Existence of Solutions of Stochastic Differential Equations related to the Bessel process.

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Abstract

For applications in finance, we study the stochastic differential equation $dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s$ with β negative, g a continuous function vanishing at zero which satisfies a Hölder condition and δ a measurable and adapted stochastic process such that $\int_0^t \delta_u du < \infty$ for all $t \in \mathbb{R}^+$.
In this paper, we recall that there exists a unique strong solution. We give a construction of this solution and we prove that it is non-negative. The method we use is based on Yamada's (1978).

Key words
Stochastic differential equation, stochastic drift, Hölder condition, weak and strong solution, Euler scheme.

1 Introduction.

Suppose that a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ is given and that a Brownian motion $(B_t)_{t\geq 0}$ is defined on it. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is supposed to satisfy the usual hypothesis.

For applications in finance, we are interested in the existence and construction of a unique strong solution of the stochastic differential equation

$$dX_s = (2\beta X_s + \delta_s)ds + g(X_s)dB_s \quad \forall s \in I\!\!R^+ \tag{1}$$

with $\beta \leq 0$ and g a function vanishing at zero which satisfies the Hölder condition

$$|g(x) - g(y)| \leq b\sqrt{|x-y|},$$

and $\delta : \Omega \times I\!\!R^+ \to I\!\!R^+$ is a measurable and adapted stochastic process such that $\int_0^t \delta_u du < \infty$ for all $t \in I\!\!R^+$.

The stochastic differential equation (1) is a particular case of the Doléans-Dade and Protter's equation

$$X_t = K_t + \int_0^t f_s\left(\cdot, X_\cdot(\cdot)\right) dZ_s \tag{2}$$

where the driving process Z is a m-dimensional semimartingale, the coefficient f is a predictable process which depends on the path of X, and K is an adapted process with right-continuous paths with left-hand limits.

In our case, $(Z_s)_{s\geq 0}$ is the two-dimensional semimartingale $(s, B_s)_{s\geq 0}$; $K_s = \int_0^s \delta_u du$ and f_s only depends on X_s, namely $f_s\left(\cdot, X_\cdot(\cdot)\right) = (2\beta X_s, g(X_s))$.

Jacod and Memin (1980) have studied the existence and uniqueness of solutions of the Doléans-Dade and Protter's equation by introducing extensions of the given probability space. If C$(I\!\!R^+, I\!\!R)$ denotes the space of continuous sample paths, $\mathcal{C}_s$ the canonical σ-field and $\mathcal{C} = (\mathcal{C}_s)_{s\geq 0}$, then we take

$$\overline{\Omega} = \Omega \times C(I\!\!R^+, I\!\!R), \quad \overline{\mathcal{F}} = \mathcal{F} \otimes \mathcal{C}, \quad \overline{\mathcal{F}}_t = \bigcap_{s>t} \left(\mathcal{F}_s \otimes \mathcal{C}_s\right).$$

Jacod and Memin have proved that on this space $(\overline{\Omega}, \overline{\mathcal{F}})$, there exists a probability measure $\overline{I\!\!P}$ such that $\tilde{X}$ is a solution on $\overline{\Omega}$, which means that $\tilde{X}$ is $\overline{\mathcal{F}}$-adapted, that Z is a semimartingale on $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{I\!\!P})$ and that if $f(\tilde{X})$ is defined by $(\omega, \omega', t) \rightsquigarrow f(\tilde{X})_t(\omega, \omega') = f_t(\omega, \tilde{X}(\omega, \omega'))$, then one has $f(\tilde{X}) \in L(Z; \overline{\Omega}, (\overline{\mathcal{F}}_t)_{t\geq 0}, \overline{I\!\!P})$ and $\tilde{X} = K + f(\tilde{X}) \cdot Z$.
This solution-measure is strong if there exists a solution-process X on the space $(\Omega, \mathcal{F}, (\mathcal{F}^{I\!\!P})_{t\geq 0}, I\!\!P)$ such that $\overline{I\!\!P} = I\!\!P \circ \varphi^{-1}$ with $\varphi : \Omega \to \overline{\Omega}$ defined

by $\varphi(\omega) = (\omega, X(\omega))$.

