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An Interest Rate Model with Upper and Lower Bounds

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Abstract: We propose a new interest rate dynamics model where the interest rates fluctuate in a bounded region. The model is characterised by five parameters which are sufficiently flexible to reflect the prediction of the future interest rates distribution. The interest rate converges in law to a Beta distribution and has transition probabilities which are represented by a series of Jacobi polynomials. We derive the moment evaluation formula of the interest rate. We also derive the arbitrage free pure discount bond price formula by a weighted series of Jacobi polynomials. Furthermore we give simple lower and upper bounds for the arbitrage free discount bond price which are tight for the narrow interest rates region case. Finally we show that the numerical evaluation procedure converges to the exact value in the limit and evaluate the accuracy of the approximation formulas for the discount bond prices.

Keywords: Bounded state space, Beta distribution, Jacobi polynomials, Discount bond price, Lower and Upper Bounds, Numerical computation.

1 Introduction

In this paper, we study a new interest rate dynamics model which is defined by the following Markov type stochastic differential equation :

$$dr_t = \alpha(r_\mu - r_t)dt + \beta\sqrt{(r_t - r_m)(r_M - r_t)}dW_t. \tag{1.1}$$

$\boldsymbol{W} = \{W_t; 0 \leq t\}$ is a standard Wiener process on the probability space $(\Omega, \mathcal{F}, P)$ equiped with the natural filteration $\{\mathcal{F}_t; 0 \leq t\}$ generated by $\boldsymbol{W}$, i.e. $\{\mathcal{F}_t; 0 \leq t\}$ satisfies the usual assumptions. We assume that α, $\beta > 0$ and $r_m < r_\mu < r_M$ which will garantee the existence of stationary distribution. This type of diffusions (called Jacobi processes) have been well studied in genetics (see Ethier-Kurtz [3] and Karlin-Taylor [5]). Also Warren-Yor [8] studied the law of functionals of Jacobi process through Girsanov and time-change techniques. Here we use this model to study the interest rate process $\{r_t\}$ which fluctuates in the bounded region $[r_m, r_M]$. The same model was studied by Fujita, see [4]. The methods used are (of course) related but also at the same time different. Fujita essetially uses the perturbation method whereas we use a spectral method. In our model, the parameter α reflects the speed of reversion to the longrun mean r_μ and β represents the scale of variance caused by the random process $\boldsymbol{W}$. Let $\sigma(x) = \beta\sqrt{(x - r_m)(r_M - x)}$. Then for any x, $y \in [r_m, r_M]$,

$$|\sigma^2(x) - \sigma^2(y)| = \beta|r_m + r_M - (x + y)||x - y| \leq \beta(r_M - r_m)|x - y|.$$

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This means that the diffusion coefficient function $\sigma(x)$ is Hölder $\frac{1}{2}$ continuous. Since the drift function $\mu(x) = \alpha(r_\mu - x)$ is Lipschitz continuous, the pathwise uniqueness of the stochastic differential equation (1.1) is guaranteed by the general uniqueness theorem ([6], Theorem 4.5, p.360). The object of this paper is to evaluate the arbitrage free discount bond price for the Jacobi-type interest rate processes. The use of such models could be questionable. In particular it is not clear how the upper and lower bounds, r_M and r_m can be fixed or derived from data. However in risk management of portfolios of interest related instruments, we might be interested in the influence of high and low interest rates. As an example we could ask whether the value (under a certain model) of the position is due to the probability that the interest rate reaches a high level. In such a case we might want to evaluate the value under a model that does not allow for such higher values of the interest rate. In risk management it is quite normal to analyse the value of a portfolio under alternative models. The model that we describe leads to tractable expressions and therefore might be of some use in practice. By carefully adapting the coefficient β we can see the CIR-model, [2] as a limit of the present model. More precisely one can easily show that if we put $r_m = 0$, if we take β_M so that $\beta_M r_M^{1/2} = \sigma_0$ and if $r_M \to +\infty$, the law of the process (1.1) tends to the CIR-model.

Let's consider the following variable transformation :

$$z_t = \frac{r_t - r_m}{r_M - r_m}. \tag{1.2}$$

Then z_t follows

$$dz_t = \alpha(\gamma - z_t)dt + \beta\sqrt{z_t(1 - z_t)}dW_t, \tag{1.3}$$

where $\gamma = \frac{r_\mu - r_m}{r_M - r_m}$. Equation (1.3) is a special case of equation (1.1). However we can derive the properties of $\{r_t; t \geq 0\}$ from those of $\{z_t; t \geq 0\}$ through the inverse transformation $r_t = r_m + (r_M - r_m)z_t$. Hence we shall consider (1.3) to study the equation (1.1).

This paper is organized as follows. In Section 2, we summarize the basic properties of the process $\{r_t\}$ defined by (1.1), e.g. the hitting probability of boundaries, transition probability density and the moment evaluation. In Section 3, we study the arbitrage free pure discount bond price and give a spectral representation by a weighted series of Jacobi polynomials. Then we consider the approximate evaluation of the bond prices by the trancated sum of a series representation and we derive lower and upper bounds in Section 4. Finally in Section 5, we show that the numerical computation scheme which enables us to evaluate the exact bond prices, converges.

2 Basic Properties of Model

2.1 Hitting Probability

First we derive the hitting probability for the boundaries 0 or 1. Suppose that the process is stopped when it hits the boundaries 0 or 1. Let τ_y be the stopping time :

$$\tau_y = \inf\{t \geq 0; z_t = y\}, \quad 0 \leq y \leq 1. \tag{2.1}$$

Warren-Yor [8] have already derived the Laplace transform of τ_y using the Jacobi process representation by Bessel processes. Here we consider the standard argument to get the hitting probabilities in simple and easy way. Let $\rho_{x,y}$ be the probability that z_t hits y in finite time when it starts from x,

$$\rho_{x,y} = P[\tau_y < \infty | Z_0 = x]. \tag{2.2}$$

Then by the general result for one dimensional diffusions, $\rho_{x,y}$ is given by (see e.g., Proposition 9.4, [1, p.419]) :

$$\rho_{x,0} = \lim_{y \to 0, z \to 0} \frac{B_{x,z}(p,q)}{B_{y,z}(p,q)}, \tag{2.3}$$

$$\rho_{x,1} = \lim_{y\to 0, z\to 0} \frac{B_{y,x}(p,q)}{B_{y,z}(p,q)}, \tag{2.4}$$

where

$$\left\{ \begin{array}{rcl} p &=& 1-\frac{2\alpha\gamma}{\beta^2} \\ q &=& 1-\frac{2\alpha(1-\gamma)}{\beta^2} \\ B_{x,y}(u,v) &=& \int_x^y z^{u-1}(1-z)^{v-1}dz. \end{array} \right. \tag{2.5}$$

For $x \in (0,1)$, $\lim_{y\to 0} B_{y,x} < \infty \Leftrightarrow p > 0$ and $\lim_{z\to 1} B_{x,z} < \infty \Leftrightarrow q > 0$. Hence from (2.3) and (2.4), we have the following conditions for the attainability of the boundary.

$$\rho_{x,0} > 0 \Leftrightarrow \lim_{y\to 0} B_{y,x}(p,q) < \infty \Leftrightarrow \frac{2\alpha\gamma}{\beta^2} < 1, \tag{2.6}$$

$$\rho_{x,1} > 0 \Leftrightarrow \lim_{z\to 1} B_{x,z}(p,q) < \infty \Leftrightarrow \frac{2\alpha(1-\gamma)}{\beta^2} < 1. \tag{2.7}$$

From (2.6) and (2.7), the boundary $\{0,1\}$ is inaccessible if and only if $\frac{\beta^2}{2\alpha} \le \gamma \le 1 - \frac{\beta^2}{2\alpha}$.

