with } \mathbb{E}\left[\Phi\left(\lambda|\xi|\right)\right] < \infty\}.$$

On L^Φ we put the Luxemburg norm

$$\|\xi\|_\Phi = \inf\left\{\lambda > 0 \mid \mathbb{E}\left[\Phi\left(\frac{|\xi|}{\lambda}\right)\right] \le 1\right\}.$$

In general the space L^∞ is not dense in L^Φ. The closure of L^∞ in L^Φ is denoted $L^{(\Phi)}$ and is called the Orlicz heart. We have

$$L^{(\Phi)} = \{\xi \mid \text{ for all } \lambda > 0 \text{ we have } \mathbb{E}\left[\Phi\left(\lambda|\xi|\right)\right] < \infty\}.$$

The dual space of $L^{(\Phi)}$ is L^Ψ but the dual of L^Φ is not necessarily L^Ψ, it may contain so called singular elements. For more information on Orlicz spaces we refer to [?].

Definition 1. A function $u\colon L^\infty \to \mathbb{R}$ is called a monetary utility function if

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(1) $u(0)=0$ and $u(\xi)\geq 0$ whenever $\xi\geq 0 a.s.$,
(2) u is concave,
(3) if $a\in\mathbb{R}$ then $u(\xi+a)=u(\xi)+a$,
(4) if ξ_n is a uniformly bounded sequence such that $\xi_n\to\xi$ in probability, then $u(\xi)\geq \limsup_n u(\xi_n)$.

The concavity of u implies $-u(-\xi)\geq u(\xi)$, hence $-u(-|\xi|)\geq u(|\xi|)$. Property 4 in the definition is equivalent to: if $\xi_n\downarrow\xi$ then $u(\xi_n)\downarrow u(\xi)$. The same property for increasing sequences is strictly stronger as we shall see later. For more information on monetary utility functions we refer to [5]. The concave function u is upper semi continuous for the weak* topology $\sigma(L^\infty,L^1)$. As Föllmer and Schied, see [?] proved, it can be described by its Fenchel-Moreau-Legendre conjugate. This function, denoted by c, has a domain included in the set $\mathcal{P}$ of probability measures absolutely continuous with respect to $\mathbb{P}$:

$$c(\mathbb{Q})=\sup\{\mathbb{E}_{\mathbb{Q}}[-\xi]\mid u(\xi)\geq 0\}.$$

The function $c\colon\mathcal{P}\to\mathbb{R}_+\cup\{+\infty\}$ is convex, lower semi continuous, $\inf_{\mathbb{Q}\in\mathcal{P}}c(\mathbb{Q})=0$ and

$$u(\xi)=\inf\left\{\mathbb{E}_{\mathbb{Q}}[\xi]+c(\mathbb{Q})\mid\mathbb{Q}\in\mathcal{P}\right\}.$$

The famous theorem of R. James allows to prove the following result, [5]

Theorem 1. *For a monetary utility function u, the following are equivalent:*

(1) *if ξ_n is a uniformly bounded sequence such that $\xi_n\to\xi$ in probability, then $u(\xi)=\lim_n u(\xi_n)$,*
(2) *for each $\xi\in L^\infty$: $u(\xi)=\min\left\{\mathbb{E}_{\mathbb{Q}}[\xi]+c(\mathbb{Q})\mid\mathbb{Q}\in\mathcal{P}\right\}$,*
(3) *for every $\infty>m\geq 0$ the set $\{\mathbb{Q}\mid c(\mathbb{Q})\leq m\}$ is weakly compact in L^1.*

Cheridito and Li showed that the above properties are also equivalent to the statement that u can be extended continuously to a locally Lipschitz function defined on $L^{(\Phi)}$. This result also can be obtained as follows.

The equivalent properties of the previous theorem mean that u when restricted to balls of L^∞ is continuous for the Mackey topology. The results of Delbaen-Pennanen then show that u is continuous for the Mackey topology on L^∞. The theorem of de la Vallée-Poussin shows that the Mackey topology is the initial topology of the mappings $L^\infty\to L^\Phi$, where Φ are Young functions. It follows that there exists a Young function Φ such that u is bounded below on a ball for the Orlicz norm $\|.\|_\Phi$. General results on convex functions then show that u is locally Lipschitz at every point of L^∞.

**** cannot be done by general topology, more is needed.

This locally Lipschitz property allows to extend the utility function u to $L^{(\Phi)}$, where it remains locally Lipschitz. So a lot can be obtained using the general theory of convex functions. The basic ingredients are the identification of the Mackey topology on balls as the convergence in probability and the de la Vallée-Poussin theorem to get the relation between the Mackey topology and Orlicz spaces.

We will give another proof of this theorem as a consequence of the approach in section 4.

2. The norm $\|.\|_u$

If we want to find a norm on L^∞ such that u is continuous for it, we should look at the set

$$B=\{\xi\mid u(-|\xi|)\geq -1\},$$

and ask that this set contains a multiple of the unit ball of this new norm. So why not try the set B itself? Because B is convex, absorbing and balanced, it can be used as a unit ball for a new norm on L^∞. The condition $u(-|\xi|) \ge -1$ is equivalent to the statement

$$\text{for all } \mathbb{Q} \in \mathcal{P} : \mathbb{E}_{\mathbb{Q}}[|\xi|] \le 1 + c(\mathbb{Q}).$$

The set $\{\xi \mid u(-|\xi|) \ge -1\}$ is the polar set of

$$S' = \left\{ \eta \mid \text{ there is } \mathbb{Q} \in \mathcal{P} \text{ with } |\eta| \le \frac{\frac{d\mathbb{Q}}{d\mathbb{P}}}{1 + c(\mathbb{Q})} \right\}.$$

The set S' is not necessarily convex and hence we will replace it by its closed convex hull S. This does not change the polar set. The Minkowski functional of B defines a norm on L^∞:

$$\|\xi\|_u = \inf\{\lambda \mid \lambda \ge 0, \xi \in \lambda B\} = \sup_{\eta \in S} \mathbb{E}[\eta\xi].$$

3. The space L^u

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