Jacod and Memin found that the classical theorem of Yamada-Watanabe still holds in this more general situation. As pathwise uniqueness is easily shown as in Karatzas-Shreve (1988) on page 291 and Revuz-Yor (1991) on page 360, it is known that the stochastic differential equation (1) has a unique strong solution as soon as δ is a measurable and adapted process such that $\int_0^t \delta_u du < \infty$ for all $t \in \mathbb{R}^+$.

We will construct a solution by the method of finite differences on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^{\mathbb{P}})_{t\geq 0}, \mathbb{P})$. We will show that the approximating solution converges in L^1-supnorm towards the solution of the stochastic differential equation (1) and we check that this solution remains positive. We remark that the convergence also holds in the $\underline{\underline{H}}^1$-norm.

The method we use is the same as Yamada's (1978). However, in his paper the proof is restricted to the case

$$dX_s = b(s, X_s)ds + \sigma(s, X_s)dB_s$$

with b only depending on the time and the process itself. Our presentation is on some instances easier than Yamada's because we can benefit from the fact that $\beta \leq 0$. The case of a random drift satisfying a Lipschitz condition can be handled in the same way but the technicalities become more involved. For applications in finance, we limit ourselves to the case at hand.

Afterwards, we extend the result to stochastic differential equations with volatility defined by $|g(x)| = k\, x^\alpha$ for $1/2 \leq \alpha \leq 1$.

2 Proof of the existence by construction.

First, we prove that for a fixed time T, there exists a solution on the interval $[\![0, T]\!]$. We start with the extra assumption that $\int_0^T \delta_u\, du \leq K$. Afterwards, we generalize the result by using the uniqueness theorem and a localisation technique.

We remark that trivially $\int_0^T \mathbb{E}[\delta_u]\, du \leq K$. Let us define a function

$$\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ : \gamma(\nu) = \sup_{0\leq s\leq t\leq s+\nu\leq T} \int_s^t \mathbb{E}[\delta_u] du.$$

Since the function $u \mapsto \mathbb{E}[\delta_u]$ is integrable over the interval $[0, T]$, we have that $\gamma(\nu)$ converges to zero as ν tends to zero.

We divide the interval $[0,T]$ in order to apply a discretisation technique, known as the Euler scheme. For each $n \geq 1$, we take a subdivision

$$0 = t_0^n < t_1^n < \cdots < t_N^n = T$$

and denote this net by $\triangle_n$. For notational use, we drop the index n.

The mesh of the net is defined as $\|\triangle_n\| = \sup_{1\leq k\leq N} |t_k - t_{k-1}|$. We are working with a sequence of nets $(\triangle_n)_n$ such that the meshes are tending to zero. There is no need to suppose that $\triangle_n \subset \triangle_{n+1}$.

The solution of the stochastic differential equation turns out to be non-negative but the approximations we will need may take negative values. We therefore put $g'(x) = g(x)$ if $x \geq 0$ and $g'(x) = 0$ if $x \leq 0$. Remark that also g' satisfies

$$|g'(x) - g'(y)| \leq b\sqrt{|x-y|}.$$

If we are working with the net $\triangle_n$, we look at $X_{\triangle_n}(t)$, which we denote by $X_n(t)$. We put $X_n(0) = x_0$. For t taken between two netpoints, e.g. $t_k \leq t \leq t_{k+1}$, $k = 0, \cdots, N-1$, we define $X_n(t)$ as follows:

$$X_n(t) \quad = \quad X_n(t_k) + 2\beta X_n(t_k)(t - t_k) + \int_{t_k}^t \delta_u du + g'(X_n(t_k))(B_t - B_{t_k}).$$

We remark that if we denote $\eta_n(t) = t_k$ for $t_k \leq t < t_{k+1}$, then the terms telescope and we may write:

$$X_n(t) = x_0 + \int_0^t 2\beta X_n\left(\eta_n(u)\right) du + \int_0^t \delta_u du + \int_0^t g'\left(X_n\left(\eta_n(u)\right)\right) dB_u.$$

Since $X_n(0) = x_0$, $X_n(0)$ is bounded in L^2. By induction it is easy to show that for all $t \in [0,T]$, $X_n(t)$ is bounded in L^2.