2.2 Transition Probability

We can derive the transition probability for (1.1) or (1.3) by spectral methods. Let $p(x,s;y,t)$ be the transition probability density for process $\{z_t; t \ge 0\}$,

$$p(x,s;y,t) = \frac{\partial}{\partial y} P[z_t \le y | z_s = x]. \tag{2.8}$$

Then $p(x,s;y,t)$ is given by [1, p.410] :

$$p(x,s;y,t) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)\pi(y)e^{-\lambda_n(t-s)}. \tag{2.9}$$

Here π is the nonnegative function which is proportional to the stationary distribution of (1.3),

$$\pi(x) = \frac{2K}{\beta^2 x(1-x)} \exp\{I(c,x)\}, \quad K > 0, \tag{2.10}$$

where

$$I(c,x) = \int_c^x \frac{2(\gamma - z)}{\beta^2 z(1-z)} dz, \quad c \in (0,1). \tag{2.11}$$

$\{(\phi_n, \lambda_n); n \ge 0\}$ are the normalized orthogonal eigenfunctions and associated eigenvalues such that

$$\int_0^1 \phi_m(x)\phi_n(x)\pi(x)dx = \left\{ \begin{array}{ll} 1, & m = n, \\ 0, & m \ne n, \end{array} \right. \tag{2.12}$$

and

$$\mathcal{L}\phi_n + \lambda_n \phi_n = 0, \quad n \ge 0, \tag{2.13}$$

where $\mathcal{L}$ is the differential operator :

$$\mathcal{L}f(x) = \alpha(\gamma - x)\frac{\partial f(x)}{\partial x} + \frac{\beta^2 x(1-x)}{2}\frac{\partial^2 f(x)}{\partial x^2}. \tag{2.14}$$

The complete orthogonal basis $\{(\phi_n, \lambda_n); n \ge 0\}$ for the operator $\mathcal{L}$ is given by Jacobi polynomials. That is :

$$J_n(x;u,v) = \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) \frac{(u+n)_k}{(v)_k} x^k, \quad n \ge 0, \tag{2.15}$$

where $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$. The properties of Jacobi polynomials are summarized as follows [7].

Proposition 2.1 *(1) For each $n \geq 0$, $J_n(x;u,v)$ is the solution of the differential equation,*

$$x(1-x)y'' + [v-(u+1)x]y' + n(u+n)y = 0, \quad y(0) = 1, \quad y'(0) = -\frac{n(u+n)}{v}. \tag{2.16}$$

(2) Let $u-v > -1$ and $v > 0$. Then the following properties hold for each m, $n \geq 0$.

$$(i) \quad \int_0^1 x^{v+k-1}(1-x)^{u-v} J_n(x;u,v)dx = 0, \quad for \quad k = 0,1,\cdots,n-1, \tag{2.17}$$

$$(ii) \quad \int_0^1 x^{v-1}(1-x)^{u-v} J_m(x;u,v) J_n(x;u,v)dx = \begin{cases} \frac{n!\Gamma(n+u-v+1)\Gamma^2(v)}{(u+2n)\Gamma(u+n)\Gamma(v+n)}, & if \quad m=n, \\ 0, & if \quad m \neq n. \end{cases} \quad \square \tag{2.18}$$

From these we can easily check the following explicit representation for (2.9) :

Theorem 2.2

$$p(x,s;y,t) = \sum_{n=0}^{\infty} k_n \psi_n(x)\psi_n(y)w(y)e^{-\lambda_n(t-s)}, \tag{2.19}$$

where

$$\begin{cases} a &= \frac{2\alpha\gamma}{\beta^2} > 0 \\ b &= \frac{2\alpha(1-\gamma)}{\beta^2} > 0 \\ \lambda_n &= \alpha n + \frac{\beta^2}{2} n(n-1) \\ k_n &= \frac{(a+b+2n-1)\Gamma(a+n)\Gamma(a+b+n-1)}{n!\Gamma(a)^2\Gamma(b+n)} \\ \psi_n(x) &= \sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \frac{(a+b+n-1)_k}{(a)_k} x^k \\ w(x) &= x^{a-1}(1-x)^{b-1}. \end{cases} \tag{2.20}$$

Especially, the stationary and limit distribution is given by the Beta distribution with density :

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} w(y). \quad \square \tag{2.21}$$

2.3 Moment Evaluation

From the transition probability (2.19), we can deduce the moments of z_t. That is

$$\begin{aligned} & E[z_t^n | z_s = z] \\ = & \int_0^1 x^n p(z,s;x,t)dx \\ = & \sum_{v=0}^{\infty} k_v \psi_v(z) e^{-\lambda_v(t-s)} \int_0^1 x^n \psi_v(x) w(x)dx \\ = & \sum_{v=0}^{n} k_v \psi_v(z) e^{-\lambda_v(t-s)} \sum_{k=0}^{v}(-1)^k \begin{pmatrix} v \\ k \end{pmatrix} \frac{\Gamma(a+b+v+k-1)\Gamma(a)}{\Gamma(a+b+v-1)\Gamma(a+k)} \int_0^1 x^{n+k} w(x)dx \\ = & \sum_{v=0}^{n} \left\{ \sum_{k=0}^{v}(-1)^k \begin{pmatrix} v \\ k \end{pmatrix} \frac{(a+b+2v-1)(a)_v(a+k)_n}{v!(b)_v(a+b+v+k-1)_{n-v+1}} \right\} \psi_v(z) e^{-\lambda_v(t-s)}. \end{aligned} \tag{2.22}$$

The third equality follows from (2.15), (2.17) and the last equality follows from (2.20) and $B_{0,1}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. From (2.15), (2.20) and (2.22), we can derive the moment formulas explicitly for n=1, 2.

Proposition 2.3 *The first and second moments of z_t are given as follows.*

$$
\begin{aligned}
E[z_t|z_s = z] &= \gamma + (z-\gamma)e^{-\alpha(t-s)}, & (2.23)\\
E[z_t^2|z_s = z] &= \tfrac{(2\alpha\gamma+\beta^2)\gamma}{2\alpha+\beta^2} + \tfrac{(2\alpha\gamma+\beta^2)\gamma}{\alpha+\beta^2}(z-\gamma)e^{-\alpha(t-s)} & (2.24)\\
&\quad + \left(z^2 - \tfrac{2\alpha\gamma+\beta^2}{\alpha+\beta^2}z + \tfrac{\alpha\gamma(2\alpha\gamma+\beta^2)}{(2\alpha+\beta^2)(\alpha+\beta^2)}\right)e^{-(2\alpha+\beta^2)(t-s)}. \quad \square
\end{aligned}
$$

Next we shall consider the moments of r_t. From (1.2) and the binomial theorem,

$$
\begin{aligned}
&E[r_t^n|r_s = r]\\
&= E[(r_m + (r_M - r_m)z_t)^n|z_s = z]\\
&= r_m^n \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{r_M - r_m}{r_m}\right)^k E[z_t^k|z_s = z], \qquad (2.25)
\end{aligned}
$$

where $z = \frac{r-r_m}{r_M-r_m}$. Substituting (2.23), (2.24) to (2.25), we get the following formulas for the moments of r_t.

