MOD-φ CONVERGENCE

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Abstract. Using Fourier analysis, we study local limit theorems in weak-convergence problems. Among many applications, we discuss random matrix theory, some probabilistic models in number theory, the winding number of complex brownian motion and the classical situation of the central limit theorem, and a conjecture concerning the distribution of values of the Riemann zeta function on the critical line.

Notation and Preliminaries

Our random variables will take values in $\mathbb{R}^d$, a fixed $d-$dimensional space, and we denote by $|t|$ the Euclidian norm in $\mathbb{R}^d$. We will use the Landau and Vinogradov notations $f = O(g)$ and $f \ll g$ in some places; these are equivalent statements, and mean that there exists a constant $c \geq 0$ such that

$$|f(x)| \leq cg(x)$$

for all x in a set X which is indicated. Any suitable value of c is called "an implied constant", and it may depend on further parameters.

For $t = (t_1, \ldots, t_d) \in \mathbb{R}^d$, and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, the inner product is denoted by $t \cdot x = t_1 x_1 + \ldots + t_d x_d$.

A sequence of probability measures $(\mu_n)_n$ on $\mathbb{R}^d$ converges weakly to a probability measure if for all bounded continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$, we have $\lim_n \int f\,d\mu_n = \int f\,d\mu$. Equivalently we can ask that the convergence holds for C^∞ functions with compact support. Lévy's theorem asserts that this is equivalent to the pointwise convergence of the characteristic functions $\int \exp(i\,t \cdot x)\,d\mu_n \to \int \exp(i\,t \cdot x)\,d\mu$. Lévy's theorem can be phrased as follows: if φ_n is the sequence of characteristic functions of probability measures μ_n, if $\varphi_n(t)$ converges pointwise to a function φ, if this convergence is continuous at the point 0, then φ is a characteristic function of a probability measure μ, μ_n converges weakly to μ and the convergence of φ_n to φ is uniform on compact sets of $\mathbb{R}^d$. We recall that the convergence is continuous if $x_n \to 0$ in $\mathbb{R}^d$ implies $\varphi_n(x_n) \to \varphi(0) = 1$.

We say that a sequence of random variables converges in law if the image measures (or laws) converge weakly. Most of the time one needs a scaling of the sequence. This is for instance the case in the central limit theorem, which in an elementary form says that for a sequence of independent identically distributed real-valued random variables, $(X_n)_n$, $\mathbb{E}[X_n] = 0, \mathbb{E}[X_n^2] = 1$, the normalised (or rescaled) sequence $\frac{X_1 + \ldots + X_n}{\sqrt{n}}$ converges weakly to the standard Gaussian law.

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In the applications below we will use different kinds of scaling. In the higher-dimensional case, we will scale the random variables using a sequence of linear isomorphisms (or non-degenerate matrices) $A_n\colon \mathbb{R}^d \to \mathbb{R}^d$. The inverse of these matrices will be denoted by $\Sigma_n\colon \mathbb{R}^d \to \mathbb{R}^d$. The transpose of a linear map or matrix A is denoted by A^*.

If B is a finite set, we note $\#B$ for the cardinality of B.

Our methods are based on Fourier analysis and we will use basic facts from this theory freely. We define the Fourier transform as is usually done in probability theory, namely

$$\hat{f}(t) = \int_{\mathbb{R}^d} \exp(i\, t \cdot x) f(x)\, dx.$$

The inversion formula is, at least when $\hat{f} \in L^1(\mathbb{R}^d)$, given by

$$f(x) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} \exp(-it \cdot x)\hat{f}(t)\, dt.$$

In particular, when μ is a probability measure with an integrable characteristic function φ, we get that μ is absolutely continuous with respect to Lebesgue measure m, and its density is given by

$$\frac{d\mu}{dm}(x) = \left(\frac{1}{2\pi}\right)^d \int \exp(-it \cdot x)\varphi(t)\, dt,$$

which is therefore continuous.

1. Introduction

In [14], the notion of *mod-gaussian convergence* was introduced: intuitively, it corresponds to a sequence of random variables X_n that – through the Fourier lens – "look like" a sum $G_n + Y_n$ where (G_n) is a sequence of gaussian variables with arbitrary variance and (Y_n) is a convergent sequence independent from G_n. However, most interest lies in cases where this simple-minded decomposition does not exist: what remains is the existence of a limiting function Φ, not necessarily a Fourier transform of a probability measure, such that the limit theorem

$$\lim_{n\to+\infty} \mathbb{E}[e^{itG_n}]^{-1}\mathbb{E}[e^{itX_n}] = \Phi(t) \tag{1}$$

holds, locally uniformly, for $t \in \mathbb{R}$.

Mod-Gaussian convergence as described in (1) in fact appears in a variety of situations: for instance it holds for some probabilistic models in number theory (see [14], [19]) as well as for some statistics related to function field L-functions ([14], [19]); as outlined in [8], it is also the good framework when combined with the theory of dependency graphs to study for instance sums of partially dependent random variables or subgraph counts in the Erdös-Rényi random graphs. Many more examples and applications (e.g. to random matrix theory or to non-commutative probability) can be found in [8, 14, 19]. It is clear that mod-Gaussian convergence implies the central limit theorem but it also implies other refinements of the central limit theorem such as precise moderate and large deviations (see [8]) or local limit theorems as explained in [19]. More precisely we showed in [19], in a multidimensional extension of (1), that if one makes extra assumptions on the rate of convergence in (1) and on the size of the limiting function Φ, one can deduce a local limit theorem for the sequence (X_n) with an error estimate. The main goal in [19] was to understand a conjecture of Ramachandra on the value distribution of the Riemann zeta function on the critical line, namely that the set of values of $\zeta(1/2+iu)$, $u \in \mathbb{R}$, is dense in $\mathbb{C}$. The methods developed in [19] allowed us to prove the analogue result for the characteristic polynomial of random unitary matrices, for finite field L-functions. As far as Ramachandra's conjecture is concerned, we were only able to show that a suitable uniform version of the Keating-Snaith moments conjecture implies it, which can be considered as a very strong assumption. The main goal of this paper is to show that Ramachandra's conjecture would follow from a much weaker assumption on the Fourier transform of $\log \zeta(1/2 + it)$ (see section 3.9 for the detailed discussion).

Our main idea is that proving Ramachandra's conjecture amounts to showing a local limit theorem for $\log \zeta(1/2 + it)$, $t \in \mathbb{R}$. Indeed in classical probability theory, it is well known that under reasonable assumptions, if a central limit theorem type result holds for sums of i.i.d. random variables $(X_n)_{n\geq 1}$, then one also has a local limit theorem, i.e. asymptotics for $\mathbb{P}(X_1 + \cdots + X_n \in B)$ for B a Jordan measurable set when $n \to \infty$. Since $\log \zeta(1/2 + it)$ satisfies a central limit theorem (Selberg's theorem), we wish to have a general framework including the classical setting of sums of i.i.d. random variables in which convergence in law (or central limit theorem) combined with some reasonable assumptions (on characteristic functions) implies a local limit theorem. In this spirit, we introduce in this paper a notion of "convergence" where the reference law is not necessarily Gaussian but a fairly general probability law, with integrable characteristic function φ. Under suitable conditions, we are able to prove a general local limit theorem which extends the result recently found in [19, Th. 4]: the local limit theorems in [19] are obtained under more restrictive conditions but they are also more accurate since they provide an error term.

To illustrate the power and flexibility of this approach, we mention a conditional result (where the condition to check is much weaker than the moments conjecture) in the direction of Ramachandra's conjecture as well as two other results which, to the best of our knowledge, are new and which outline the variety of situations in which our result can be applied:

Theorem 1. *If for all $k > 0$ there exists $C_k \geq 0$ such that*

$$\left|\frac{1}{T}\int_0^T \exp(it \cdot \log \zeta(1/2 + iu))du\right| \leq \frac{C_k}{1 + |t|^4(\log \log T)^2} \tag{2}$$

for all $T \geq 1$ and t with $|t| \leq k$, then for any bounded Jordan-measurable subset $B \subset \mathbb{C}$, we have

$$\lim_{T\to+\infty} \frac{\frac{1}{2}\log\log T}{T} m(u \in [0, T] \mid \log \zeta(1/2 + iu) \in B) = \frac{m(B)}{2\pi}.$$

Remark 1. The above theorem shows that the method developed in this paper can be considered as a new systematic method of interest in number theoretical problems dealing with sets of zero density. We also illustrate this fact with some more simple examples. Other applications to finite field L-functions can be found in [19].

Theorem 2 (Local limit theorem for the winding number of complex Brownian motion)**.** *For $u \geq 0$, let θ_u denote the argument or winding number of a complex brownian motion W_u such that $W_0 = 1$. Then for any real numbers $a < b$, we have*

$$\lim_{u\to\infty} \frac{\log u}{2} \mathbb{P}\left[a < \theta_u < b\right] = \frac{1}{\pi}(b - a).$$

This is proved in Section 3.2.

Theorem 3 (Local limit theorem for unitary matrices)**.** *For $n \geq 1$, let g_n denote a random matrix which is Haar-distributed in the unitary group $U(n)$. Then for any bounded Borel subset $B \subset \mathbb{C}$ with boundary of Lebesgue measure 0, and for any $b \in \mathbb{C}$, we have*

$$\lim_{n\to+\infty} \Big(\frac{\log n}{2}\Big)\mathbb{P}\left[\log\det(1-g_n) - \frac{\log n}{2}\ b \in B\right] = \frac{1}{2\pi}e^{-|b|^2/2}m(B),$$

where $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{C}$.

This, together with similar facts for the other families of classical compact groups, is proved in Section 3.7.

We emphasize that the last two applications are examples; this paper contains quite a few more, and it seems certain that many more interesting convergence theorems can be proved or understood using the methods of this paper.

2. Mod$-\varphi$ convergence

2.1. **Definition.** We now explain our generalization of the definition in [14]. First of all, we fix $d \geq 1$ and a probability measure μ on $\mathbb{R}^d$. We then assume given a sequence (X_n) of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $\mathbb{R}^d$. We define φ_n to be the characteristic function of X_n. We now consider the following properties:

- **H1.** The characteristic function φ of the probability measure μ is integrable; in particular, μ has a density $d\mu/dm$, with respect to Lebesgue measure m.
- **H2.** There exists a sequence of linear automorphisms $A_n \in \mathrm{GL}_d(\mathbb{R})$, with inverses $\Sigma_n = A_n^{-1}$, such that Σ_n converges to 0 and $\varphi_n(\Sigma_n^* t)$ converges continuously at 0 (or what is equivalent: uniformly on compact sets) to $\varphi(t)$. In other words, the renormalized random variables $\Sigma_n(X_n)$ converge in law to μ.
- **H3.** For all $k \geq 0$, the sequence

  $$f_{n,k} = \varphi_n(\Sigma_n^* t)\mathbf{1}_{|\Sigma_n^* t|\leq k}$$

  is *uniformly integrable* on $\mathbb{R}^d$; since $f_{n,k}$ are uniformly bounded in L^1 and L^∞ (for fixed k), this is equivalent to the statement that, for all $k \geq 0$, we have

$$\lim_{a\to+\infty}\sup_{n\geq 1}\int_{|t|\geq a}|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|\Sigma_n^* t|\leq k}m(dt) = 0. \tag{3}$$

Remark 2. Property **H1** excludes discrete probability laws, such as Poisson random variables. However, similar ideas do apply for such cases. We refer to [18] (for the case of Poisson distributions) and to [2] (for much more general discrete distributions, where earlier work of Hwang [11] is also relevant) for these developments.

Remark 3. Property **H3** will typically be established by proving an estimate of the type

$$|\varphi_n(\Sigma_n^* t)| \leq h(t) \tag{4}$$

for all $n \geq 1$ and all $t \in \mathbb{R}^d$ such that $|\Sigma_n^* t| \leq k$, where $h \geq 0$ is an integrable function on $\mathbb{R}^d$ (which may depend on k).