We now prove that there exists an explicit bound, independent of n and t, for $I\!\!E\left[\,|X_n(\eta_n(t))|\,\right]$ with t such that $t_k \leq t \leq t_{k+1}$:

$$\begin{aligned} & I\!\!E\left[\,|X_n(\eta_n(t))|\,\right] \\ \leq \quad & x_0 + |2\beta| \int_0^t I\!\!E\left[\,|X_n(\eta_n(u))|\,\right] du \\ & + \int_0^t I\!\!E[\delta_u] du + I\!\!E\left[\left|\int_0^{t_k} g'\left(X_n(\eta_n(u))\right) dB_u\right|\right]. \end{aligned}$$

Since the L^2-norm exceeds the L^1-norm, we find by stochastic calculus and by the hypothesis that $g'(x) \leq b\sqrt{x} \leq b(1+x)$:

$$I\!\!E\left[|X_n(\eta_n(t))|\right] \leq \left(x_0 + \int_0^t I\!\!E[\delta_u] du + b\right) + (|2\beta| + b) \int_0^t I\!\!E\left[|X_n(\eta_n(u))|\right] du.$$

By Gronwall's inequality, we find that

$$\mathbb{E}\left[\,|X_n(\eta_n(t))|\,\right] \leq \left(x_0 + \int_0^t \mathbb{E}[\delta_u]du + b\right) e^{(|2\beta|+b)t} = G_t \leq G_T.$$

This upperbound is independent of n and t.

Moreover, we can find a bound for the norm $\|X_n(t) - X_n(\eta_n(t))\|_1$:

$$\mathbb{E}\left[\,|X_n(t) - X_n(\eta_n(t))|\,\right] \leq |2\beta| G_T \| \triangle_n \| + \gamma(\| \triangle_n \|) + b\sqrt{G_T}\sqrt{\| \triangle_n \|}.$$

Let us denote the right-hand side by $H_T(n)$. This bound is independent of t and tends to zero for n tending to infinity.

We will use these intermediate results to prove that $(X_n)_{n\geq 1}$ is a Cauchy sequence in $L^1_{C([0,T])} = \{f : \Omega \to C([0,T]) \mid f \text{ is Bochner integrable}\}$, with $C([0,T])$ the space of continuous functions from $[0,T]$ to $\mathbb{R}$. We will indeed show that if n and n' tend to infinity:

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_n(t) - X_{n'}(t)|\right] \longrightarrow 0.$$

The method we use is the same as Yamada's. We introduce a sequence of functions $\varphi_m(u),\ m = 1, 2, \cdots \in C^2((-\infty,\infty))$ tending to $|u|$ in an appropriate manner.

First, we search for a sequence of numbers $1 = a_0 > a_1 > \cdots > a_m > 0$ such that

$$\int_{a_1}^{a_0} \frac{du}{b\,u} = 1, \cdots, \int_{a_m}^{a_{m-1}} \frac{du}{b\,u} = m.$$

Obviously $a_m \longrightarrow 0$ for m going to infinity.
We define $\varphi_m(u),\ m = 1, 2, \cdots$ by $\varphi_m(u) = \Phi_m(|u|)$ with $\Phi_m(u)$ defined on $[0,\infty)$, $\Phi_m \in C^2([0,\infty))$ and $\Phi_m(0) = 0$ such that:

- $\Phi"_m(u) = \begin{cases} 0 & 0 \leq u \leq a_m \\ \text{between } 0 \text{ and } \frac{2}{mub} & a_m < u < a_{m-1} \\ 0 & u \geq a_{m-1} \end{cases}$
- $\int_{a_m}^{a_{m-1}} \Phi"_m(u)\,du = 1$
- $\Phi"_m(u)$ a continuous function.