Corollary 2.4 *The first and second moments of r_t are given as follows.*

$$
\begin{aligned}
E[r_t|r_s = r] &= r_\mu + (r-r_\mu)e^{-\alpha(t-s)}, & (2.26)\\
E[r_t^2|r_s = r] &= r_\mu^2 + \frac{\beta^2}{2\alpha+\beta^2}(r_\mu - r_m)(r_M - r_\mu)\\
&\quad + \frac{2(r-r_\mu)}{\alpha+\beta^2}\left(\frac{\beta^2(r_m+r_M)}{2} + \alpha r_\mu\right)e^{-\alpha(t-s)} & (2.27)\\
&\quad + \frac{1}{2\alpha+\beta^2}\left(\frac{2\alpha(r-r_\mu)}{\alpha+\beta^2}\left(\alpha(r-r_\mu) - \beta^2\left(\frac{r_m+r_M}{2} - r\right)\right)\right.\\
&\qquad \left. -\beta^2(r-r_m)(r_M-r)\right)e^{-(2\alpha+\beta^2)(t-s)}. \quad \square
\end{aligned}
$$

3 Arbitrage Free Pure Discount Bond Price

We assume that the probability measure P is already the risk neutral measure. Then the arbitrage free pure discount bond price at time t which pays \$ 1 at maturity T, is given by:

$$
B(r,t,T) = E\left[\exp\left(-\int_t^T r_u du\right)\middle| r_t = r\right]. \qquad (3.1)
$$

In general, it is difficult to get the exact evaluation of (3.1). However using the transition probability (2.19), we can represent (3.1) by a weighted series of Jacobi polynomials.

Theorem 3.1

$$
\begin{aligned}
&B(r,t,T)\\
&= e^{-r_m(T-t)}\left\{\begin{array}{l} 1 + \sum_{n=1}^{\infty}(r_M - r_m)^n \\ \sum_{(v_n,\cdots,v_1)\in\mathcal{V}^n} \psi_{v_n}\left(\frac{r-r_m}{r_M-r_m}\right)\prod_{j=n}^{1} k_{v_j} q(v_j, v_{j-1}) I_{t,T}^{(n)}(\lambda_{v_n},\cdots,\lambda_{v_1}) \end{array}\right\}, \qquad (3.2)
\end{aligned}
$$

where

$$
\mathcal{V}^n = \{(v_n,\cdots,v_1)\in \boldsymbol{Z}_+^n;\ |v_j - v_{j-1}| \le 1,\ 1\le j\le n,\ v_0 = 0\}, \qquad (3.3)
$$

$$
\begin{aligned}
q(v_j, v_{j-1}) &= \left\{\begin{array}{ll} \frac{(2v(a+b+v-1)+a(a+b-2))\Gamma^2(a)v!\Gamma(b+v)}{(a+b+2v)(a+b+2v-1)(a+b+2v-2)\Gamma(a+v)\Gamma(a+b+v-1)}, & \textit{if}\ \ v_j = v_{j-1},\\ -\frac{v!\Gamma^2(a)\Gamma(b+v)}{(a+b+2v-1)(a+b+2v-2)(a+b+2v-3)\Gamma(a+v-1)\Gamma(a+b+v-2)}, & \textit{if}\ \ |v_j - v_{j-1}| = 1, \end{array}\right.\\
&\qquad v = v_j \vee v_{j-1}, \qquad (3.4)
\end{aligned}
$$

$$
I_{t,T}^{(n)}(\lambda_n,\cdots,\lambda_1) = \int_t^T\int_{s_n}^T\cdots\int_{s_2}^T \exp\{-\sum_{j=n}^{1}\lambda_j(s_j - s_{j+1})\}ds_1\cdots ds_n,\ s_{n+1} = t. \qquad (3.5)
$$

Proof. From (1.2),

$$\begin{aligned} B(r,t,T) &= E\left[\exp\left\{-\int_t^T r_u du\right\}\Big| r_t = r\right] \\ &= e^{-r_m(T-t)}E_{z,t}\left[\exp\left\{-(r_M - r_m)\int_t^T z_u du\right\}\right] \\ &= e^{-r_m(T-t)}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n(r_M-r_m)^n}{n!}E_{z,t}\left[\left(\int_t^T z_u du\right)^n\right]\right\}, \end{aligned}$$

where $z = \frac{r-r_m}{r_M-r_m}$ and $E_{z,t}[\,\cdot\,] = E[\,\cdot\,|z_t = z]$. Substituting the equality :

$$E_{z,t}\left[\left(\int_t^T z_u du\right)^n\right] = n!\int_t^T\int_{s_n}^T\cdots\int_{s_2}^T E_{z,t}[z_{s_n}z_{s_{n-1}}\cdots z_{s_1}]ds_1\cdots ds_{n-1}ds_n,$$

we get

$$B(r,t,T) = e^{-r_m(T-t)}\left\{1+\sum_{n=1}^{\infty}(-1)^n(r_M-r_m)^n\int_t^T\cdots\int_{s_2}^T E_{z,t}[z_{s_n}\cdots z_{s_1}]ds_1\cdots ds_n\right\}. \tag{3.6}$$

For the given $(s_n,\cdots,s_1)$,

$$E_{z,t}[z_{s_n}\cdots z_{s_1}] = \int_0^1\cdots\int_0^1 z_n\cdots z_1\prod_{j=n}^{1}p(z_{j+1},s_{j+1};z_j,s_j)dz_1\cdots dz_n, \tag{3.7}$$

where $s_{n+1} = t$, $z_{n+1} = z$. Since the probability density in (3.7) is given by

$$\begin{aligned} &\prod_{j=n}^{1}p(z_{j+1},s_{j+1};z_j,s_j) \\ &= \prod_{j=n}^{1}\sum_{v_j=0}^{\infty}k_{v_j}\psi_{v_j}(z_{j+1})\psi_{v_j}(z_j)w(z_j)e^{-\lambda_{v_j}(s_j-s_{j+1})} \\ &= \sum_{(v_n,\cdots,v_1)\in Z_+^n}\psi_{v_n}(z)\prod_{j=n}^{1}k_{v_j}\psi_{v_j}(z_j)\psi_{v_{j-1}}(z_j)w(z_j)e^{-\lambda_{v_j}(s_j-s_{j+1})}, \quad v_0 = 0, \end{aligned}$$

we have

$$E_{z,t}[z_{s_n}\cdots z_{s_1}] = \sum_{(v_n,\cdots,v_1)\in Z_+^n}\psi_{v_n}(z)\prod_{j=n}^{1}k_{v_j}e^{-\lambda_{v_j}(s_j-s_{j+1})}\int_0^1 z_j\psi_{v_j}(z_j)\psi_{v_{j-1}}(z_j)w(z_j)dz_j. \tag{3.8}$$

Furthermore as shown in Lemma A.1,

$$\int_0^1 z_j\psi_{v_j}(z_j)\psi_{v_{j-1}}(z_j)w(z_j)dz_j = \begin{cases} q(v_j,v_{j-1}), & \text{if } |v_j-v_{j-1}|\le 1, \\ 0, & \text{if } |v_j-v_{j-1}|\ge 2. \end{cases} \tag{3.9}$$

Then from (3.8) and (3.9), we get

$$E_{z,t}[z_{s_n}\cdots z_{s_1}] = \sum_{(v_n,\cdots,v_1)\in\mathcal{V}^n}\psi_{v_n}(z)\left(\prod_{j=n}^{1}k_{v_j}q(v_j,v_{j-1})\right)\exp\left\{-\sum_{j=n}^{1}\lambda_{v_j}(s_j-s_{j+1})\right\}. \tag{3.10}$$

Combining (3.6) and (3.10), we obtain (3.2). □

4 Approximate Evaluation

From Theorem 3.1, we can approximate the arbitrage free pure discount bond price by the trancated sum of series (3.2). Let

$$
\begin{aligned}
&B^{(j)}(r,t,T)\\
&= e^{-r_m(T-t)}\left\{\begin{array}{l} 1+\sum_{n=1}^{j}(r_M-r_m)^n \\ \sum_{(v_n,\cdots,v_1)\in\mathcal{V}^n}\psi_{v_n}\left(\frac{r-r_m}{r_M-r_m}\right)\prod_{j=n}^{1}k_{v_j}q(v_j,v_{j-1})I_{t,T}^{(n)}(\lambda_{v_n},\cdots,\lambda_{v_1}) \end{array}\right\},\quad 0\le j.
\end{aligned}
\tag{4.1}
$$

After the "tedious" but elementary calculation, we get the following results for $0 \le j \le 2$. The proof is left to the reader.