We give a name to sequences with these properties:

Definition 1 (Mod-φ convergence)**.** If μ is a probability measure on $\mathbb{R}^d$ with characteristic function φ, X_n is a sequence of $\mathbb{R}^d$-valued random variables with characteristic functions φ_n, if the properties **H1**, **H2**, **H3** hold, we say that there is *mod*$-\varphi$ *convergence* for the sequence X_n.

Below, we will comment further on the hypotheses, and in particular give equivalent formulations of **H3**. In Section 3.1, we also explain the relation with conditions arising in classical convergence theorems.

To make the link with the original definition in [14], i.e., the assumption that a limit formula like (1) holds, we observe that mod-φ convergence will hold when **H1** is true and we have

- **H2'.** There exists a sequence of linear automorphisms $A_n \in \mathrm{GL}_d(\mathbb{R})$, with inverses $\Sigma_n = A_n^{-1}$, such that Σ_n converges to 0, and there exists a continuous function $\Phi : \mathbb{R}^d \to \mathbb{C}$ such that for arbitrary $k > 0$

$$\varphi_n(t) = \Phi(t)\varphi(A_n^* t)(1 + o(1)) \tag{5}$$

uniformly for t such that $|\Sigma_n^* t| \leq k$.

In many applications considered in this paper (not all), this stronger condition holds, or is expected to hold. It is very likely that, when this is the case, the "limiting function" Φ also carries significant information, as discussed already in special cases in [14, §4].

2.2. **Local limit theorem.** We now state and prove our main result, which is a local limit theorem that shows that, when mod$-\varphi$ convergence holds, the expectations $\mathbb{E}\left[f(X_n)\right]$ (for reasonable functions f) are well-controlled: they behave like

$$|\det(A_n)|^{-1} \frac{d\mu}{dm}(0) \int_{\mathbb{R}^d} f \, dm$$

as n goes to infinity. The proof of our main result (Theorem 5) is based on the following approximation theorem, which is also stated in a work by Bretagnolle and Dacunha-Castelle [5]. Since their proof is only sketched there in dimension 1 and since we are not entirely satisfied with the arguments for the lower bound, we provide here a detailed proof.

Theorem 4. *Suppose $f\colon \mathbb{R}^d \to \mathbb{R}$ is a continuous function with compact support. Then for each $\eta > 0$ we can find two integrable functions $g_1, g_2\colon \mathbb{R}^d \to \mathbb{R}$ such that*

1. *$\widehat{g_1}, \widehat{g_2}$ have compact support,*
2. *$g_2 \leq f \leq g_1$,*
3. *$\int_{\mathbb{R}^d} (g_1 - g_2)(t)\, dt \leq \eta$.*

Proof. We prove the theorem for f^+ and f^- separately and hence without loss of generality we may assume $f \geq 0$. Now let $k > 0$ be such that the support of f, denoted *suppf* is included in $[-k, k]^d$.

We first explain how to construct g_1. Let $\varepsilon > 0$ and take

$$\lambda = \varepsilon \mathbf{1}_{[-k-2,k+2]^d}.$$

For $R > 0$, we then define the kernel K_R as follows:

$$K_R(x) = C_4 \prod_{j=1}^{d} \frac{\sin^4(Rx_j)}{R^3 x_j^4},$$

where C_4 is a normalizing constant independent of R, chosen so that $\int_{\mathbb{R}^d} K_R(x)dx = 1$. The Fourier transform of K_R is a convolution of

$$\prod_{j=1}^{d}\left(1-\frac{|t_j|}{2R}\right)^{+}$$

with itself and hence has support in $[-4R, 4R]^d$. Next we consider the convolution product $K_R \star (f+\lambda)$. Since K_R is an approximation of the identity and since f and λ are continuous on $[-k-2, k+2]^d$, we have that

$$K_R \star (f+\lambda) \to f+\lambda$$

uniformly on $[-k-1, k+1]^d$. Consequently for R large enough and $x \in [-k-2, k+2]^d$, we have

$$K_R \star (f+\lambda)(x) \geq (f+\lambda)(x) - \varepsilon \geq f(x).$$

Outside $[-k-1, k+1]^d$ we have $K_R \star (f+\lambda)(x) \geq 0 = f(x)$. Summarizing, we have for R large enough

$$g_1 = K_R \star (f+\lambda) \geq f.$$

To find g_2 we could start with $f-\lambda$ but there is no guarantee that $K_R \star (f-\lambda) \leq 0$ outside $[-k-1, k+1]^d$. Consequently we need to introduce an extra correction. Let us observe that for all R large enough, we already have for $x \in [-k-1, k+1]^d$:

$$K_R \star (f-\lambda)(x) \leq (f-\lambda)(x) + \varepsilon \leq f(x).$$

The correction we make uses the fact that $K_R \star f$ is small outside $[-k-1, k+1]^d$. Let us define the kernel H_R as follows:

$$H_R = C_2 \prod_{j=1}^{d} \frac{\sin^2(Rx_j)}{Rx_j^2},$$

where C_2 is a normalizing constant, independent of R, chosen such that $\int_{\mathbb{R}^d} H_R(x)dx = 1$.

Now set $\theta = \varepsilon \mathbf{1}_{[-k,k]^d}$ and consider $H_R \star \theta$ for R large enough. We claim that outside $[-k-1, k+1]^d$, we have

$$K_R \star (f-\lambda) - H_R \star \theta \leq 0.$$

Indeed this can be proved by direct calculation:

$$K_R \star f - \ H_R \star \theta \leq K_R \star ||f||_\infty \mathbf{1}_{[-k,k]^d} - H_R \star \varepsilon \mathbf{1}_{[-k,k]^d}.$$

So we only need to show that on the complement of $[-k-1, k+1]^d$,

$$||f||_\infty K_R \star \mathbf{1}_{[-k,k]^d} - \varepsilon H_R \star \mathbf{1}_{[-k,k]^d} \leq 0.$$

Now take $x \notin [k-1, k+1]^d$ and for simplicity assume that $|x_1| > k+1$. We then get

$$\begin{aligned}||f||_\infty \int_{\mathbb{R}^d} K_R(x-y)\mathbf{1}_{[-k,k]^d}(y)dy - \varepsilon \int_{\mathbb{R}^d} H_R(x-y)\mathbf{1}_{[-k,k]^d}(y)dy \\ = \int_{[-k,k]^d} \left(||f||_\infty K_R(x-y) - \varepsilon H_R(x-y)\right) dy.\end{aligned}$$

But for $|x_1| \geq k+1$ and $y \in [-k, k]^d$ the integrand is negative. Indeed,

$$
\begin{aligned}
& C_4||f||_\infty \prod_{j=1}^d \frac{\sin^4 R(x_j-y_j)}{R^3(x_j-y_j)^4} - C_2\varepsilon \prod_{j=1}^d \frac{\sin^2 R(x_j-y_j)}{R(x_j-y_j)^2} \\
\leq\ & C_4||f||_\infty \frac{\sin^4 R(x_1-y_1)}{R^3(x_1-y_1)^4} \prod_{j\geq 2} \frac{\sin^2 R(x_j-y_j)}{R(x_j-y_j)^2} - C_2\varepsilon \prod_{j=1}^d \frac{\sin^2 R(x_j-y_j)}{R(x_j-y_j)^2} \\
\leq\ & \prod_{j\geq 2} \frac{\sin^2 R(x_j-y_j)}{R(x_j-y_j)^2} \left(C_4||f||_\infty \frac{\sin^4 R(x_1-y_1)}{R^3(x_1-y_1)^4} - C_2\varepsilon \frac{\sin^2 R(x_1-y_1)}{R(x_1-y_1)^2} \right).
\end{aligned}
$$

Now note that the second factor has the same sign as

$$
\left(C_4||f||_\infty \frac{\sin^2 R(x_1-y_1)}{R^2(x_1-y_1)^2} - C_2\varepsilon \right).
$$

Now, since $|y_1| \leq k$ and $|x_1| > k+1$, we have that $(x_1-y_1)^2 \geq 1$. Therefore, this last factor is smaller than

$$
C_4||f||_\infty \frac{1}{R^2} - C_2\varepsilon
$$

which is negative for R large enough. Summarizing we have for R large enough

$$
g_2 = K_R \star (f-\lambda) - H_R \star \theta.
$$

We have thus constructed g_1 and g_2, such that $g_2 \leq f \leq g_1$ with $\hat{g}_1$ and $\hat{g}_2$ compactly supported. And clearly g_1 and g_2 are integrable.

Finally we check that

$$
\begin{aligned}
\int_{\mathbb{R}^d} (g_1-g_2)(x)dx &= \int_{\mathbb{R}^d} \left(K_R \star (f+\lambda)(x) - (K_R \star (f-\lambda)(x) - H_R \star \theta(x))\right) dx \\
&= \int_{\mathbb{R}^d} \left(2K_R \star \lambda(x) + H_R \star \theta(x)\right) dx \\
&\leq 2\int_{\mathbb{R}^d} \lambda(x)dx + \int_{\mathbb{R}^d} \theta(x)dx \\
&\leq 2\varepsilon(2(k+2))^d + \varepsilon(2k)^d,
\end{aligned}
$$

and this completes the proof.

□

Theorem 5 (Local limit theorem for mod-φ convergence)**.** *Suppose that mod–φ convergence holds for the sequence X_n. Then we have*

$$
|\det(A_n)|\,\mathbb{E}\left[f(X_n)\right] \to \frac{d\mu}{dm}(0) \int_{\mathbb{R}^d} f\, dm,
$$

for all continuous functions with compact support. Consequently we also have

$$
|\det(A_n)|\,\mathbb{P}\left[X_n \in B\right] \to \frac{d\mu}{dm}(0)\, m(B). \tag{6}
$$

for relatively compact Borel sets $B \subset \mathbb{R}^d$ with $m(\partial B) = 0$, or in other words for bounded Jordan-measurable sets $B \subset \mathbb{R}^d$.

Proof. We first assume that f is continuous, bounded, integrable and that $\hat{f}$ has compact support; using Theorem 4, the case of a general continuous function with compact support will follow easily. We write

$$\mathbb{E}[f(X_n)] = \int_{\mathbb{R}^d} f(x) d\mu_n(x)$$

where μ_n is the law of X_n. Applying the Parseval formula transforms this into

$$\mathbb{E}[f(X_n)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \varphi_n(t) \hat{f}(-t)\, dt.$$

By the linear change of variable $t = \Sigma_n^* s$, we get

$$\mathbb{E}[f(X_n)] = (2\pi)^{-d} |\det(\Sigma_n)| \int_{\mathbb{R}^d} \varphi_n(\Sigma_n^* s) \hat{f}(-\Sigma_n^* s)\, ds.$$

Now fix k so that the support of $\hat{f}$ is contained in the ball of radius k; we then have

$$\mathbb{E}[f(X_n)] = (2\pi)^{-d} |\det(\Sigma_n)| \int_{|\Sigma_n^* s| \le k} \varphi_n(\Sigma_n^* s) \hat{f}(-\Sigma_n^* s)\, ds.$$

The integrand converges pointwise to $\varphi(s)\hat{f}(0)$ according to the assumption **H2**. The condition **H3** of uniform integrability then implies the convergence in L^1. One can see this quickly in this case: for any $\varepsilon > 0$, and for any $a > 0$ large enough, we have

$$\int_{|s|>a} |\varphi_n(\Sigma_n^* s) \mathbf{1}_{|\Sigma_n^* s| \le k} \hat{f}(-\Sigma_n^* s)|\, ds \le \|\hat{f}\|_\infty \int_{|s|>a} |\varphi_n(\Sigma_n^* s) \mathbf{1}_{|\Sigma_n^* s| \le k}|\, ds < \varepsilon$$

for all n by (3). On $|s| \le a$, the pointwise convergence is dominated by $\|\hat{f}\|_\infty \mathbf{1}_{|s| \le a}$, hence

$$\int_{|s| \le a} \varphi_n(\Sigma_n^* s) \mathbf{1}_{|\Sigma_n^* s| \le k} \hat{f}(-\Sigma_n^* s)\, ds \to \hat{f}(0) \int_{|s| \le a} \varphi(s) ds.$$

For a large enough, this is $\hat{f}(0) \int \varphi$, up to error ε, hence we get the convergence

$$\int_{|\Sigma_n^* s| \le k} \varphi_n(\Sigma_n^* s) \hat{f}(-\Sigma_n^* s)\, ds \to \hat{f}(0) \int_{\mathbb{R}^d} \varphi(s) ds.$$

Finally, this leads to

$$|\det(A_n)| \mathbb{E}[f(X_n)] \to (2\pi)^{-d} \hat{f}(0) \int_{\mathbb{R}^d} \varphi(s) ds = \frac{d\mu}{dm}(0) \int_{\mathbb{R}^d} f(s) ds,$$

which concludes the proof for f integrable and with $\hat{f}$ with compact support.