If we integrate $\Phi"_m(u)$, we obtain:

$$\Phi'_m(u) = \begin{cases} 0 & 0 \leq u \leq a_m \\ \text{between } 0 \text{ and } 1 & a_m < u < a_{m-1} \\ 1 & u \geq a_{m-1} \end{cases}$$

Φ is then defined as the integral of Φ'.

Remark that $|u| - a_{m-1} \leq \varphi_m(u)$. Consequently, we have that $|X_n(t) - X_{n'}(t)| \leq a_{m-1} + \varphi_m\left(X_n(t) - X_{n'}(t)\right)$. We use this property to estimate the L^1-norm $\|X_n(t) - X_{n'}(t)\|_1$.

By Itô's lemma we obtain as far as the integrals exist (which will be proved below):

$$
\begin{aligned}
&I\!\!E\left[\varphi_m\left(X_n(t) - X_{n'}(t)\right)\right] \\
&= I\!\!E\left[\int_0^t \varphi'_m\left(X_n(u) - X_{n'}(u)\right) 2\beta\left(X_n(\eta_n(u)) - X_{n'}(\eta_{n'}(u))\right) du\right] \\
&\quad + I\!\!E\left[\int_0^t \varphi'_m\left(X_n(u) - X_{n'}(u)\right)\left(g'\left(X_n(\eta_n(u))\right) - g'\left(X_{n'}(\eta_{n'}(u))\right)\right) dB_u\right] \\
&\quad + \frac{1}{2} I\!\!E\left[\int_0^t \varphi''_m\left(X_n(u) - X_{n'}(u)\right)\left(g'\left(X_n(\eta_n(u))\right) - g'\left(X_{n'}(\eta_{n'}(u))\right)\right)^2 du\right].
\end{aligned}
$$

Let us investigate the first term. Since $|\varphi'_m| \leq 1$ a.e. and $I\!\!E\left[\,|X_n(\eta_n(u))|\,\right]$ is bounded independent of n and t, the integral exists. Using the facts that $|\varphi'_m| \leq 1$ and that φ is decreasing for $x \leq 0$ and increasing for $x \geq 0$, we find that

$$
\begin{aligned}
&I\!\!E\left[2\beta \int_0^t \varphi'_m\left(X_n(u) - X_{n'}(u)\right)\left(X_n(\eta_n(u)) - X_{n'}(\eta_{n'}(u))\right) du\right] \\
&\leq |2\beta| \int_0^t I\!\!E\left[\,|X_n(\eta_n(u)) - X_n(u)|\,\right] du + |2\beta| \int_0^t I\!\!E\left[\,|X_{n'}(u) - X_{n'}(\eta_{n'}(u))|\,\right] du \\
&\leq |2\beta|\, T\, H_T(n) + |2\beta|\, T\, H_T(n').
\end{aligned}
$$

We now treat the second term. Calculating the square of the L^2-norm, it is easy to prove that

$$
\left(\int_0^t \varphi'_m\left(X_n(u) - X_{n'}(u)\right)\left(g'\left(X_n(\eta_n(u))\right) - g'\left(X_{n'}(\eta_{n'}(u))\right)\right) dB_u\right)_{t \geq 0}
$$

is a martingale, bounded in L^2. Therefore the second term equals zero.