Proposition 4.1 *The truncated sums of (3.2) up to second order are given as follows.*

$$
B^{(0)}(r,t,T) = e^{-r_m(T-t)}, \tag{4.2}
$$

$$
B^{(1)}(r,t,T) = e^{-r_m(T-t)}\left(1-(r_\mu-r_m)(T-t)-(r-r_\mu)\frac{1-e^{-\alpha(T-t)}}{\alpha}\right), \tag{4.3}
$$

$$
\begin{aligned}
&B^{(2)}(r,t,T)\\
&= e^{-r_m(T-t)}\left\{\begin{array}{l}
1-(r_\mu-r_m)(T-t)-(r-r_\mu)\frac{1-e^{-\alpha(T-t)}}{\alpha}+\frac{1}{2}(r_\mu-r_m)^2(T-t)^2\\
+\left(\frac{\beta^2}{2\alpha+\beta^2}(r_\mu-r_m)(r_M-r_\mu)+(r_\mu-r_m)(r-r_\mu)\right)\frac{T-t}{\alpha}\\
+\left(\begin{array}{c}\frac{\alpha}{\alpha+\beta^2}(r-r_\mu)^2+2\left(\frac{\beta^2}{\alpha+\beta^2}\right)^2(r-r_\mu)\left(\frac{r_m+r_M}{2}-r_\mu\right)\\ -\frac{\beta^2}{\alpha+\beta^2}(r_\mu-r_m)(r_M-r_\mu)\end{array}\right)\frac{1-e^{-\alpha(T-t)}}{\alpha^2}\\
-\left(\frac{\alpha}{\alpha+\beta^2}(r_\mu-r_m)(r-r_\mu)+\frac{\beta^2}{\alpha+\beta^2}(r-r_\mu)(r_M-r_\mu)\right)\frac{(T-t)e^{-\alpha(T-t)}}{\alpha}\\
-\left(\begin{array}{c}\frac{\alpha}{\alpha+\beta^2}\frac{\beta^2}{2\alpha+\beta^2}(r_\mu-r_m)(r_M-r_\mu)\\ +\frac{\alpha}{\alpha+\beta^2}(r-r_\mu)^2-\frac{\beta^2}{\alpha+\beta^2}(r-r_m)(r_M-r)\end{array}\right)\frac{1-e^{-(2\alpha+\beta^2)(T-t)}}{(\alpha+\beta^2)(2\alpha+\beta^2)}
\end{array}\right\}.
\end{aligned}
$$

$\square$ (4.4)

From (4.2) through (4.4), we see that the volatility coefficient β appeared only for $j = 2$. This means that we should use at least $B^{(2)}(r,t,T)$ to take into account the volatility effect in the approximation. Also $B^{(0)}$ is an obvious upper bound of the discount bond price for any initial interest rate. However we can derive more sophisticated upper bounds as will be shown in Theorem 4.2.

We proposed the approximation formula (4.1) assuming that the higher order terms are negligible. Hence we should consider the lower and upper bounds to evaluate the error magnitude. Since the interest rate distribution has bounded support $[r_m, r_M]$, we can derive both the lower and upper bounds.

Theorem 4.2 *(1) Lower bound :*

$$
\exp\left\{-r_\mu(T-t)-(r-r_\mu)\frac{1-e^{-\alpha(T-t)}}{\alpha}\right\} \le B(r,t,T). \tag{4.5}
$$

(2) Upper Bound :

$$
\begin{aligned}
B(r,t,T) \le\ &\left(1-\gamma-(z-\gamma)\frac{1-e^{-\alpha(T-t)}}{\alpha(T-t)}\right)e^{-r_m(T-t)}\\
&+\left(\gamma+(z-\gamma)\frac{1-e^{-\alpha(T-t)}}{\alpha(T-t)}\right)e^{-r_M(T-t)},
\end{aligned}
\tag{4.6}
$$

where $z = \frac{r-r_m}{r_M-r_m}$.

Proof. Let X be a random variable with mean x_μ and with a distribution supported by $[x_m, x_M]$. Then for any convex function f, the following inequalities are obvious consequences of Jensen's inequality.

$$f(x_\mu) \le E[f(X)] \le \frac{x_M - x_\mu}{x_M - x_m} f(x_m) + \frac{x_\mu - x_m}{x_M - x_m} f(x_M). \tag{4.7}$$

The left hand side is attained when X is degenerated to the mean x_μ and the right hand side is attained when X follows two point distribution such that $X \in \{x_m, x_M\}$ and $E[X] = x_\mu$. Since $f(x) = e^{-x}$ is a convex function, we can apply (4.7) for $X = \int_t^T r_u du$. From (2.26),

$$\begin{aligned} E\left[\int_t^T r_u du \middle| r_t = r\right] &= \int_t^T (r_\mu + (r - r_\mu)e^{-\alpha(u-t)})du \\ &= r_\mu(T-t) + (r - r_\mu)\frac{1 - e^{-\alpha(T-t)}}{\alpha}. \end{aligned} \tag{4.8}$$

Furthermore since $r_u \in [r_m, r_M]$ for all u, $r_m(T-t) \le \int_t^T r_u du \le r_M(T-t)$. This together with (4.7) and (4.8), we get (4.5) and (4.6). □

Remark 4.3 *The lower bound (4.5) is valid for all the linear mean reversion type interest rate models, e.g. [2]. However (4.5) is expecially useful in our model since the lower bound seems to be close to the exact value for reasonable parameters. This comes from the boundedness of the fluctuation. We will study this effect numerically in the next section.*

5 Numerical Computation

For the sophisticated approximate evaluation of discount bond price (3.1), we can use a numerical computation method. Let us define the discrete state space $\mathcal{S}^{(\Delta,\epsilon,c_m,c_M)} = \{r_n^{(\Delta,\epsilon,c_m,c_M)}; n \in I^{(\Delta,\epsilon,c_m,c_M)}\}$ and the indices set $I^{(\Delta,\epsilon,c_m,c_M)} = \{-N_m^{(\Delta,\epsilon,c_m)}, -N_m^{(\Delta,\epsilon,c_m)} + 1, \cdots, 0, \cdots, N_M^{(\Delta,\epsilon,c_M)} - 1, N_M^{(\Delta,\epsilon,c_M)}\}$ for $(\Delta, \epsilon, c_m, c_M) \in \boldsymbol{R}_+^4$ by

$$r_n^{(\Delta,\epsilon,c_m,c_M)} = \begin{cases} r_\mu + (r_M - r_\mu)(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)} - n}, & \text{if } 1 \le n \le N_M^{(\Delta,\epsilon,c_M)}, \\ r_\mu, & \text{if } n = 0, \\ r_\mu - (r_\mu - r_m)(1-\alpha\Delta)^{N_m^{(\Delta,\epsilon,c_M)} + n}, & \text{if } -N_m^{(\Delta,\epsilon,c_m)} \le n \le -1, \end{cases} \tag{5.1}$$