Now if f is continuous with compact support, we use Theorem 4: by linearity, we can assume f to be real-valued, and then, given $\eta > 0$ and $g_2 \le f \le g_1$ as in the approximation theorem, we have

$$|\det(A_n)| \mathbb{E}[g_2(X_n)] \le |\det(A_n)| \mathbb{E}[f(X_n)] \le |\det(A_n)| \mathbb{E}[g_1(X_n)],$$

and hence

$$|\det(A_n)| \mathbb{E}[g_2(X_n)] - \frac{d\mu}{dm}(0) \int g_2 - \frac{d\mu}{dm}(0) \int (g_1 - g_2) \le |\det(A_n)| \mathbb{E}[f(X_n)] - \frac{d\mu}{dm}(0) \int f$$

and

$$|\det(A_n)|\mathbb{E}[f(X_n)] - \frac{d\mu}{dm}(0)\int f \leq |\det(A_n)|\mathbb{E}[g_1(X_n)] - \frac{d\mu}{dm}(0)\int g_1 + \frac{d\mu}{dm}(0)\int (g_1 - g_2)$$

and hence

$$\limsup_n ||\det(A_n)|\mathbb{E}[f(X_n)] - \frac{d\mu}{dm}(0)\int f| \leq \eta$$

which proves the result since $\eta > 0$ is arbitrary. The proof of (6) is performed in standard ways. ∎

Remark 4. To illustrate why our results are generalisations of the local theorems, let us analyse a particularly simple situation. We assume that $d = 1$ and that the random variables X_n have characteristic functions φ_n such that $\varphi_n(t/b_n)$ converge to $\varphi(t)$ in $L^1(\mathbb{R})$, with $b_n \to +\infty$ (such situations are related, but less general, than the classical results discussed in Section 3.1 or in [5] and [26].) In that case, the density functions f_n of X_n/b_n exist, are continuous and converge (in $L^1(\mathbb{R})$ and uniformly) to a continuous density function f. For a bounded interval (α, β), we obtain

$$\begin{aligned} b_n\mathbb{P}[X_n \in (\alpha, \beta)] &= b_n\mathbb{P}[X_n/b_n \in (\alpha/b_n, \beta/b_n)] \\ &= b_n \int_{\alpha/b_n}^{\beta/b_n} f_n(x)\, dx \to f(0)(\beta - \alpha), \end{aligned}$$

by elementary calculus.

It may be worth remarking explicitly that it is quite possible for this theorem to apply in a situation where the constant $\frac{d\mu}{dm}(0)$ is zero. In this case, the limit gives some information, but is not as precise as when the constant is non-zero. For instance, consider the characteristic function $\varphi(t) = 1/(1 - it)^2$, which corresponds to the sum $E_1 + E_2$ of two independent exponential random variables with density $e^{-x}dx$ on $[0, +\infty[$. An easy computation shows that the density for φ itself is xe^{-x} (supported on $[0, +\infty[$), and for $X_n = n(E_1 + E_2)$, we have mod-φ convergence with $A_n t = nt$, leading to the limit

$$\lim_{n \to +\infty} n\mathbb{P}[\alpha < X_n < \beta] = 0$$

for all $\alpha < \beta$. Note that any other limit $c(\beta - \alpha)$ would not make sense here, since X_n is always non-negative, whereas there is no constraint on the signs of α and β.

However, in similar cases, the following general fact will usually lead to more natural results:

Proposition 1 (Mod-φ convergence and shift of the mean)**.** *Let $d \geq 1$ be an integer, and let (X_n) be a sequence of $\mathbb{R}^d$-valued random variables such that there is mod-φ convergence with respect to the linear maps A_n. Let $\alpha \in \mathbb{R}^d$ be arbitrary, and let $\alpha_n \in \mathbb{R}^d$ be a sequence of vectors such that*

$$\lim_{n \to +\infty} \Sigma_n \alpha_n = \alpha, \tag{7}$$

for instance $\alpha_n = A_n \alpha$. Then the sequence $Y_n = X_n - \alpha_n$ satisfies mod-ψ convergence with parameters A_n for the characteristic function

$$\psi(t) = \varphi(t)e^{-it\cdot\alpha}.$$

In particular, for any continuous function f on $\mathbb{R}^d$ with compact support, we have

$$\lim_{n\to+\infty} |\det(A_n)|\mathbb{E}[f(X_n-\alpha_n)] = \frac{d\mu}{dm}(\alpha)\int_{\mathbb{R}^d} f(x)dx,$$

where μ is the probability measure with characteristic function φ, and for any bounded Jordan-measurable subset $B\subset\mathbb{R}^d$, we have

$$\lim_{n\to+\infty} |\det(A_n)|\mathbb{P}[X_n-\alpha_n\in B] = \frac{d\mu}{dm}(\alpha)m(B). \tag{8}$$

Proof. This is entirely elementary: ψ is of course integrable and since

$$\mathbb{E}[e^{itY_n}] = \varphi_n(t)e^{-it\cdot\alpha_n},$$

we have $\mathbb{E}[e^{i\Sigma_n^* t\cdot Y_n}] = \varphi_n(\Sigma_n^* t)e^{-it\cdot\Sigma_n\alpha_n}$, which converges locally uniformly to $\psi(t)$ by our assumption (7). Since the modulus of the characteristic function of Y_n is the same, at any point, as that of X_n, Property **H3** holds for (Y_n) exactly when it does for (X_n), and hence mod-ψ convergence holds. If $h = d\mu/dm$, the density of the measure with characteristic function ψ is $g(x) = h(x+\alpha)$, and therefore the last two limits hold by Theorem 5. ☐

In the situation described before the statement, taking $\alpha_n = cn$ with $c>0$ leads to the (elementary) statement

$$\lim_{n\to+\infty} n\mathbb{P}[\alpha+cn<X_n<\beta+cn] = ce^{-c}(\beta-\alpha).$$

Even when the density of μ does not vanish at 0, limits like (8) are of interest for all $\alpha\neq 0$.

Another easy and natural extension of the local limit theorem involves situations where a further linear change of variable is performed:

Proposition 2 (Local limit theorem after linear change of variable)**.** *Suppose that (X_n) satisfies mod-φ convergence relative to A_n and Σ_n. Suppose that (T_n) is a sequence of linear isomorphisms $T_n\colon\mathbb{R}^d\to\mathbb{R}^d$ such that $\Sigma_nT_n\to 0$. Suppose also that the following balancedness condition holds: there is a constant C such that $|(\Sigma_nT_n)^*t|\leq 1$ implies that $|\Sigma_n^*t|\leq C$. Then the sequence $T_n^{-1}X_n$ also satisfies the conditions of the theorem, and in particular for any bounded Jordan measurable set B we have*

$$\frac{|\det(A_n)|}{|\det(T_n)|}\,\mathbb{P}\left[X_n\in T_nB\right]\to\frac{d\mu}{dm}(0)\,m(B).$$

Proof. Let us put $\tilde{X}_n = T_n^{-1}X_n$ and $\tilde{\varphi}_n(t) = \mathbb{E}[\exp(i\,t.T_n^{-1}X_n)] = \varphi_n((T_n^{-1})^*t)$. Clearly the sequence Σ_nT_n tends to zero and $\Sigma_nT_n(\tilde{X}_n) = \Sigma_nX_n$ tends to μ in law. The only remaining thing to verify is the uniform integrability condition. Let us look at

$$\tilde{\varphi}_n((\Sigma_nT_n)^*t)\mathbf{1}_{|(\Sigma_nT_n)^*t|\leq k} = \varphi_n((\Sigma_n)^*t)\mathbf{1}_{|(\Sigma_nT_n)^*t|\leq k}.$$

Because of the balancedness condition we get that

$$\mathbf{1}_{|(\Sigma_nT_n)^*t|\leq k}\leq\mathbf{1}_{|(\Sigma_n)^*t|\leq Ck}.$$

The rest is obvious. ☐

Remark 5. The balancedness condition is always satisfied if $d=1$. In dimension $d>1$, there are counterexamples. In case the ratio of the largest singular value of Σ_n to its smallest singular value is bounded, the balancedness condition is satisfied (this is an easy linear algebra exercise). See also [19] for the use of such conditions in mod-Gaussian convergence. To see that for $d = 2$ it is not necessarily satisfied take the following sequences:

$$\Sigma_n = \begin{pmatrix} n^{-1/4} & 0 \\ 0 & n^{-1/2} \end{pmatrix}, \qquad T_n = \begin{pmatrix} 0 & n^{-1} \\ n^{1/4} & 0 \end{pmatrix}.$$

2.3. **Conditions ensuring mod-φ convergence.** We now derive other equivalent conditions, or sufficient ones, for mod-φ convergence. First of all, the conditions **H1**, **H2**, **H3** have a probabilistic interpretation. We suppose $d = 1$ to keep the presentation simple. Instead of taking the indicator function $\mathbf{1}_{|\Sigma_n^* t| \le k}$, we could have taken the triangular function Δ_k defined as $\Delta_k(0) = 1, \Delta_k(2k) = 0 = \Delta(-2k)$, $\Delta_k(x) = 0$ for $|x| \ge 2k$ and Δ_k is piecewise linear between the said points. The function Δ_1 is the characteristic function of a random variable Y (taken independent of the sequence X_n). Hence we get that the sequence X_n satisfies $\mathbf{H1}, \mathbf{H2}, \mathbf{H3}$ if and only if for each $k \ge 1$, the characteristic functions of $Z_n = \Sigma_n \left(X_n + \frac{1}{k} Y \right)$ converge in $L^1(\mathbb{R}^d)$ to φ. Indeed the characteristic function of Z_n equals $\varphi_n(\Sigma_n t) \Delta_k(\Sigma_n t)$. There is no need to use the special form of the random variable Y.

In fact we have the following:

Theorem 6. *Suppose that for the sequence X_n the conditions* **H1, H2** *hold. The condition* **H3** *holds as soon as there is a random variable, V, independent of the sequence X_n such that for each $\varepsilon > 0$, $\mathbb{E}\left[\exp\left(it\Sigma_n\left(X_n + \varepsilon V\right)\right)\right]$ tends to φ in $L^1(\mathbb{R}^d)$.*

Proof. Let $\psi(t) = \mathbb{E}\left[\exp\left(it \cdot V\right)\right]$ be the characteristic function of V. The hypothesis of the theorem is equivalent to the property that for each $\varepsilon > 0$, the sequence

$$\varphi_n(\Sigma_n^* t)\psi(\varepsilon \Sigma_n^* t),$$

is uniformly integrable. Let $\delta > 0$ be such that for $|t| \le \delta$, $|\psi(t)| \ge 1/2$. Then the uniform integrability of the above mentioned sequence implies for each $\varepsilon > 0$, the uniform integrability of the sequence

$$|\varphi_n(\Sigma_n^* t)| \mathbf{1}_{\varepsilon |\Sigma_n^* t| \le \delta} \le 2 \left| \varphi_n(\Sigma_n^* t) \psi(\varepsilon \Sigma_n^* t) \right|.$$

This ends the proof. ∎

Remark 6. We suppose that $d = 1$. For higher dimensions the discussion can be made along the same lines but it is much more tricky. Polya's theorem says that if $\gamma \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function such that $\gamma(0) = 1$, $\lim_{x \to +\infty} \gamma(x) = 0$, then there is a random variable Y (which can be taken to be independent of the sequence X_n, such that $\varphi_Y(t) = \gamma(|t|)$. The characteristic function of $\Sigma_n(X_n + Y)$ is then $\gamma(|\Sigma_n t|)\varphi_n(\Sigma_n t)$ and hence is a uniformly integrable sequence. Since $\|\Sigma_n\| \to 0$, we see that $\Sigma_n(X_n + Y)$ tends in law to μ with characteristic function φ. The convergence is much stronger than just weak convergence. In fact the random variables $\Sigma_n(X_n + Y)$ have densities and because the characteristic functions tend in L^1 to φ, the densities of $\Sigma_n(X_n + Y)$ converge to the density of μ in the topology of $L^1(\mathbb{R})$.