It remains to look at the last term. The integral exists since $\varphi''_m(u) \leq \frac{2}{m\,|u|\,b}$ and $\sup_u |\varphi''_m(u)| \leq \frac{2}{m\, a_m\, b}$. Furthermore,

$$
\begin{aligned}
&\frac{1}{2} I\!\!E\left[\int_0^t \varphi''_m\left(X_n(u) - X_{n'}(u)\right)\left(g'\left(X_n(\eta_n(u))\right) - g'\left(X_{n'}(\eta_{n'}(u))\right)\right)^2 du\right] \\
&\leq \frac{3}{2} I\!\!E\left[\int_0^t \frac{2}{m\,|X_n(u) - X_{n'}(u)|\,b} b^2\, |X_n(u) - X_{n'}(u)|\, du\right]
\end{aligned}
$$

$$+\frac{3}{2}\|\varphi"_m\|b^2 I\!E\left[\int_0^t (|(X_n(\eta_n(u)) - X_n(u)| + |(X_{n'}(\eta_{n'}(u)) - X_{n'}(u)|)\, du\right]$$

$$\leq \quad \frac{3T\,b}{m} + \frac{3}{2}\|\varphi"_m\|\, b^2 T\, H_T(n) + \frac{3}{2}\|\varphi"_m\|\, b^2 T\, H_T(n').$$

For given ε, let m be such that $0 < a_{m-1} < \frac{\varepsilon}{3}$ and $\frac{3T\,b}{m} < \frac{\varepsilon}{3}$. For this fixed m, $\|\varphi"_m\|$ is known to be bounded. So, we can look for a n_0 such that $(H_T(n) + H_T(n'))\,(\frac{3}{2}\|\varphi"_m\|b^2 + |2\beta|)T < \frac{\varepsilon}{3}$ for all $n,\ n' \geq n_0$ and for all $t \leq T$.

Summarizing, there exists a n_0 such that for all $n, n' \geq n_0$ and for all $t \leq T$

$$I\!E\,[\,|X_n(t) - X_{n'}(t)|\,] \leq a_{m-1} + I\!E\,[\varphi_m(X_n(t) - X_{n'}(t))] < \varepsilon.$$

We completed the proof that $(X_n)_{n\geq 1}$ is a Cauchy sequence in $L^1([0,T]\times\Omega)$. Since this space is complete, there is a process X in $L^1([0,T]\times\Omega)$ such that

$$\lim_{n\to\infty} X_n(t,\omega) = X(t,\omega)$$

Obviously, we also obtain in L^1

$$\lim_{n\to\infty} X_n(\eta_n(t),\omega) = X(t,\omega).$$

Thus, there exists a subsequence, still denoted by n, such that $\lim_{n\to\infty} X_n(t,\omega) = X(t,\omega)$ and $\lim_{n\to\infty} X_n(\eta_n(t),\omega) = X(t,\omega)$ a.e. for the measure $du \times dI\!P$.

We now proceed with the proof and try to estimate $I\!E\left[\sup_{0\leq t\leq T}|X_n(t) - X_{n'}(t)|\,\right]$ for n and n' tending to infinity (along the chosen subsequence):

$$I\!E\left[\sup_{0\leq t\leq T}|X_n(t) - X_{n'}(t)|\right]$$

$$\leq \quad |2\beta|\int_0^T I\!E\,[|X_n(\eta_n(u)) - X(u)|]\,du + |2\beta|\int_0^T I\!E\,[|X(u) - X_{n'}(\eta_{n'}(u))|]\,du$$

$$+\left\|\sup_{0\leq t\leq T}\left|\int_0^t g'(X_n(\eta_n(u))) - g'(X_{n'}(\eta_{n'}(u)))\,dB_u\right|\right\|_2.$$

By the previous results, the first two terms clearly tend to zero for n and n' tending to infinity. By the martingale inequality and the hypothesis that $g'(x) \leq b\sqrt{x}$, this is also true in case of the last term.