where

$$N_x^{(\Delta,\epsilon,c)} = \left\lceil \frac{\log\{(1+\epsilon)\beta c \eta_x \sqrt{\Delta}\}}{\log(1-\alpha\Delta)} \right\rceil, \quad x = m, M, \tag{5.2}$$

$$\eta_x = \begin{cases} \sqrt{\frac{\gamma}{1-\gamma}}, & \text{if } x = M, \\ \sqrt{\frac{1-\gamma}{\gamma}}, & \text{if } x = m, \end{cases} \tag{5.3}$$

$$\lceil x \rceil = \inf\{n \in \boldsymbol{Z}; n \ge x\}.$$

We can easily check that

$$\lim_{\Delta \to 0} N_x^{(\Delta,\epsilon,c)} = \infty, \tag{5.4}$$

$$\lim_{\Delta \to 0} \sup_{-N_m^{(\Delta,\epsilon,c_m)} \le n \le N_M^{(\Delta,\epsilon,c_M)} - 1} |r_{n+1}^{(\Delta,\epsilon,c_m,c_M)} - r_n^{(\Delta,\epsilon,c_m,c_M)}| = 0. \tag{5.5}$$

Let us define the drift and diffusion term for this discrete time state space :

$$\begin{aligned} \mu_n^{(\Delta,\epsilon,c_m,c_M)} &= \alpha(r_\mu - r_n^{(\Delta,\epsilon,c_m,c_M)})\Delta 1\{|n| \ge 2\} \\ &= \begin{cases} -\alpha\Delta(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)} - n}(r_M - r_\mu), & \text{if } n \ge 2, \\ 0, & \text{if } |n| \le 1, \\ \alpha\Delta(1-\alpha\Delta)^{N_m^{(\Delta,\epsilon,c_m)} + n}(r_\mu - r_m), & \text{if } n \le -2, \end{cases} \end{aligned} \tag{5.6}$$

and

$$
\begin{aligned}
&\sigma_n^{(\Delta,\epsilon,c_m,c_M)}\\
&= \beta\sqrt{(r_n^{(\Delta,\epsilon,c_m,c_M)}-r_m)(r_M-r_n^{(\Delta,\epsilon,c_m,c_M)})\Delta}\\
&= \begin{cases}
\beta(r_M-r_m)\sqrt{\begin{array}{l}(1-\gamma)(\gamma+(1-\gamma)(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)}-n})\\ \times(1-(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)}-n})\Delta\end{array}}, & \text{if } n\geq 1,\\
\beta(r_M-r_m)\sqrt{\gamma(1-\gamma)\Delta}, & \text{if } n=0,\\
\beta(r_M-r_m)\sqrt{\begin{array}{l}\gamma(1-\gamma+\gamma(1-\alpha\Delta)^{N_m^{(\Delta,\epsilon,c_m)}+n})\\ \times(1-(1-\alpha\Delta)^{N_m^{(\Delta,\epsilon,c_m)}+n})\Delta\end{array}}, & \text{if } n\leq -1.
\end{cases}
\end{aligned}
\tag{5.7}
$$

(5.6) and (5.7) correspond to the discretized value of the drift and diffusion coefficient functions μ, σ for all $n\in I^{(\Delta,\epsilon,c_m,c_M)}$ except for $\mu_{\pm 1}$. From the definition of state space, we can easily deduce

$$
\mu_n^{(\Delta,\epsilon,c_m,c_M)} = \begin{cases}
r_{n-1}^{(\Delta,\epsilon,c_m,c_M)}-r_n^{(\Delta,\epsilon,c_m,c_M)}, & \text{if } n\geq 2,\\
0, & \text{if } |n|\leq 1,\\
r_{n+1}^{(\Delta,\epsilon,c_m,c_M)}-r_n^{(\Delta,\epsilon,c_m,c_M)}, & \text{if } n\leq -2.
\end{cases}
\tag{5.8}
$$

This together with the next lemma play a key role to construct a sequence of Markov chains which converge to (1.1).

Lemma 5.1 *For sufficiently small $\Delta>0$, there exist (n_1,n_2) for $n\in I^{(\Delta,\epsilon,c_m,c_M)}$ such that*

$$
\begin{aligned}
\textit{Case I}: n>0;\quad & 0\leq n_1\leq n\leq n_2\leq N_M^{(\Delta,\epsilon,c_M)},\\
& r_n^{(\Delta,\epsilon,c_m,c_M)}+\frac{\sqrt{\alpha(1-\gamma)}}{(1+\epsilon)\beta}\sigma_n^{(\Delta,\epsilon,c_m,c_M)}\leq r_{n_2}^{(\Delta,\epsilon,c_m,c_M)}, && (5.9)\\
& r_{(n-1)\vee 1}^{(\Delta,\epsilon,c_m,c_M)}-c_M\sigma_n^{(\Delta,\epsilon,c_m,c_M)}\geq r_{n_1}^{(\Delta,\epsilon,c_m,c_M)}. && (5.10)\\
\textit{Case II}: n<0;\quad & -N_m^{(\Delta,\epsilon,c_m)}\leq n_2\leq n\leq n_1\leq 0,\\
& r_{(n+1)\wedge -1}^{(\Delta,\epsilon,c_m,c_M)}+c_m\sigma_n^{(\Delta,\epsilon,c_m,c_M)}\leq r_{n_1}^{(\Delta,\epsilon,c_m,c_M)}, && (5.11)\\
& r_n^{(\Delta,\epsilon,c_m,c_M)}-\frac{\sqrt{\alpha\gamma}}{(1+\epsilon)\beta}\sigma_n^{(\Delta,\epsilon,c_m,c_M)}\geq r_{n_2}^{(\Delta,\epsilon,c_m,c_M)}. && (5.12)\\
\textit{Case III}: n=0;\quad & n_1=-1,\ n_2=1,\\
& r_n^{(\Delta,\epsilon,c_m,c_M)}+c_M\sigma_n^{(\Delta,\epsilon,c_m,c_M)}\leq r_{n_2}^{(\Delta,\epsilon,c_m,c_M)}, && (5.13)\\
& r_n^{(\Delta,\epsilon,c_m,c_M)}-c_m\sigma_n^{(\Delta,\epsilon,c_m,c_M)}\geq r_{n_1}^{(\Delta,\epsilon,c_m,c_M)}. && (5.14)
\end{aligned}
$$

Proof. Since the proof is similar, we prove only (5.9) and (5.10). If $n=N_M^{(\Delta,\epsilon,c_M)}$, then $\sigma_n^{(\Delta,\epsilon,c_m,c_M)}=0$ and hence there exist $n_1=n-1$ and $n_2=n$ which satisfy (5.9) and (5.10). Let us assume $0<n<N_M^{(\Delta,\epsilon,c_M)}$. Then

$$
\begin{aligned}
&\lim_{\Delta\to 0}\sup_{0<n<N_M^{(\Delta,\epsilon,c_M)}}\frac{\sqrt{\alpha(1-\gamma)}}{(1+\epsilon)\beta}\frac{\sigma_n^{(\Delta,\epsilon,c_m,c_M)}}{r_M-r_n^{(\Delta,\epsilon,c_m,c_M)}}\\
&= \lim_{\Delta\to 0}\sup_{0<n<N_M^{(\Delta,\epsilon,c_M)}}\frac{1}{1+\epsilon}\sqrt{\left(\frac{1}{1-(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)}}}-(1-\gamma)\right)\alpha\Delta}\\
&= \lim_{\Delta\to 0}\frac{1}{1+\epsilon}\sqrt{1-(1-\gamma)\alpha\Delta}\\
&= \frac{1}{1+\epsilon}<1.
\end{aligned}
$$