Remark 7. Adding a random variable Y can be seen as a regularisation (mollifier). Indeed adding an independent random variable leads to a convolution for the densities. In our context this means that the distribution of X_n is convoluted with an integrable kernel (the density of Y). The regularity of the law of Y is then passed to the law of $X_n + Y$. In probability theory such a mollifier is nothing else than adding an independent random variable with suitable properties.

We can go one step further and replace the condition for each k by a condition where we use just one random variable. This is the topic of the next theorem.

Theorem 7. *Suppose that the hypotheses* **H1, H2** *hold. Then* **H3** *is also equivalent to either of the following:*

(1) *There exists a non-increasing function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$, such that $\gamma(0) = 1$, $0 < \gamma \leq 1$,*

$$\lim_{x \to +\infty} \gamma(x) = 0$$

and such that the sequence $\gamma(|\Sigma_n^* t|)\varphi(\Sigma_n^* t)$ is uniformly integrable.

(2) *There exists a non-increasing convex function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$, such that $\gamma(0) = 1$, $0 < \gamma \leq 1$,*

$$\lim_{x \to +\infty} \gamma(x) = 0$$

and such that the sequence $\gamma(|\Sigma_n^* t|)\varphi(\Sigma_n^* t)$ is uniformly integrable.

Proof. It is quite clear that (1) or (2) imply **H3**, since for any $k > 0$, we obtain

$$|\varphi_n(\Sigma_n^* t)| \mathbf{1}_{|\Sigma_n^* t| \leq k} \leq \frac{1}{\gamma(k)} \gamma(|\Sigma_n^* t|)|\varphi(\Sigma_n^* t)|,$$

and therefore the desired uniform integrability.

For the reverse, it is enough to show that **H3** implies (2), since (1) is obviously weaker. For $x \geq 0$ we define

$$g(x) = \sup_n \|\varphi_n(\Sigma_n^* t)\mathbf{1}_{|\Sigma_n^* t| \leq x+1}\|_{L^1(\mathbb{R}^d)}.$$

The function g is clearly non-decreasing. Let us first observe that there is a constant $C > 0$ such that for x big enough, $\exp\left(-\int_0^x g(s)\, ds\right) \leq \exp(-Cx)$. We define

$$\gamma(x) = \alpha \int_x^\infty \exp\left(-\int_0^u g(s)\, ds\right) du,$$

where α is chosen so that $\gamma(0) = 1$ (since the integrals converge, this function is well defined).

The function γ is also convex and tends to zero at ∞. Furthermore

$$\gamma(x) \leq \alpha \int_x^\infty \frac{g(u)}{g(x)} \exp\left(-\int_0^u g(s)\, ds\right) du \leq \frac{\alpha}{g(x)} \exp\left(-\int_0^x g(s)\, ds\right)$$

from which it follows that $\int_0^\infty \gamma(x) g(x)\, dx < \infty$. We now claim that $\gamma(|\Sigma_n^* t|)\varphi_n(\Sigma_n^* t)$ is uniformly integrable. Because the sequence is uniformly bounded we only need to show that for each $\varepsilon > 0$ there is a k such that

$$\int_{\mathbb{R}^d} \gamma(|\Sigma_n^* t|)|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|t| \geq k}\, dt \leq \varepsilon$$

for all $n \geq 1$.

For k, K integers, we split the integral as follows

$$\begin{aligned}\int_{\mathbb{R}^d} \gamma(|\Sigma_n^* t|)|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|t|\geq k}\, dt &\leq \int_{\mathbb{R}^d} \gamma(|\Sigma_n^* t|)|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|t|\geq k}\mathbf{1}_{|\Sigma_n^* t|\leq K}\, dt \\ &\quad + \int_{\mathbb{R}^d} \gamma(|\Sigma_n^* t|)|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|t|\geq k}\mathbf{1}_{|\Sigma_n^* t|> K}\, dt \\ &\leq \int_{\mathbb{R}^d} |\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|t|\geq k}\mathbf{1}_{|\Sigma_n^* t|\leq K}\, dt \\ &\quad + \int_{\mathbb{R}^d} \gamma(|\Sigma_n^* t|)|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|\Sigma_n^* t|> K}\, dt.\end{aligned}$$

The last term is dominated as follows:

$$\begin{aligned}\int_{\mathbb{R}^d} \gamma(|\Sigma_n^* t|)|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|\Sigma_n^* t|> K}\, dt &\leq \sum_{l\geq K} \gamma(l) \int_{l\leq |\Sigma_n^* t|\leq l+1} |\varphi_n(\Sigma_n^* t)|\, dt \\ &\leq \sum_{l\geq K} \gamma(l) g(l) \\ &\leq \alpha \sum_{l\geq K} \exp\left(-\int_0^l g(s)\, ds\right),\end{aligned}$$

which can be made smaller than $\varepsilon/2$ by taking K big enough. Once K fixed we use the uniform integrability of the sequence $\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|\Sigma_n^* t|\leq K}$ and take k big enough so that we get for each n:

$$\int_{\mathbb{R}^d} |\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|t|\geq k}\mathbf{1}_{|\Sigma_n^* t|\leq K}\, dt \leq \varepsilon/2.$$

This completes the proof. □

In particular, we get a sufficient condition:

Corollary 1. *Suppose that the sequence X_n satisfies the following:*

1. **H1, H2** *hold;*
2. *There is a non-decreasing function $c\colon \mathbb{R} \to \mathbb{R}_+$, $c(0) = 1$ as well as an integrable function $h\colon \mathbb{R}^d \to \mathbb{R}_+$ such that $|\varphi_n(t)| \leq h(A_n^* t)\, c(|t|)$ for all n and $t \in \mathbb{R}^d$.*

Then the property **H3** *holds as well.*

Proof. This is clear from the previous theorem, since $|\varphi_n(t)| \leq h(A_n^* t)\, c(|t|)$ for all t implies that for all t we have

$$\frac{|\varphi_n(\Sigma_n^* t)|}{c(|\Sigma_n^*(t)|)} \leq h(t),$$

which verifies (1) in Theorem 7. □

3. Applications

In this section, we collect some examples of mod-φ convergence, for various types of limits φ, and therefore derive local limit theorems. Some of these results are already known, and some are new. It is quite interesting to see all of them handled using the relatively elementary framework of the previous section. The coming subsections are mostly independent of each other; the first few are of probabilistic nature, while the last ones involve arithmetic considerations.

3.1. **The Central Limit Theorem and convergence to stable laws.** In this section we suppose that $(X_n)_{n\geq 1}$ is a sequence of independent identically distributed random variables. The central limit theorem deals with convergence in law of expressions of the form $\frac{X_1+\ldots+X_n}{b_n} - a_n$, where b_n are normalising constants. We will suppose without further notice that the random variables are *symmetric* so that we can suppose $a_n = 0$. The possible limit laws have characteristic functions of the form $\exp(-c\,|u|^p)$, where $0 < p \leq 2$ and where $c > 0$. For information regarding this convergence we refer to Loève [21]. The basis for the theory is Karamata's theory of regular variation. In this section we are interested in expressions of the form $\lim b_n \mathbb{P}[(X_1 + \ldots + X_n) \in B]$ for suitably bounded Borel sets B.

For the case $\mathbb{E}[X^2] < \infty$, the problem was solved by Shepp [24]. The multidimensional square integrable case was solved by Borovkov and Mogulskii [4] and Mogulskii [23]. The case $p < 2$ was solved by Stone [26] and at the same by Bretagnolle and Dacunha-Castelle [5] (see also Ibragimov and Linnik [13]). Such theorems are known as *local limit theorems*.

Theorem 8. *Suppose that the non-lattice random variable X is symmetric and that it is in the domain of attraction of a stable law with exponent p. More precisely we suppose that $\frac{X_1+\ldots X_n}{b_n}$ converges in law to a probability distribution with characteristic function* $\exp(-|t|^p)$, $0 < p \leq 2$. *Then for Jordan-measurable bounded Borel sets, we have*

$$\lim_{n\to+\infty} b_n \mathbb{P}[(X_1 + \ldots + X_n) \in B] = c_p\, m(B),$$

where $c_p = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-|t|^p)\, dt$. Suppose moreover that $0 < \tau_n \to +\infty$ in such a way that $\frac{b_n}{\tau_n} \to +\infty$, then

$$\lim_{n\to+\infty} \frac{b_n}{\tau_n} \mathbb{P}\left[\frac{X_1 + \ldots + X_n}{\tau_n} \in B\right] = c_p\, m(B).$$

In order to prove this theorem, we first observe that, when **H1** and **H2** are satisfied, the condition **H3** of uniform integrability is equivalent with classical conditions that arise in the current context.

Theorem 9. *Under the hypotheses* **H1, H2**, *the hypothesis* **H3** *is equivalent to the validity of the following two conditions:*

- **H3'.** *For all $k \geq \varepsilon > 0$, we have*

$$\lim_{n\to+\infty} |\det(A_n)| \int_{\varepsilon\leq|t|\leq k} |\varphi_n(t)|\, dt = 0.$$

- **H4'.** *For all $\eta > 0$. there is $a \geq 0, \varepsilon > 0$ such that*

$$\limsup_{n\to+\infty} \int_{a\leq|t|;|\Sigma_n^* t|\leq\varepsilon} |\varphi_n(\Sigma_n^* t)|\, dt \leq \eta.$$

Proof. First suppose that **H3** holds, i.e., for each $k > 0$, $\varphi_n(\Sigma_n^* t)\mathbf{1}_{|\Sigma_n^* t|\leq k}$ is uniformly integrable. Since $\Sigma_n \to 0$, we immediately get

$$\lim_n \int_{\varepsilon\leq|t|\leq k} |\det(A_n)\varphi_n(t)|\, dt = \int_{\varepsilon\leq|\Sigma_n^* s|\leq k} |\varphi_n(\Sigma_n^* s)|\, ds \to 0,$$

which is **H3'**. To establish **H4'**, let us first remark that (using **H2**) we have

$$\lim_n \varphi_n(\Sigma_n^* t)\mathbf{1}_{|\Sigma_n^* t|\leq k} = \varphi(t)$$

for all t. Then we take $a > 0$ such that for given $\eta > 0$ we have

$$\int_{|t|\geq a} |\varphi(t)|\, dt \leq \eta.$$

Take now $\varepsilon > 0$ and observe that by uniform integrability

$$\lim_n \int_{a\leq|t|;|\Sigma_n^* t|\leq\varepsilon} |\varphi_n(\Sigma_n^* t)|\, dt = \int_{|s|\geq a} |\varphi(s)|\, ds \leq \eta.$$

Now we proceed to the converse and we suppose that **H1, H2, H3', H4'** hold. We first show that given $\eta > 0$, the sequence $\varphi_n(\Sigma_n^* t)\mathbf{1}_{|\Sigma_n^* t|\leq k}$ has up to η all its mass on a ball of radius a. Given $\eta > 0$ we can find $a, \varepsilon > 0$ such that

$$\limsup_n \int_{a\leq|t|;|\Sigma_n^* t|\leq\varepsilon} |\varphi_n(\Sigma_n^* t)|\, dt \leq \eta.$$

Then according to **H3'** and **H4'**, we can find n_0 such that for all $n \geq n_0$ we have

$$\int_{a\leq|t|;|\Sigma_n^* t|\leq\varepsilon} |\varphi_n(\Sigma_n^* t)|\, dt \leq 2\eta,$$

$$\int_{\varepsilon\leq|\Sigma_n^* t|\leq k} |\varphi_n(\Sigma_n^* t)|\, dt \leq \eta.$$

Increasing a allows us to suppose that the same inequalities hold for all $n \geq 1$. So we get that

$$\int_{|\Sigma_n^* t|\leq k;|t|\geq a} |\varphi_n(\Sigma_n^* t)|\, dt \leq 3\eta.$$

Since the sequence is uniformly bounded we have proved uniform integrability. ☐

Proof of Theorem 8. We have here $\varphi_n = \psi^n$ where ψ is the characteristic function of a random variable in the domain of attraction of a stable law. Property **H3'** follows since the sequence φ_n tends to zero exponentially fast, uniformly on compact sets of $\mathbb{R}^d \setminus \{0\}$. Moreover, Property **H4'** is known as Gnedenko's condition (see Gnedenko and Kolmogorov [9] or the discussion of I_2 (resp. I_3) in Ibragimov and Linnik [13, p. 123]) (resp. [13, p. 127]). Thus the hypotheses **H1**, **H2**, **H3'**, **H4'** are fulfilled in this setting. ☐

Remark 8. (1) The proof of Property **H4'** is based on the regular variation of ψ around 0. The fact that regular variation is needed suggest that it is difficult to get a more abstract version of this property.