This completes the proof that $(X_n)_{n\geq 1}$ is a Cauchy sequence in $L^1_{C([0,T])}$. Since $L^1_{C([0,T])}$ is complete and since the norm is finer than the $L^1([0,T]\times\Omega)$ norm, the sequence converges to a process $du \times dI\!P$ a.e. equal to X and therefore still denoted by X. Thus $X_n(t)$ is a.e. uniformly convergent on [0,T] and

$$\lim_{n\to\infty} I\!E\left[\sup_{0\leq t\leq T}|X_n(t) - X(t)|\right] = 0.$$

We will now show that

$$X(t) = X_0 + \int_0^t \delta_u\,du + \int_0^t 2\beta X_u\,du + \int_0^t g'(X_u)dB_u.$$

Indeed,

$$\begin{aligned} &I\!\!E\left[\sup_{0\le t\le T}\left|X(t) - X_0 - \int_0^t \delta_u\,du - \int_0^t 2\beta X_u\,du - \int_0^t g'(X_u)dB_u\right|\right] \\ &= I\!\!E\left[\sup_{0\le t\le T}\left|X(t) - X_n(t) + \int_0^t 2\beta(X_n(\eta_n(u)) - X_u)du \right.\right. \\ &\qquad \left.\left. + \int_0^t \left(g'(X_n(\eta_n(u))) - g'(X_u)\right)dB_u\right|\right] \end{aligned}$$

and the result follows by the triangular inequality and the previous calculations.

We now prove that $X(t)$ is a non-negative process. Therefore, we introduce stopping-times:

$$\begin{aligned} \tau_1 &= \inf\{t \mid X_t < -\varepsilon\} \wedge T \\ \sigma_1 &= \sigma_1^1 \wedge \sigma_1^2 \wedge T \\ &\quad \text{with } \sigma_1^1 = \inf\{t \mid X_t < -2\varepsilon\} \\ &\quad \text{and } \sigma_1^2 = \inf\{t > \tau_1 \mid X_t = 0\} \end{aligned}$$

Trivially $\tau_1 < \sigma_1 \le T$. Let us define the set $A = \{\inf_{u\le T} X_u < -2\varepsilon\}$. On this set, $\sigma_1^2 < \sigma_1^1 \le T$ a.e., since if $\sigma_1^1 < \sigma_1^2$, then $\sigma_1 = \sigma_1^1$ and consequently $X_{\sigma_1} - X_{\tau_1} = -\varepsilon < 0$. But on the other hand,

$$X_{\sigma_1} - X_{\tau_1} = \int_{\tau_1}^{\sigma_1} \delta_u\,du + \int_{\tau_1}^{\sigma_1} 2\beta X_u\,du \ge 0.$$

We conclude that on A, $X_{\sigma_1} = 0$.
Let us define some more stopping-times:

$$\begin{aligned} \tau_2 &= \inf\{t > \sigma_1 \mid X_t < -\varepsilon\} \wedge T \\ \sigma_2 &= \sigma_2^1 \wedge \sigma_2^2 \wedge T \\ &\quad \text{with } \sigma_2^1 = \inf\{t \mid X_t < -2\varepsilon\} \\ &\quad \text{and } \sigma_2^2 = \inf\{t > \tau_2 \mid X_t = 0\} \end{aligned}$$

Analogously on A, $\tau_1 < \sigma_1 < \tau_2 < \sigma_2 \le T$ and $X_{\sigma_2} = 0$. Therefore, we can repeat this reasoning and conclude that on the set A, there exists a strict increasing sequence of stopping-times:$\tau_1 < \sigma_1 < \cdots < \tau_n < \sigma_n < \cdots \le T$.
Since all stopping-times are smaller than T, this sequence converges to a limit μ. All subsequences have to converge to the same limit μ. Thus $(\tau_n)_n \uparrow \mu$ and

$(\sigma_n)_n \uparrow \mu$. But for all n, $X_{\tau_n} = -\varepsilon$ and $X_{\sigma_n} = 0$ on A. Consequently $\mathbb{P}[A] = 0$ or equivalently $\mathbb{P}[\inf_{u \leq T} X_u < -2\varepsilon] = 0$. Since this is true for all $\varepsilon > 0$, we have proved that $\mathbb{P}[\inf_{u \leq T} X_u < 0] = 0$.

Because $X(t)$ is a non-negative process, we can replace g' by g:

$$X(t) = X_0 + \int_0^t \delta_u \, du + \int_0^t 2\beta X_u \, du + \int_0^t g(X_u) dB_u.$$

We have proved that there exists a solution of the stochastic differential equation (1) on the stochastic interval $[\![0, T]\!]$ under the assumption that $\int_0^T \delta_u du \leq K$.