Hence (5.9) holds. On the other hand,

$$
\lim_{\Delta\to 0}\sup_{0<n<N_M^{(\Delta,\epsilon,c_M)}}\frac{c_M\sigma_n^{(\Delta,\epsilon,c_m,c_M)}+r_n^{(\Delta,\epsilon,c_m,c_M)}-r_{(n-1)\vee 1}^{(\Delta,\epsilon,c_m,c_M)}}{r_n^{(\Delta,\epsilon,c_m,c_M)}-r_0^{(\Delta,\epsilon,c_m,c_M)}}
$$

$$= \lim_{\Delta\to 0} \sup_{0<n<N_M^{(\Delta,\epsilon,c_M)}} c_M\beta \sqrt{\begin{array}{c}\left(\frac{\gamma}{(1-\gamma)(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)}-n}}+1\right)\\ \times\left(\frac{1}{(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)}-n}}-1\right)\Delta\end{array} + \alpha\Delta 1\{n>1\}}$$

$$= \lim_{\Delta\to 0} c_M\beta\sqrt{\left(\frac{\gamma}{(1-\gamma)(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)}-1}}+1\right)\left(\frac{1}{(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M)}-1}}-1\right)\Delta}$$

$$= \lim_{\Delta\to 0} c_M\beta\sqrt{\frac{\gamma}{1-\gamma}\left(\frac{(1-\alpha\Delta)^{\epsilon'}}{(1+\epsilon)\beta c_M\eta_M\sqrt{\Delta}}\right)^2\Delta}$$

$$= \frac{1}{1+\epsilon} < 1.$$

In the third equality, $\epsilon' \in [0,1)$ is determined by the truncation operator $\lceil\cdot\rceil$. Hence (5.10) holds. $\square$

Let us now for each given $\Delta > 0$, define a Markov chain . Let $\epsilon > 0$ and

$$c_m^* = \max\left\{1, \frac{(1+\epsilon)\beta}{\sqrt{\alpha\gamma}}\right\}, \tag{5.15}$$

$$c_M^* = \max\left\{1, \frac{(1+\epsilon)\beta}{\sqrt{\alpha(1-\gamma)}}\right\}. \tag{5.16}$$

Then we can construct the discrete time Markov chain $\boldsymbol{X}^{(\Delta)} = \{X_u^{(\Delta)}; u = 0, \Delta, 2\Delta, \cdots\}$ on the state space $\mathcal{S}^{(\Delta,\epsilon,c_m^*,c_M^*)}$ with the indices set $I^{(\Delta,\epsilon,c_m^*,c_M^*)}$. The transition probability is stationary and is defined by

$$\begin{aligned} p_{i,j}^{(\Delta)} &= Pr[X_{t+\Delta}^{(\Delta)} = r_j^{(\Delta,\epsilon,c_m^*,c_M^*)} | X_t^{(\Delta)} = r_i^{(\Delta,\epsilon,c_m^*,c_M^*)}] \\ &= \begin{cases} \frac{r_{j_2(i)}-r_{j_1(i)}}{r_{j_3(i)}-r_{j_1(i)}} \frac{\sigma_i^2}{(r_{j_2(i)}-r_{j_1(i)})(r_{j_3(i)}-r_{j_2(i)})}, & \text{if } j = j_3(i), \\ 1 - \frac{\sigma_i^2}{(r_{j_2(i)}-r_{j_1(i)})(r_{j_3(i)}-r_{j_2(i)})}, & \text{if } j = j_2(i), \\ \frac{r_{j_3(i)}-r_{j_2(i)}}{r_{j_3(i)}-r_{j_1(i)}} \frac{\sigma_i^2}{(r_{j_2(i)}-r_{j_1(i)})(r_{j_3(i)}-r_{j_2(i)})}, & \text{if } j = j_1(i), \\ 0, & \text{elsewhere,} \end{cases} \end{aligned} \tag{5.17}$$

where

$$j_1(i) = \begin{cases} \left(N_M^{(\Delta,\epsilon,c_M)} - \left\lceil \frac{\log\left\{(1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)}-i+1} - \frac{c_M^*}{r_M-r_\mu}\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right\}}{\log(1-\alpha\Delta)} \right\rceil\right) \vee 0, & \text{if } 1 < i \le N_M^{(\Delta,\epsilon,c_M)}, \\ -1\{i=0\}, & \text{if } |i| \le 1, \\ \left(\left\lceil \frac{\log\left\{(1-\alpha\Delta)^{N_m^{(\Delta,\epsilon,c_M^*)}+i+1} - \frac{c_m^*}{r_\mu-r_m}\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right\}}{\log(1-\alpha\Delta)} \right\rceil - N_m^{(\Delta,\epsilon,c_m^*)}\right) \wedge 0, & \text{if } -N_m^{(\Delta,\epsilon,c_m^*)} \le i < -1, \end{cases}$$

$$j_2(i) = \begin{cases} i-1, & \text{if } 1 < i \le N_M^{(\Delta,\epsilon,c_M)}, \\ i, & \text{if } |i| \le 1, \\ i+1, & \text{if } -N_m^{(\Delta,\epsilon,c_m^*)} \le i < -1, \end{cases}$$

$$
j_3(i) = \begin{cases} N_M^{(\Delta,\epsilon,c_M)} - \left\lfloor \dfrac{\log\left\{ (1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)}-i} + \frac{\sqrt{\alpha(1-\gamma)}}{(r_M-r_\mu)(1+\epsilon)\beta}\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)} \right\}}{\log(1-\alpha\Delta)} \right\rfloor, & \text{if } 1 \le i \le N_M^{(\Delta,\epsilon,c_M)}, \\ 1, & \text{if } i = 0, \\ \left\lfloor \dfrac{\log\left\{ (1-\alpha\Delta)^{N_m^{(\Delta,\epsilon,c_M^*)}+i} + \frac{\sqrt{\alpha\gamma}}{(r_\mu-r_m)(1+\epsilon)\beta}\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)} \right\}}{\log(1-\alpha\Delta)} \right\rfloor - N_m^{(\Delta,\epsilon,c_m^*)}, & \text{if } -N_m^{(\Delta,\epsilon,c_m^*)} \le i \le -1, \end{cases}
$$

$$
\lfloor x \rfloor = \sup\{n \in \boldsymbol{Z}; n \le x\}. \tag{5.18}
$$

From Lemma 5.1, (5.18) is well defined on $I^{(\Delta,\epsilon,c_m^*,c_M^*)}$. We can easily check that the following properties hold for the transition probabilities (5.17).

$$
(1) \quad \sum_{j \in I^{(\Delta,\epsilon,c_m^*,c_M^*)}} p_{i,j}^{(\Delta)} = 1, \tag{5.19}
$$

$$
(2) \quad \sum_{j \in I^{(\Delta,\epsilon,c_m^*,c_M^*)}} (r_j^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_i^{(\Delta,\epsilon,c_m^*,c_M^*)}) p_{i,j}^{(\Delta)} = \mu_i^{(\Delta,\epsilon,c_m^*,c_M^*)}, \tag{5.20}
$$

$$
(3) \quad \sum_{j \in I^{(\Delta,\epsilon,c_m^*,c_M^*)}} (r_j^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_i^{(\Delta,\epsilon,c_m^*,c_M^*)})^2 p_{i,j}^{(\Delta)} = \left(\mu_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)^2 + \left(\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)^2. \tag{5.21}
$$

Furthermore we can show that for sufficiently small Δ, the $p_{i,j}$ are nonnegative.