(2) Taking the most classical case where $p = 2$ and (X_n) independent and identically distributed, it is easy to check that the stronger condition **H2'** (i.e., (5)) is *not* valid, except if the X_n are themselves Gaussian random variables. Thus the setting in this paper is a genuine generalization of the original mod-Gaussian convergence discussed in [14].

3.2. **The winding number of complex Brownian motion.** We take a complex Brownian Motion W, starting at 1. Of course we can also see W as a two-dimensional real BM. With probability one, the process W will never attain the value 0 and hence, by continuous extension or lifting, we can define the argument θ. We get $W_u = R_u \exp(i\theta_u)$ where $\theta_0 = 0$ and $R_u = |W_u|$. The process θ is called the *winding number*, see [22]. Spitzer in [25] computed the law of θ_u and gave its Fourier transform, and a more precise convergence result was given in [3].

The characteristic function is given by

$$\mathbb{E}\left[\exp(it\theta_u)\right] = \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{1}{4u}\right)^{1/2}\exp\left(-\frac{1}{4u}\right)\left(I_{(|t|-1)/2}\left(\frac{1}{4u}\right) + I_{(|t|+1)/2}\left(\frac{1}{4u}\right)\right)$$

where $I_\nu(z)$ denotes the I-Bessel function, which can be defined by its Taylor expansion

$$I_\nu(z) = \sum_{m\geq 0}\frac{1}{m!\Gamma(\nu+m+1)}\left(\frac{z}{2}\right)^{\nu+2m}.$$

Using elementary properties of Bessel functions, Spitzer deduced that $\frac{2\theta_u}{\log u}$ converges to a Cauchy law with characteristic function $\varphi(t) = \exp(-|t|)$ and density $\frac{1}{\pi}\frac{1}{1+t^2}$.

Theorem 10 (Mod-Cauchy convergence of the winding number)**.** *For any sequence (u_n) of positive real numbers tending to infinity, the sequence $X_n = \theta_{u_n}$ satisfies mod-φ convergence with $d = 1$, $\varphi(t) = \exp(-|t|)$, $A_n(t) = A_n^*(t) = (\log u_n)t/2$.*

In particular, for any real numbers $a < b$, we have

$$\lim_{u\to\infty}\frac{\log u}{2}\,\mathbb{P}\left[a < \theta_u < b\right] = \frac{1}{\pi}(b-a).$$

Although this is a very natural statement, we have not found this local limit theorem in the literature.

Proof. The conditions **H1** and **H2** of mod-φ convergence are clear, the second by Spitzer's Theorem. To check the uniform integrability condition **H3**, we take $k \geq 1$ and we proceed to bound

$$|\varphi_n(\Sigma_n^* t)|\mathbf{1}_{|\Sigma_n^* t|\leq k}$$

for $t \geq 0$. But if $|\Sigma_n^* t| \leq k$, we have

$$-\frac{1}{2} \leq \frac{|\Sigma_n^* t| \pm 1}{2} \leq \frac{k+1}{2}$$

and $0 \leq \frac{1}{4u_n} \leq 1$ for n large enough. The Taylor series expansion shows immediately that there exists $C_k \geq 0$ such that

$$|I_\nu(z)| \leq C_k$$

uniformly for ν real with $-1/2 \leq \nu \leq \frac{k+1}{2}$ and $z \in \mathbb{C}$ with $|z| \leq 1$, so that for $|\Sigma_n^* t| = 2|t|/(\log u_n) \leq k$, we have

$$|\varphi_n(\Sigma_n^* t)| \leq B_k u_n^{-1/2} \leq B_k \exp\Big(-\frac{|t|}{k}\Big)$$

where $B_k = C_k(\pi/2)^{1/2}$. This gives the desired uniform integrability, in the form (4). ☐

3.3. **"Relaxed" Poisson variables.** We present here a special case of a phenomenon which is related to Poisson approximation and therefore probably quite general: if P_n, $n \geq 1$, denotes a Poisson-distributed random variable with parameter λ_n going to infinity, the sequence

$$X_n = \frac{P_n - \lambda_n}{\lambda_n^{1/3}}$$

satisfies mod$-\varphi$ convergence with $d = 1$, $\varphi(t) = e^{-t^2/2}$ (i.e., for a standard gaussian) and $A_n t = \lambda_n^{1/6} t$. Indeed, **H2** holds because

$$\varphi_n(t) = e^{-i\lambda_n^{2/3}t} \exp(\lambda_n(e^{it/\lambda_n^{1/3}} - 1)) = \exp(-\lambda_n^{1/3}t^2/2)\exp(-it^3/6)(1 + o(1))$$

as n tends to infinity. Moreover, the next term in the expansion of the exponential $e^{it/\lambda_n^{1/3}}$ shows that

$$\left|\varphi_n\left(\frac{t}{n^{1/6}}\right)\right| = e^{-t^2/2}\left(1 + O\left(\frac{|t|^4}{\lambda_n}\right)\right)$$

and the uniform integrability condition **H3** therefore holds even for the range $|t| \leq \lambda_n^{1/4}$. (Except for **H3**, this example was considered in [18, Prop. 2.4].)

As a consequence, we get the local limit

$$\lim_{n\to+\infty} \lambda_n^{1/6}\mathbb{P}[a\lambda_n^{1/3} + \lambda_n < P_n < b\lambda_n^{1/3} + \lambda_n] = \frac{b-a}{\sqrt{2\pi}}$$

for any fixed $a < b$.

Remark 9. Using the formula [18, (4.9)], we see that the same mod-φ convergence property holds when P_n is replaced with ω_n defined as the number of cycles in the decomposition in cycles of a uniformly chosen random permutation in the symmetric group on n letters, with $\lambda_n = \log n$. These are well-known (see, e.g., [1, 18]) to be well-approximated by Poisson variables with these parameters.

3.4. **Dedekind Sums.** In this section we give an application to Dedekind sums. Our limit theorems are based on the estimates in Vardi's paper [27]. Let us recall the definition of Dedekind sums. We recall the standard notation

$$\begin{aligned}\lfloor x \rfloor &= \sup\{n \in \mathbb{Z} \mid n \leq x\},\\ ((x)) &= x - \lfloor x \rfloor - 1/2, \text{ if } x \notin \mathbb{Z}\\ &= 0 \text{ if } x \in \mathbb{Z}.\end{aligned}$$

For natural numbers $0 < d < c$ with $\gcd(d, c) = 1$, the Dedekind sum is defined as

$$s(d,c) = \sum_{0<k<c} \left(\left(\frac{kd}{c}\right)\right)\left(\left(\frac{d}{c}\right)\right).$$

For every $N \in \mathbb{N}$ we define the finite probability space:

$$\begin{aligned}\Omega_N &= \{(d,c) \mid 0 < d < c < N; \gcd(d,c) = 1\},\\ \mathbb{P}_N[A] &= \frac{\#A}{\#\Omega_N} \text{ the normalised counting measure},\\ X_N(d,c) &= s(d,c).\end{aligned}$$

The distribution of X_N is symmetric as easily seen by using the measure preserving transformation $(d,c) \to (c-d,c)$. It is well known that $\#\Omega_N / \left(\frac{3N^2}{\pi^2}\right) \to 1$, see e.g section 3.4 of this paper. Vardi [27, Prop. 2] proved an asymptotic formula which implies the following:

Proposition 3. *For $0 \leq |t| \leq 1/4$ we have that:*

$$|\varphi_N(2\pi t)| \leq C N^{-|t|} + o(N^{-1/3}),$$

where C is an absolute constant and where the last term is uniform in t.

Remark 10. The result of [27] actually gives the same result for larger values of t, but the error term is only smaller than the main term when $|t| < 2/3$.

As a consequence of the same proposition in [27], we get that for $t \in \mathbb{R}$:

$$\varphi_N\left(\frac{2\pi t}{\log N}\right) \to \varphi(t) = \exp(-|t|),$$

the characteristic function of a standard Cauchy random variable with density $\frac{1}{\pi(1+x^2)}$.

The bound given by Vardi does not allow to show a mod$-\varphi$ (in this case “mod-Cauchy”) convergence, but it suffices to obtain the following weaker statement:

Proposition 4. *For any sequence (τ_N) such that $\tau_N \to +\infty$ and $\frac{\log N}{\tau_N} \to +\infty$, the sequence $\frac{X_N}{\tau_N}$ satisfies mod-φ convergence with $A_N t = \frac{\log N}{2\pi\tau_N} t$. Hence, for every bounded Jordan-measurable set $B \subset \mathbb{R}$, we have*

$$\frac{\log N}{2\pi\tau_N}\mathbb{P}_N\left[\frac{X_N}{\tau_N} \in B\right] \to \frac{1}{\pi} m(B).$$

Proof. We only have to show that for each k, the sequence

$$\varphi_N(2\pi t/\log N)\mathbf{1}_{\frac{2\pi|t|\tau_N}{\log N} \leq k}$$

is uniformly integrable. This is seen as follows: if $\frac{2\pi|t|\tau_N}{\log N} \leq k$ and if N is big enough, then $|t|/\log N \leq 1/2$. Consequently for N large enough (depending on k), we get

$$\left|\varphi_N(2\pi t/\log N)\mathbf{1}_{\frac{2\pi|t|\tau_N}{\log N} \leq k}\right| \leq C\exp(-|t|) + \psi_N(t),$$

where $\|\psi_N\|_1 \leq C N^{-1/3}\log N$. This implies uniform integrability of the sequence. $\square$

Remark 11. The more precise local limit theorem

$$\frac{\log N}{2\pi}\mathbb{P}_N\left[X_N \in B\right] \to \frac{1}{\pi} m(B)$$

is in fact valid, as proved by Bruggeman [6]. Our methods, only using mod-φ convergence, do not seem to lead to this result.

3.5. **The ζ-distribution.** The $\zeta-$distributions are purely atomic, infinitely divisible, probability distributions, denoted μ^σ, which were considered by Khintchine and studied in more detail in [20].