Let us now look at the general case with the local assumption $\int_0^T \delta_u \, du < \infty$ a.e.. We define the sequence $(\sigma_n)_{n \geq 1}$ by $\sigma_n = \inf\{t \mid \int_0^t \delta_u \, du \geq n\} \wedge T$. We denote $\delta_u 1\!\!1_{[\![0,\sigma_n]\!]}$ by $\delta_u^{(n)}$. Since $\int_0^T \delta_u^{(n)} \, du \leq K$, the stochastic differential equation

$$dX_s^{(n)} = \left(2\beta X_s^{(n)} + \delta_s^{(n)}\right) + g(X_s^{(n)}) dB_s$$

has a unique solution $X^{(n)}$ on $[\![0, \sigma_n]\!]$. But on $[\![0, \sigma_n]\!]$, all $X^{(k)}, k \geq n$ are equal by the uniqueness of the solution of the stochastic differential equation (1). Since $\bigcup [\![0, \sigma_n]\!] \supset [\![0, T]\!]$, the result holds under the local assumption $\int_0^T \delta_u \, du < \infty$.

By the same reasoning, we find that on each interval $[\![0, l]\!]$ with $l > 0$, there exists a solution $X^{(l)}$. By uniqueness, the solutions $(X^{(l)})_{l > 0}$ have to be extensions of each other.

q.e.d.

Remark: The approximating solution converges in the $\underline{\underline{H}}^1$-norm towards the solution of the stochastic differential equation.
Indeed, let us recall from Protter (1990) that for a continuous semimartingale Z with $Z_0 = 0$, the $\underline{\underline{H}}^1$-norm is defined by:

$$\|Z\|_{\underline{\underline{H}}^1} = \| \, [N, N]_\infty^{1/2} + \int_0^\infty |dA_s| \, \|_{L^1}$$

where $Z = N + A$ is the decomposition of Z in its martingale part N and its predictable part A.
If we take for Z the difference $X_n - X_{n'}$ (as defined in the proof of the theorem), then one easily shows that $(X_n)_{n \geq 1}$ is a Cauchy sequence in the space of continuous semimartingales with the $\underline{\underline{H}}^1$-norm, which is complete. Since the $\underline{\underline{H}}^1$-norm is stronger than the L^1-supnorm (also denoted as the $\underline{\underline{S}}^1$-norm), $(X_n)_{n \geq 1}$ converges also in the $\underline{\underline{H}}^1$-norm towards X, the solution of the stochastic differential equation.

Remark: We can extend the result to stochastic differential equations with volatility defined by $|g(x)| = kx^\alpha$ for $\frac{1}{2} \leq \alpha \leq 1$. We define the sequence $(\sigma_h)_{h\geq 1}$ by $\sigma_h = \inf\{t \mid X_t^{(h)} \geq h\}$.The stochastic differential equation

$$dX_s^{(h)} = (2\beta X_s^{(h)} + \delta_s)ds + g_h(X_s^{(h)})dB_s$$

with $g_h(x) = kx^\alpha$ for $x \leq h$ and $g_h(x) = kh^\alpha$ for $x \geq h$ has a unique solution on the stochastic interval $[\![0, \sigma_h]\!]$ since the function g_h satisfies the Hölder condition $|g_h(x) - g_h(y)| \leq b_h\sqrt{x-y}$.
But on $[\![0, \sigma_h]\!]$ all solutions $X^{(m)}, m \geq h$ have to be the same by uniqueness. Since one can easily show that $I\!P\left[\sup_{0\leq t\leq T} X_t^{(h)} \geq h\right]$ converges to zero for h going to infinity, $\bigcup[\![0, \sigma_h]\!]=[\![0, \infty)$. Thus also in this case, there exists a unique strong solution on $I\!R^+ \times \Omega$.

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