Lemma 5.2 *There exists $\delta > 0$ such that*

$$
p_{i,j}^{(\Delta)} \ge 0, \quad \textit{for all} \ \ i, j \in I^{(\Delta,\epsilon,c_m^*,c_M^*)}, \ 0 \le \Delta \le \delta. \tag{5.22}
$$

Proof. From (5.17),

$$
\begin{aligned}
& p_{i,j}^{(\Delta)} \ge 0 \\
& \Leftrightarrow \left(r_{j_2(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_{j_1(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)\left(r_{j_3(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_{j_2(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)}\right) \ge \left(\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)^2.
\end{aligned} \tag{5.23}
$$

Suppose $1 < i \le N_M^{(\Delta,\epsilon,c_M^*)}$. Then

$$
\begin{aligned}
& \left(r_{j_2(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_{j_1(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)\left(r_{j_3(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_{j_2(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)}\right) \\
= \ & (r_M - r_\mu)^2 \left((1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - j_2(i)} - (1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - j_1} \right) \\
& \left((1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - j_3(i)} - (1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - j_2} \right) \\
= \ & (r_M - r_\mu)^2 (1-\alpha\Delta)^{-\epsilon''} \\
& \left((1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - i + 1} - \left((1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - i + 1} - \frac{c_M^*}{r_M - r_\mu}\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)} \right)(1-\alpha\Delta)^{\epsilon'} \right) \\
& \left(\left((1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - i} + \frac{\sqrt{\alpha(1-\gamma)}}{(r_M - r_\mu)(1+\epsilon)\beta}\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)} \right) - (1-\alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - i + 1 + \epsilon''} \right).
\end{aligned}
$$

In the last equality, $\epsilon', \epsilon'' \in [0, 1)$ are determined by the truncation operators $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$. From Lemma 5.1, we deduce

$$
r_{i-1}^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_\mu - c_M^* \sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)} = (r_M - r_\mu)\left(1 - \alpha\Delta)^{N_M^{(\Delta,\epsilon,c_M^*)} - i + 1} - \frac{c_M^*}{r_M - r_\mu}\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right) \ge 0.
$$

Therefore

$$
\left(r_{j_2(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_{j_1(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)\left(r_{j_3(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)} - r_{j_2(i)}^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)
$$

$$
\begin{aligned}
&\geq c_M \frac{\sqrt{\alpha(1-\gamma)}}{(1+\epsilon)\beta} \left(\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)^2 \\
&\geq \left(\sigma_i^{(\Delta,\epsilon,c_m^*,c_M^*)}\right)^2 .
\end{aligned}
$$

This together with (5.23) gives (5.22). We can handle the cases $-N_m^{(\Delta,\epsilon,c_m^*)} \leq i < -1$ and $|i| \leq 1$ in the same way. □

The Markov chain $\tilde{\boldsymbol{X}}^{(\Delta)}$ converges to the process $\{r_t; 0 \leq t\}$ defined by (1.1).

Theorem 5.3 *Consider the continuous time Markov chain* $\tilde{\boldsymbol{X}}^{(\Delta)}$ *defined by*

$$
\tilde{X}_t^{(\Delta)} = X_{n_t \Delta}^{(\Delta)} \tag{5.24}
$$

where $n_t = \lfloor \frac{t}{\Delta} \rfloor$. *Then*

$$
\{\tilde{X}_t^{(\Delta)}; 0 \leq t\} \stackrel{law}{\to} \{r_t; 0 \leq t\} \quad as \quad \Delta \to 0. \tag{5.25}
$$

Proof. From (5.20) and (5.21), we see that the mean and the quadratic variational process of $\tilde{X}_t^{(\Delta)}$ converge to those of $\{r_t\}$. Also the transition jump size goes to 0 as $\Delta \to 0$. Hence from the martingale central limit theorem ([3], Theorem5.1, p.354), (5.25) holds. □

A Appendix

Lemma A.1 *For any* $m, n \in \boldsymbol{Z}$, $0 \leq m \leq n$ *and* $u, v \in \boldsymbol{R}$, $u > 0$, $u - v > -1$,

$$
\begin{aligned}
&\int_0^1 x^v (1-x)^{u-v} J_m(x;u,v) J_n(x;u,v) dx \\
&= \begin{cases} 0, & \text{if } 0 \leq m \leq n-2, \\ -\frac{n!\Gamma(n+u-v+1)\Gamma^2(v)}{(u+2n)(u+2n-1)(u+2n-2)\Gamma(u+n-1)\Gamma(v+n-1)}, & \text{if } 0 \leq m = n-1, \\ \frac{(2n(u+n)+v(u-1))\Gamma^2(v) n! \Gamma(n+u-v+1)}{(u+2v+1)(u+2n)(u+2n-1)\Gamma(u+n)\Gamma(v+n)}, & \text{if } 0 \leq m = n. \end{cases}
\end{aligned} \tag{A.1}
$$

Proof. From (2.15) and (2.17), we deduce

$$
\begin{aligned}
&\int_0^1 x^v (1-x)^{u-v} J_m(x;u,v) J_n(x;u,v) dx \\
&= \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} \frac{(u+m)_k}{(v)_k} \int_0^1 x^{v+k} (1-x)^{u-v} J_n(x;u,v) dx \\
&= \sum_{k=n-1}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} \frac{(u+m)_k}{(v)_k} \int_0^1 x^{v+k} (1-x)^{u-v} J_n(x;u,v) dx,
\end{aligned} \tag{A.2}
$$

where $\sum_{k=n-1}^{m} = 0$ for $n \geq m+2$.
Case I: $0 \leq m \leq n-2$. (A.1) follows from (A.2) directly.
Case II: $0 \leq m = n-1$. First we prove the following equation.

$$
\begin{aligned}
&\sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \frac{(u+n)_k}{(v)_k} \frac{\Gamma(v+n+k)\Gamma(u-v+1)}{\Gamma(u+n+k+1)} \\
&= \frac{(-1)^n n! \Gamma(u-v+n+1)\Gamma(v)}{\Gamma(u+2n+1)}, \quad v > 0, \ u-v > -1.
\end{aligned} \tag{A.3}
$$

From (2.15), (2.17) and (2.18) for $m = n$,

$$
\int_0^1 x^{v-1} (1-x)^{u-v} J_n^2(x;u,v) dx
$$

$$
\begin{aligned}
&= \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(u+n)_k}{(v)_k} \int_0^1 x^{v+k-1}(1-x)^{u-v} J_n(x;u,v)dx \\
&= (-1)^n \frac{(u+n)_n}{(v)_n} \int_0^1 x^{v+n-1}(1-x)^{u-v} J_n(x;u,v)dx \\
&= (-1)^n \frac{(u+n)_n}{(v)_n} \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(u+n)_k}{(v)_k} \int_0^1 x^{v+n+k-1}(1-x)^{u-v}dx \\
&= (-1)^n \frac{\Gamma(u+2n)\Gamma(v)}{\Gamma(u+n)\Gamma(v+n)} \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(u+n)_k}{(v)_k} \frac{\Gamma(v+n+k)\Gamma(u-v+1)}{\Gamma(u+n+k+1)} \\
&= \frac{n!\Gamma(u-v+n+1)\Gamma^2(v)}{(u+2n)\Gamma(u+n)\Gamma(v+n)}. \qquad \text{(A.4)}
\end{aligned}
$$