The measure μ^σ is defined for $\sigma > 1$ as the measure supported on the points $\{-\log(n) \mid n \in \mathbb{N}; n \geq 1\}$, such that

$$\mu^\sigma(-\log n) = \frac{n^{-\sigma}}{\zeta(\sigma)}$$

for $n \geq 1$. Its characteristic function is then given by

$$\varphi^\sigma(t) = \sum_{n\geq 1} \frac{n^{-\sigma}}{\zeta(\sigma)} e^{it(-\log n)} = \frac{\zeta(\sigma+it)}{\zeta(\sigma)}.$$

The limit of interest here is when $\sigma \downarrow 1$. Since the zeta function can be written

$$\zeta(s) = \frac{\zeta^*(s)}{s-1},$$

where $\zeta^*(s)$ defines an entire function of $s \in \mathbb{C}$ (i.e., the zeta function has only a simple pole with residue 1 at $s = 1$), the behavior of $\varphi^\sigma(t)$ is easy to understand, namely

$$\varphi^\sigma(t) = \frac{\zeta(\sigma+it)}{\zeta(\sigma)} = \frac{1}{1+\frac{it}{\sigma-1}} \frac{\zeta^*(\sigma+it)}{\zeta^*(\sigma)}.$$

Thus, if X^σ are random variables with law μ^σ, we see that $(\sigma-1)X^\sigma$ converges in law to a "negative" exponential distribution supported on $]-\infty, 0]$ with density e^x. The characteristic function is not integrable, hence we cannot apply our results. To work around this, we consider independent copies X_1^σ, X_2^σ of random variables having the law μ^σ, and define

$$Y^\sigma = X_1^\sigma - X_2^\sigma.$$

These random variables have characteristic function given by

$$|\varphi^\sigma(t)|^2 = \frac{|\zeta(\sigma+it)|^2}{\zeta(\sigma)^2} = \frac{1}{1+\frac{t^2}{(\sigma-1)^2}} \frac{|\zeta^*(\sigma+it)|^2}{\zeta^*(\sigma)^2}$$

and hence $(\sigma-1)Y^\sigma$ converges in law, as $\sigma \downarrow 1$, to a double exponential (or Laplace) distribution, with characteristic function $\varphi(t) = \frac{1}{1+t^2}$ and density $\frac{1}{2}\exp(-|x|)$. Thus conditions **H1** and **H2** are now satisfied (in the version of a continuous limit $\sigma \downarrow 1$). Moreover, if $1 < \sigma \leq 2$ and t ranges over the set where $|(\sigma-1)t| \leq k$, for $k > 0$ fixed, the values of

$$\frac{|\zeta^*(\sigma+it(\sigma-1))|^2}{\zeta^*(\sigma)^2}$$

vary in a bounded set. This shows that **H3** also holds, and we can apply Theorem 5; it follows that

$$\lim_{\sigma\downarrow 1} \frac{1}{\sigma-1} \mathbb{P}[a < Y^\sigma < b] = \frac{1}{2}(b-a).$$

for all $-\infty < a < b < +\infty$. We can make this limit explicit: indeed, Y^σ takes values of the form $\log(k) - \log(n) = \log(k/n)$ where $k, n \geq 1$. The probability that $Y^\sigma = \log(k/n)$ for $(k,n) = 1$ (i.e. k and n are coprime) is easily seen to be

$$\mathbb{P}[Y^\sigma = \log(k/n)] = \sum_{m\geq 1} \frac{(km)^{-\sigma}(nm)^{-\sigma}}{\zeta(\sigma)^2} = k^{-\sigma}n^{-\sigma}\frac{\zeta(2\sigma)}{\zeta(\sigma)^2}.$$

Hence the limit becomes, for $0 < \alpha < \beta$:

$$\frac{\zeta(2\sigma)}{(\sigma-1)\zeta(\sigma)^2} \sum_{\substack{(k,n)=1 \\ \alpha<k/n<\beta}} k^{-\sigma}n^{-\sigma} \to \frac{1}{2}\log\left(\frac{\beta}{\alpha}\right)$$

as $\sigma \downarrow 1$, which is equivalent (since $\zeta(2) = \pi^2/6$ and $\zeta(\sigma) \sim (\sigma - 1)^{-1}$) to

$$(\sigma - 1) \sum_{\substack{(k,n)=1 \\ \alpha < k/n < \beta}} k^{-\sigma} n^{-\sigma} \to \frac{3}{\pi^2} \log\left(\frac{\beta}{\alpha}\right).$$

We could not find any reference to this statement, so it might be new (although it could certainly be proved with more traditional methods.)

3.6. **Squarefree integers.** This section is motivated by a recent paper of Cellarosi and Sinai [7], who discuss a natural probabilistic model of random squarefree integers. As we will see, some of its properties fall into the framework of mod-φ convergence, with a very non-standard characteristic function φ.

The set-up, in a slightly different notation than the one used in [7], is the following. We fix a probability space Ω that is big enough to carry independent copies of random variables η_p, with index p running over the prime numbers, with the following distribution laws:

$$\mathbb{P}[\eta_p = 1] = \mathbb{P}[\eta_p = -1] = \frac{p}{(p+1)^2}, \qquad \mathbb{P}[\eta_p = 0] = \frac{p^2+1}{(p+1)^2}.$$

We consider the random variables

$$X_n = \sum_{p \leq p_n} \eta_p \log p, \qquad Q_n = \exp(X_n)$$

for $n \geq 1$, where p_n is the n-th prime number.

The link with [7] is the following: in the notation of [7, Th. 1.1], the distribution of X_n is the same as that of the difference

$$(\zeta_n - \zeta_n') \log p_n$$

of two independent copies ζ_n and ζ_n' of the random variables variables

$$\zeta_n = \sum_{p \leq p_n} \nu_p \log p$$

of [7, Th. 1.1], where the ν_p are independent Bernoulli variables with

$$\mathbb{P}[\nu_p = 0] = \frac{1}{p+1}, \qquad \mathbb{P}[\nu_p = 1] = \frac{p}{p+1}.$$

These random variables ν_p are very natural in studying squarefree numbers. Indeed, a simple computation shows that ν_p is the limit in law, as $x \to +\infty$, of the Bernoulli random variables $\nu_{p,x}$ defined by

$$\mathbb{P}[\nu_{p,x} = 1] = \frac{\#\{n \leq x \mid n \text{ squarefree and divisible by } p\}}{\#\{n \leq x \mid n \text{ squarefree}\}}.$$

for fixed p.

By definition, the support of the values of $\exp(\zeta_n)$ is the set of squarefree integers only divisible by primes $p \leq p_n$, and for Q_n, it is the set of rational numbers $x = a/b$ where a, $b \geq 1$ are coprime integers, both squarefree, and both divisible only by primes $p \leq p_n$. It is natural to see them as giving probabilistic models of these numbers. We obtain mod-φ convergence for X_n:

Theorem 11. *Let*

$$\varphi(t) = \exp\left(-4\int_0^1 \sin^2\left(\frac{tv}{2}\right)\frac{dv}{v}\right)$$

for $t \in \mathbb{R}$. Then φ is an integrable characteristic function of a probability distribution on $\mathbb{R}$, and the sequence (X_n) satisfies mod-φ convergence with $d = 1$ and $A_n(t) = A_n^*(t) = (\log p_n)t$.

The proof is quite similar in principle to arguments in [7], though our presentation is more in the usual style of analytic number theory.

We start with the easiest part of this statement:

Lemma 1. *We have $\varphi \in L^1(\mathbb{R})$, and in fact*

$$|\varphi(t)| \leq C|t|^{-2} \tag{9}$$

for $|t| \geq 1$ and some constant $C \geq 0$.

Remark 12. The characteristic function of the limit in law of the (non-symmetrised) random variables ζ_n used in [7, Th. 1.1] only decays as t^{-1} when $|t| \to +\infty$, and hence is not integrable, which prevents us from applying our results directly to those variables. As we will see, this is quite delicate: changing the constant 4 to a constant < 2 would lead to a failure of this property.

Below, we will see that Theorem 5 *is not valid* for the variables $(\log p_n)\zeta_n$.

Proof. Integration by parts gives that

$$\int_0^t \frac{\sin^2 x}{x}\,dx = \frac{1}{2}\log t + b(t)$$

where $b(t)$ tends to a constant for $t \to \infty$. From here we deduce that

$$4\int_0^1 \sin^2\left(\frac{tv}{2}\right)\frac{dv}{v} = 2\log|t| + c(t)$$

where $c(t)$ remains bounded. As a result we get (9), which proves that $\varphi \in L^1$ since the function is continuous. (Alternatively, the addicted reader can check that

$$\varphi(t) = \exp(-2\gamma - 2\log|t| + 2\mathrm{Ci}(t)) \tag{10}$$

where γ is the Euler constant and $\mathrm{Ci}(t)$ is the cosine integral function, and use the properties of the latter.) $\square$

Proof of Theorem 11. Let

$$Y_n = \frac{1}{\log p_n}X_n$$

and let ψ_n be the characteristic function of Y_n, which we proceed to compute.

With $x = x_n = p_n$, we first have

$$\psi_n(t) = \mathbb{E}\left[\exp(itY_n)\right] = \prod_{p \leq x} \mathbb{E}\left[\exp\left(i\frac{\log p}{\log x} t\eta_p\right)\right]$$
$$= \prod_{p \leq x} \left(\frac{p^2+1}{(p+1)^2} + \frac{2p}{(p+1)^2} \cos\left(t\frac{\log p}{\log x}\right)\right)$$
$$= \prod_{p \leq x} \left(1 - \frac{2p}{(p+1)^2}\left(1 - \cos\left(t\frac{\log p}{\log x}\right)\right)\right)$$
$$= \prod_{p \leq x} \left(1 - \frac{4p}{(p+1)^2} \sin^2\left(\frac{t}{2}\frac{\log p}{\log x}\right)\right)$$
$$(11) \qquad = \exp\left(\sum_{p \leq x} \log\left(1 - \frac{4p}{(p+1)^2} \sin^2\left(\frac{t}{2}\frac{\log p}{\log x}\right)\right)\right)$$

for all $t \in \mathbb{R}$. Now we assume $t \neq 0$ (since for $t = 0$, the values are always 1). We first show pointwise, locally uniform, convergence.

The idea to see the limit emerge in the sum over p is quite simple. First of all, we can expand the logarithm in Taylor series. We have

$$\lim_{x \to +\infty} \sum_{k \geq 2} \sum_{p \leq x} p^{-k} \Big|\sin^2\Big(\frac{t \log p}{2 \log x}\Big)\Big| = 0,$$

for t in a bounded set, by dominated convergence. This allows us to restrict our attention to

$$(12) \qquad -4 \sum_{p \leq x} p^{-1} \sin^2\Big(\frac{t \log p}{2 \log x}\Big)$$

(we also used the fact that $4p/(p+1)^2$ is equal to $4/p$ up to terms of order p^{-2}.) Now, for $p \leq y$, where $y \leq x^{1/|t|}$ is a further parameter (assuming, as we can, that this is ≥ 2), we also have

$$\Big|\sum_{p \leq y} p^{-1} \sin^2\Big(\frac{t \log p}{2 \log x}\Big)\Big| \leq \Big(\frac{t}{2 \log x}\Big)^2 \sum_{p \leq y} p^{-1} (\log p)^2 \ll t^2 \frac{(\log y)^2}{(\log x)^2}.$$

Thus, for if we select $y = y(x) \leq x^{1/|t|}$ tending to infinity slowly enough that $\log y = o(\log x)$, this also converges to 0 as $x \to +\infty$, and what remains is

$$-4 \sum_{y(x) \leq p \leq x} p^{-1} \sin^2\Big(\frac{t \log p}{2 \log x}\Big).$$

We can now perform "back-and-forth" summation by parts using the Prime Number Theorem to see that this is

$$-4 \int_{y(x)}^{x} u^{-1} \sin^2\Big(\frac{t \log u}{2 \log x}\Big) \frac{du}{\log u} + o(1)$$

as $x \to +\infty$ (apply Lemma 2 below with $B = 2$ and with the function

$$f(u) = \frac{1}{u} \sin^2\Bigl(\frac{t \log u}{2 \log x}\Bigr)$$

with

$$f'(u) = -\frac{1}{u^2} \sin^2\Bigl(\frac{t \log u}{2 \log x}\Bigr) + \frac{t}{u^2(\log x)} \sin\Bigl(\frac{t \log u}{2 \log x}\Bigr) \cos\Bigl(\frac{t \log u}{2 \log x}\Bigr),$$

which satisfies

$$|f(u)| \leq u^{-1}, \qquad |f'(u)| \leq \Bigl(1 + \frac{t}{\log x}\Bigr)u^{-2} \ ; \tag{13}$$

the integral error term in Lemma 2 is then dominated by the tail beyond $y(x)$ of the convergent integral

$$\int_2^{+\infty} \frac{du}{u(\log u)^2},$$

and the result follows). Performing the change of variable

$$v = \frac{\log u}{\log x},$$

we get the integral

$$-4 \int_{\log(y(x))/(\log x)}^1 \sin^2\Bigl(\frac{tv}{2}\Bigr) \frac{dv}{v},$$

which converges to $\varphi(t)$ as $x \to +\infty$.