Rearranging the last equality, we obtain (A.3). Next we shall show (A.1) for Case II. From (A.2),

$$
\begin{aligned}
&\int_0^1 x^v(1-x)^{u-v} J_{n-1}(x;u,v) J_n(x;u,v)dx \\
&= (-1)^{n-1} \frac{(u+n-1)_{n-1}}{(v)_{n-1}} \int_0^1 x^{v+n-1}(1-x)^{u-v} J_n(x;u,v)dx \\
&= (-1)^{n-1} \frac{(u+n-1)_{n-1}}{(v)_{n-1}} \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(u+n)_k}{(v)_k} \int_0^1 x^{v+n+k-1}(1-x)^{u-v}dx \\
&= (-1)^{n-1} \frac{(u+n-1)_{n-1}}{(v)_{n-1}} \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(u+n)_k}{(v)_k} \frac{\Gamma(v+n+k)\Gamma(u-v+1)}{\Gamma(u+n+k+1)} \\
&= (-1)^{n-1} \frac{(u+n-1)_{n-1}}{(v)_{n-1}} \frac{(-1)^n n!\Gamma(u-v+n+1)\Gamma(v)}{\Gamma(u+2n+1)} \\
&= -\frac{n!\Gamma(u-v+n+1)\Gamma^2(v)}{(u+2n)(u+2n-1)(u+2n-2)\Gamma(u+n-1)\Gamma(v+n-1)},
\end{aligned}
$$

where the fourth equality follows from (A.3).

Case III : $0 \leq m = n$. First we prove the following equation by induction with respect to n.

$$
\begin{aligned}
&\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{(u+n)_k}{(v)_k} \frac{\Gamma(v+n+k+1)\Gamma(u-v+1)}{\Gamma(u+n+k+2)} \qquad \text{(A.5)} \\
&= \frac{(-1)^n(n+1)!(v+n)\Gamma(u-v+n+1)\Gamma(v)}{\Gamma(u+2n+2)}, \quad v>0,\ u-v>-1.
\end{aligned}
$$

We can easily check that (A.5) holds for $n=0$. Suppose that (A.5) holds for n. Then

$$
\begin{aligned}
&\sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} \frac{(u+n+1)_k}{(v)_k} \frac{\Gamma(v+n+k+2)\Gamma(u-v+1)}{\Gamma(u+n+k+3)} \\
&= \sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} \frac{(u+n+1)_k}{(v)_k} \frac{\Gamma(v+n+k+1)\Gamma(u-v+1)}{\Gamma(u+n+k+2)} \left(1-\frac{u-v+1}{u+n+k+2}\right) \\
&= \sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} \frac{(u+n+1)_k}{(v)_k} \frac{\Gamma(v+n+k+1)\Gamma(u-v+1)}{\Gamma(u+n+k+2)} \\
&\quad -\sum_{k=0}^{n+1}(-1)^k \left(\binom{n}{k}+\binom{n}{k-1}\right) \frac{(u+n+1)_k}{(v)_k} \frac{\Gamma(v+n+k+1)\Gamma(u-v+2)}{\Gamma(u+n+k+3)}. \qquad \text{(A.6)}
\end{aligned}
$$

From (A.3) and the assumption of induction for n,

$$
(A.6) \quad = \quad \frac{(-1)^{n+1}(n+1)!\Gamma(u-v+n+2)\Gamma(v)}{\Gamma(u+2n+3)}
$$

$$
\begin{aligned}
&-\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \frac{(u+n+1)_k}{(v)_k} \frac{\Gamma(v+n+k+1)\Gamma(u-v+2)}{\Gamma(u+n+k+3)} \\
&+\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \frac{(u+n+1)_{k+1}}{(v)_{k+1}} \frac{\Gamma(v+n+k+2)\Gamma(u-v+2)}{\Gamma(u+n+k+4)} \\
= \ & \frac{(-1)^{n+1}(n+1)!\Gamma(u-v+n+2)\Gamma(v)}{\Gamma(u+2n+3)} \\
&+\frac{(-1)^{n+1}(n+1)!(v+n)\Gamma(u-v+n+2)\Gamma(v)}{\Gamma(u+2n+3)} \\
&+\frac{u+n+1}{v}\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix} \frac{(u+n+2)_k}{(v+1)_k} \frac{\Gamma(n+k+v+2)\Gamma(u-v+2)}{\Gamma(u+n+k+4)} \\
= \ & \frac{(-1)^{n+1}(n+1)!\Gamma(u-v+n+2)\Gamma(v)(v+n+1)}{\Gamma(u+2n+3)} \\
&+\frac{u+n+1}{v}\frac{(-1)^{n}(n+1)!(v+n+1)\Gamma(u-v+n+2)\Gamma(v+1)}{\Gamma(u+2n+4)} \\
= \ & \frac{(-1)^{n+1}(n+2)!(v+n+1)\Gamma(u-v+n+2)\Gamma(v)}{\Gamma(u+2n+4)}.
\end{aligned}
$$

Hence (A.5) holds for $n+1$. Next we shall show (A.1) for Case III. From (2.15), (A.2) and (A.4),

$$
\begin{aligned}
&\int_0^1 x^v(1-x)^{u-v}J_n^2(x;u,v)dx \\
= \ & (-1)^{n-1}n\frac{(u+n)_{n-1}}{(v)_{n-1}}\int_0^1 x^{v+n-1}(1-x)^{u-v}J_n(x;u,v)dx \\
&+(-1)^n\frac{(u+n)_n}{(v)_n}\int_0^1 x^{v+n}(1-x)^{u-v}J_n(x;u,v)dx \\
= \ & (-1)^{n-1}n\frac{(u+n)_{n-1}}{(v)_{n-1}}\frac{(-1)^n n!\Gamma(u-v+n+1)\Gamma(v)}{\Gamma(u+2n+1)} \\
&+(-1)^n\frac{\Gamma(u+2n)\Gamma(v)}{\Gamma(u+n)\Gamma(v+n)}\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix}\frac{(u+n)_k}{(v)_k}\int_0^1 x^{v+n+k}(1-x)^{u-v}dx \\
= \ & -\frac{n\Gamma(u-v+n+1)\Gamma^2(v)n!}{\Gamma(u+n)\Gamma(v+n-1)(u+2n)(u+2n-1)} \\
&+(-1)^n\frac{\Gamma(u+2n)\Gamma(v)}{\Gamma(u+n)\Gamma(v+n)}\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix}\frac{(u+n)_k}{(v)_k}\frac{\Gamma(v+n+k+1)\Gamma(u-v+1)}{\Gamma(u+n+k+2)}dx \\
= \ & -\frac{n\Gamma(u-v+n+1)\Gamma^2(v)n!}{\Gamma(u+n)\Gamma(v+n-1)(u+2n)(u+2n-1)} \\
&+(-1)^n\frac{\Gamma(u+2n)\Gamma(v)}{\Gamma(u+n)\Gamma(v+n)}\frac{(-1)^n(n+1)!(v+n)\Gamma(u-v+n+1)\Gamma(v)}{\Gamma(u+2n+2)} \\
= \ & \frac{(2n(u+n)+v(u-1))\Gamma^2(v)n!\Gamma(n+u-v+1)}{(u+2v+1)(u+2n)(u+2n-1)\Gamma(u+n)\Gamma(v+n)},
\end{aligned}
$$

where the fourth equality follows from (A.5). □

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