To conclude the proof of Theorem 11, we will prove the following inequality, which guarantees the uniform integrability condition **H3**: for any $k \geq 1$ and t, n with $|t| \leq k(\log x) = k(\log p_n)$, we have

$$\psi_n(t) \ll |\varphi(t)| \exp(C \log\log 3|t|) \tag{14}$$

which gives the desired result since we know from (9) that φ decays like $|t|^{-2}$ at infinity.

We can assume that $|t| \geq 2$. Now we start with the expression (11) again and proceed to deal with the sum over $p \leq x$ in the exponential using roughly the same steps as before. To begin with, we may again estimate the sum (12) only, since the contribution of the others terms is bounded uniformly in t and x:

$$\Bigl|\sum_{k\geq 2} \sum_{p \leq x} p^{-k} \sin^2\Bigl(\frac{t \log p}{2 \log x}\Bigr)\Bigr| \leq \sum_{k \geq 2} \sum_{p} p^{-k},$$

which is a convergent series. After exponentiation, these terms lead to a fixed multiplicative factor, which is fine for our target (14).

We next deal with the small primes in (12); since $|t| \leq k \log x$, the sine term may not lead to any decay, but we still can bound trivially

$$\Bigl|\sum_{p \leq y} p^{-1} \sin^2\Bigl(\frac{t \log p}{2 \log x}\Bigr)\Bigr| \leq \sum_{p \leq y} p^{-1} \ll \log\log y$$

for any $y \leq x$ (by a standard estimate). We select $y = |t| \geq 2$, and this becomes a factor of the type

$$\exp(C \log\log t)$$

(after exponentiating), which is consistent with (14).

We now apply Lemma 2 again, writing more carefully the resulting estimate, namely

$$-4\sum_{y<p\leq x} p^{-1}\sin^2\Bigl(\frac{t\log p}{2\log x}\Bigr) = -4\int_y^x u^{-1}\sin^2\Bigl(\frac{t\log u}{2\log x}\Bigr)\frac{du}{\log u} + O\Bigl(\frac{1+k}{(\log y)^2}\Bigr)$$
$$= -4\int_{\log y/\log x}^1 v^{-1}\sin^2\Bigl(\frac{tv}{2}\Bigr)dv + O\Bigl(\frac{1+k}{(\log y)^2}\Bigr)$$

(using the bound (13)), with an absolute implied constant. The remainder here is again fine, since $y = |t| \geq 2$ by assumption.

Now, to conclude, we need only estimate the missing part of the target integral (which runs from 0 to 1) in this expression, namely

$$\int_0^{\log y/\log x} v^{-1}\sin^2\Bigl(\frac{tv}{2}\Bigr)dv.$$

We write

$$\int_0^{\log y/\log x} v^{-1}\sin^2\Bigl(\frac{tv}{2}\Bigr)dv = \int_0^{|t|^{-1}} v^{-1}\sin^2\Bigl(\frac{tv}{2}\Bigr)dv + \int_{|t|^{-1}}^{\log y/\log x} v^{-1}\sin^2\Bigl(\frac{tv}{2}\Bigr)dv$$

where the first terms is bounded by

$$(t/2)^2\int_0^{|t|^{-1}} vdv \leq 1,$$

and the second by

$$\int_{|t|^{-1}}^{\log y/\log x} v^{-1}dv = \log\Bigl(\frac{|t|\log y}{\log x}\Bigr) \leq \log(k\log y) = \log(k\log|t|).$$

Putting the inequalities together, we have proved (14), and hence Theorem 11. □

Here is the standard lemma from prime number theory that we used above, which expresses the fact that for primes sufficiently large, the heuristic – due to Gauss – that primes behave like positive numbers with the measure $du/(\log u)$ can be applied confidently in many cases.

Lemma 2. *Let $y \geq 2$ and let f be a smooth function defined on $[y, +\infty[$. Then for any $A > 1$, we have*

$$\sum_{y\leq p\leq x} f(p) = \int_y^x f(u)\frac{du}{\log u} + O\Bigl(\frac{x|f(x)|}{(\log x)^A} + \frac{y|f(y)|}{(\log y)^A} + \int_y^x |f'(u)|\frac{udu}{(\log u)^A}\Bigr)$$

where the sum is over primes and the implied constant depends only on A.

We give the proof for completeness.

Proof. We use summation by parts and the Prime Number Theorem, which is the case $f(x) = 1$, in the strong form

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\Bigl(\frac{x}{(\log x)^A}\Bigr)$$

for $x \geq 2$ and any $A > 1$, with an implied constant depending only on A (this is a consequence of the error term in the Prime Number Theorem due to de la Vallée Poussin, see e.g.[12, Cor. 5.29]); this leads to

$$\sum_{y \leq p \leq x} f(p) = f(x)\pi(x) - f(y)\pi(y) - \int_y^x f'(u)\pi(u)du,$$

and after inserting the above asymptotic formula for $\pi(x)$ and $\pi(u)$, we can revert the integration by parts to recover the main term, while the error terms lead to the result. $\square$

We now derive arithmetic consequences of Theorem 11. Applying Theorem 2, we get

$$\lim_{n \to +\infty} (\log p_n) \mathbb{P}\left[X_n \in (a,b)\right] = (b-a)\eta \tag{15}$$

where

$$\eta = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-4 \int_0^1 \frac{\sin^2 \frac{tv}{2}}{v} dv\right) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{2(\mathrm{Ci}(t)-\gamma)} \frac{dt}{|t|^2},$$

the last expression coming from (10). Using the relation between φ and the Dickman-de Bruijn function ρ, namely

$$\varphi(t) = \psi(t)\psi(-t)$$

where $\psi(t)$ is the Fourier transform of $e^{-\gamma}\rho(u)$ (this follows from [7, Th. 1.1, p. 5]), one gets

$$\eta = e^{-2\gamma} \int_{\mathbb{R}} \rho(u)^2 du = 0.454867\ldots$$

(the numerical computation was done using Sage).

This arithmetic application could certainly be proved with more traditional methods of analytic number theory, when expressed concretely as giving the asymptotic behavior as $n \to +\infty$ of

$$\sum_{\alpha < r/s < \beta} \mathbb{P}\left[X_n = \frac{r}{s}\right],$$

but it is nevertheless a good illustration of the general probabilistic framework of mod-φ convergence with an unusual characteristic function.

Although our theorem does not apply for the random model of [7] itself, it is quite easy to understand the behavior of the corresponding probabilities in that case. Indeed, denoting

$$Y_n = \exp((\log p_n)\zeta_n),$$

which takes squarefree values, we have

$$\mathbb{P}[Y_n < e^a] = \frac{1}{Z_x} \sum_{\substack{k < e^a \\ p \mid k \Rightarrow p \leq x}} \frac{\mu^2(k)}{k}$$

for any fixed $a \in \mathbb{R}$, where $x = p_n$, $\mu^2(k)$ is the indicator function of squarefree integers and Z_x is the normalizing factor given by

$$Z_x = \prod_{p \leq x} (1 + p^{-1}).$$

For x large enough and a fixed, the second condition is vacuous, and hence this is

$$\frac{1}{Z_x}\sum_{k<e^a}\frac{\mu^2(k)}{k}.$$

As observed in [7, (3)], we have $Z_x \sim e^{\gamma}\zeta(2)^{-1}\log x$, and hence we get

$$\lim_{n\to+\infty}(\log p_n)\mathbb{P}[Y_n < e^a] = \zeta(2)e^{-\gamma}\sum_{k<e^a}\frac{\mu^2(k)}{k}.$$

When a is large, this is equivalent to $e^{-\gamma}a$ (another easy fact of analytic number theory), which corresponds to the local limit theorem like (15), but we see that for fixed a, there is a discrepancy.

There is one last interesting feature of this model: the analogue of Theorem 11 for polynomials over finite fields *does not hold*, despite the many similarities that exist between integers and such polynomials (see, e.g., [18] for instances of these similarities in related probabilistic contexts.)

Precisely, let $q > 1$ be a power of a prime number and $\mathbb{F}_q$ a finite field with q elements. For irreducible monic polynomials $\pi \in \mathbb{F}_q[X]$, we suppose given independent random variables η_π, η'_π such that by

$$\mathbb{P}[\eta_\pi = \pm 1] = \mathbb{P}[\eta'_\pi = \pm 1] = \frac{|\pi|}{(|\pi|+1)^2}, \qquad \mathbb{P}[\eta_\pi = 0] = \mathbb{P}[\eta'_\pi = 0] = \frac{|\pi|^2+1}{(|\pi|+1)^2}$$

where $|\pi| = q^{\deg(\pi)}$. Then for $n \geq 1$, let $\hat{X}_n$ be the random variable

$$\sum_{\deg(\pi)\leq n}(\deg\pi)(\eta_\pi - \eta'_\pi),$$

where the sum runs over all irreducible monic polynomials of degree at most n. Then we claim that **H1**, **H2** hold for $\hat{X}_n$, with the same characteristic function $\varphi(t)$ as in Theorem 11, and $A_n t = nt$, but there is no mod$-\varphi$ convergence.

This last part at least is immediate: **H3** fails by contraposition because the local limit theorem for

$$\lim_{n\to+\infty} n\mathbb{P}[a < \hat{X}_n < b]$$

is not valid! Indeed, $\hat{X}_n$ is now integral-valued, and if $]a,b[\cap\mathbb{Z} = \emptyset$, the probability above is always 0, whereas the expected limit $(b-a)\eta$ is not.

We now check **H2** in this case. Arguing as in the beginning of the proof of Theorem 11, we get

$$\mathbb{E}[e^{it\hat{X}_n/n}] = \prod_{\deg(\pi)\leq n}\left(1 - \frac{4|\pi|}{(|\pi|+1)^2}\sin^2\left(\frac{\deg(\pi)t}{2n}\right)\right).$$

Expanding the logarithm once more, we see that it is enough to prove that (locally uniformly in t) we have

$$\lim_{n\to+\infty}\exp\left(-4\sum_{\deg(\pi)\leq n}\frac{1}{|\pi|}\sin^2\left(\frac{\deg(\pi)t}{2n}\right)\right) = \varphi(t).$$

We arrange the sum according to the degree of π, obtaining

$$\sum_{\deg(\pi)\leq n} \frac{1}{|\pi|} \sin^2\left(\frac{\deg(\pi)t}{2n}\right) = \sum_{j=1}^{n} \frac{1}{q^j} \sin^2\left(\frac{jt}{2n}\right) \Pi_q(j)$$

where $\Pi_q(j)$ is the number of monic irreducible polynomials of degree j in $\mathbb{F}_q[X]$. The well-known elementary formula of Gauss and Dedekind for $\Pi_q(j)$ shows that

$$\Pi_q(j) = \frac{q^j}{j} + O(q^{j/2})$$

for q fixed and $j \geq 1$, and hence we can write the sum as

$$\sum_{\deg(\pi)\leq n} \frac{1}{|\pi|} \sin^2\left(\frac{\deg(\pi)t}{2n}\right) = \sum_{j=1}^{n} \frac{1}{j} \sin^2\left(\frac{jt}{2n}\right) + O\left(\sum_{j=1}^{n} q^{-j/2} \sin^2\left(\frac{tj}{2n}\right)\right).$$

As n goes to infinity, the second term converges to 0 by the dominated convergence theorem, while the first is a Riemann sum (with steps $1/n$) for the integral

$$\int_0^1 \sin^2\left(\frac{tv}{2}\right) \frac{dv}{v},$$

and hence we obtain the desired limit. (This is somewhat similar to [1, Prop. 4.6].)

Remark 13. A more purely probabilistic example of the same phenomenon arises as follows: define

$$\tilde{X}_n = \sum_{j=1}^{n} (D_j - E_j)$$

where (D_j, E_j) are globally independent random variables with distribution

$$\mathbb{P}[E_j = j] = \mathbb{P}[D_j = j] = \frac{1}{j}, \qquad \mathbb{P}[E_j = 0] = \mathbb{P}[D_j = 0] = 1 - \frac{1}{j}.$$

Then the sequence $(\tilde{X}_n)$ also satisfies **H1** and **H2** for the same characteristic function $\varphi(t)$ (by very similar arguments), and does not satisfy **H3** since $\tilde{X}_n$ is integral-valued.

3.7. **Random Matrices.** Some of the first examples of mod-Gaussian convergence are related to the "ensembles" of random matrices corresponding to families of compact Lie groups, as follows from the work of Keating and Snaith [15], [16]. Using this, and our main result, we can deduce quickly some local limit theorems for values of the characteristic polynomials of such random matrices.

We consider the three standard families of compact matrix groups, which we will denote generically by $\mathbf{G}_n$, where $\mathbf{G}$ is either U (unitary matrices of size n), USp (symplectic matrices of size $2n$) or SO (orthogonal matrices of determinant 1 and size[1] $2n$). In each case, we consider $\mathbf{G}_n$ as a probability space by putting the Haar measure μ_n on $\mathbf{G}_n$, normalized so that $\mu_n(\mathbf{G}_n) = 1$. The relevant random variables (X_n) are defined as suitably centered values

[1] The odd case could be treated similarly.

of the characteristic polynomial $\det(T - g_n)$ where g_n is a $\mathbf{G}_n$-valued random variable which is μ_n-distributed. Precisely, define

$$\alpha_n = \begin{cases} 0 & \text{if } \mathbf{G} = U, \\ \frac{1}{2}\log(\pi n/2) & \text{if } \mathbf{G} = USp, \\ \frac{1}{2}\log(8\pi/n) & \text{if } \mathbf{G} = SO, \end{cases}$$

and consider $X_n = \log\det(1 - g_n) - \alpha_n$; this is real-valued except for $\mathbf{G} = U$, in which case the determination of the logarithm is obtained from the standard Taylor series at $z = 1$.

Now define the linear maps

$$A_n(t) = \begin{cases} \left(\frac{\log n}{2}\right)^{1/2}(t_1, t_2) & \text{if } \mathbf{G} = U, \\ \left(\log\left(\frac{n}{2}\right)\right)^{1/2} t & \text{otherwise,} \end{cases}$$

and their inverses Σ_n (these are diagonal so $A_n^* = A_n$, $\Sigma_n^* = \Sigma_n$).

Finally, let φ be the characteristic function of a standard complex (if $\mathbf{G} = U$) or real gaussian random variable (if $\mathbf{G} = USp$ or SO); in particular **H1** is true. It follows from the work of Keating and Snaith that in each case $\varphi_n(\Sigma_n t)$ converges continuously to $\varphi(t)$, i.e., that **H2** holds. In fact, in each case, there is a continuous (in fact, analytic) limiting function $\Phi_{\mathbf{G}}(t)$ such that

$$\varphi_n(t) = \varphi(A_n^* t)\Phi_{\mathbf{G}}(t)(1 + o(1))$$

for any fixed t, as n goes to infinity. These are given by

$$\Phi_{\mathbf{G}}(t) = \begin{cases} \frac{G(1+\frac{it_1-t_2}{2})G(1+\frac{it_1+t_2}{2})}{G(1+it_1)} & \text{if } \mathbf{G} = U, \\ \frac{G(3/2)}{G(3/2+it)} & \text{if } \mathbf{G} = USp, \\ \frac{G(1/2)}{G(1/2+it)} & \text{if } \mathbf{G} = SO, \end{cases}$$

in terms of the Barnes G-function. Detailed proofs can be found in [19, §3, Prop. 12, Prop. 15], and from the latter arguments, one obtains uniform estimates

$$|\varphi_n(t)| \leq C|\Phi_{\mathbf{G}}(t)\varphi(A_n^* t)|$$

for all t such that $|t| \leq n^{1/6}$, where C is an absolute constant. This immediately gives the uniform integrability for $\varphi_n(\Sigma_n^* t)\mathbf{1}_{|\Sigma_n t| \leq k}$ since $|\Sigma_n^* t|$ is only of logarithmic size with respect to n. In other words, we have checked **H3**, and hence there is mod-φ convergence.

Consequently, applying Theorem 5, we derive the local limit theorems (already found in [19]):

Theorem 12. *For $\mathbf{G} = U$, USp or SO, for any bounded Jordan-measurable set $B \subset \mathbb{R}$ or $\mathbb{C}$, the latter only for $\mathbf{G} = U$, we have*

$$\lim_{n\to+\infty} |\det(A_n)| \quad \mu_n(g \in \mathbf{G}_n \mid \log\det(1 - g) - \alpha_n \in B) = \frac{m(B)}{(2\pi)^{d/2}}$$

with $d = 2$ for $\mathbf{G} = U$ and $d = 1$ otherwise.

Theorem 3, stated in the introduction, is the special case $\mathbf{G} = U$, enhanced by applying Proposition 1.

As in [19, §4], one can derive arithmetic consequences of these local limit theorems, involving families of L-functions over finite fields, by appealing to the work of Katz and Sarnak.

The interested readers should have no difficulty checking this using the detailed results and references in [19].

Instead, we discuss briefly a rather more exotic type of random matrices, motivated by the recent results in [17] concerning certain averages of L-functions of Siegel modular forms. In [17, Rem. 1.3], the following model is suggested: let $G_n = SO_{2n}(\mathbb{R})$, with Haar measure μ_n, and consider the measure

$$\nu_n(g) = \frac{1}{2}\det(1-g)d\mu_n(g)$$

on G_n. The density $\det(1-g)$ is non-negative on G_n (because eigenvalues of a matrix in $SO(2n,\mathbb{R})$ come in pairs $e^{i\theta}$, $e^{-i\theta}$, and $(1-e^{i\theta})(1-e^{-i\theta}) \geq 0$); the fact that this is a probability measure will be explained below. In probabilistic terms, this is the "size-biased" version of μ_n.

Theorem 13. *Let $X_n = \log\det(1-\tilde{g}_n) - \frac{1}{2}\log(32\pi n)$, where $\tilde{g}_n$ is a G_n-valued random variable distributed according to ν_n. Let φ be the characteristic function of a standard real gaussian. Then we have mod-φ convergence with $A_n t = (\log\frac{n}{2})^{1/2}t$, and in particular*

$$\lim_{n\to+\infty}\sqrt{\log\frac{n}{2}}\quad \nu_n\left(g \in \mathbf{G}_n \mid \log\det(1-g) - \frac{1}{2}\log(32\pi n) \in B\right) = \frac{m(B)}{\sqrt{2\pi}}.$$

Proof. The characteristic function of $Y_n = \log\det(1-\tilde{g}_n)$ is half of the value at $s = 1+it$ of the Laplace transform $\mathbb{E}[e^{s\log\det(1-g_n)}]$, where g_n is Haar-distributed. The latter is computed for all complex s in [15, (56)], and we get

$$2\mathbb{E}[e^{itY_n}] = \mathbb{E}[e^{(1+it)\log\det(1-g_n)}] = 2^{2n(1+it)}\prod_{1\leq j\leq n}\frac{\Gamma(j+n-1)\Gamma(j+it+1/2)}{\Gamma(j-1/2)\Gamma(j+it+n)}.$$

At this point, the reader may check easily (by recurrence on n if needed) that this gives the right values $\mathbb{E}[e^{itY_n}] = 1$ for $t = 0$, confirming the normalizing factor $1/2$ used in the definition of ν_n.

To go further, we transform the right-hand side into values of the Barnes function $G(z)$, as in [19, §4.3], to get

$$2\mathbb{E}[e^{itY_n}] = 2^{2n(1+it)}\frac{G(1/2)}{G(3/2+it)}\frac{G(2n)G(n+3/2+it)G(1+it+n)}{G(n)G(n+1/2)G(2n+1+it)}.$$

Applying $\Gamma(z)G(z) = G(z+1)$, we transform this into

$$\begin{aligned}2\mathbb{E}[e^{itY_n}] = 2^{2n(1+it)}\frac{G(1/2)}{G(3/2+it)}&\frac{\Gamma(it+n)\Gamma(it+n+1/2)}{\Gamma(it+2n)}\\ &\times\frac{G(2n)G(n+it)G(1/2+it+n)}{G(n)G(n+1/2)G(2n+it)}\end{aligned}$$

and the last ratio of Barnes functions (together with the factor 2^{2nit}) is exactly the one handled in [19, Prop. 17, (4)]. With the asymptotic formula that follows, the Legendre duplication formula and $\Gamma(1/2) = \sqrt{\pi}$, we deduce

$$2\mathbb{E}[e^{itY_n}] = 2\frac{G(3/2)}{G(3/2+it)}\frac{\Gamma(2it+2n)}{\Gamma(it+2n)}\left(\frac{n}{2}\right)^{-t^2/2}\left(\frac{8\pi}{n}\right)^{it/2}(1+o(1))$$

uniformly for $|t| \leq n^{1/6}$. Since

$$\frac{\Gamma(2it+2n)}{\Gamma(it+2n)} = (2n)^{it}(1+o(1))$$

in this range, we get

$$\mathbb{E}[e^{itY_n}] = \frac{G(3/2)}{G(3/2+it)}\varphi(A_n^* t)(32\pi n)^{it/2}(1+o(1)),$$

uniformly for $|t| \leq n^{1/6}$, and the result follows. $\square$

The most obvious feature of this exotic model of orthogonal matrices is the "shift" of the average; whereas, for Haar-distributed $g \in SO_{2n}(\mathbb{R})$, the value $\log\det(1-g)$ is typically small (mean about $\log\sqrt{8\pi/n}$), it becomes typically large (mean $\log\sqrt{32\pi n}$, of similar order of magnitude as the mean for a symplectic matrix of the same size) when g is considered to be distributed according to ν_n. This is consistent with the discussion in [17, Rem. 1.3], especially since the "limiting function" that appears here is Φ_{USp}.

3.8. **Stochastic model of the Riemann zeta function.** The following "naive" model of the Riemann zeta function on the critical line is surprisingly helpful. The basic ingredient is a sequence of iid variables $Y_p\colon \Omega \to \mathbb{T}$ where $\mathbb{T}$ is the unit circle in $\mathbb{C}$ and the variables Y_p are uniformly distributed over $\mathbb{T}$. For notational ease the sequence is ordered by the prime numbers. In what follows p will always denote a prime number. The random variables we consider are constructed as follows. First we take finite products $Z_n = \prod_{p\leq n}\left(1-\frac{Y_p}{\sqrt{p}}\right)$. If we replace the factors Y_p by $\exp(ipt)$, then the product appears in the study of the Riemann ζ-function. An easy application of Weyl's lemma on uniform distributions shows that $(\exp(ipt))_{p\leq n}$ defined on $[0,T]$ (with normalised Lebesgue measure) tend (as $T\to\infty$) to $(Y_p)_{p\leq n}$. The random variables X_n are then defined as minus the logarithm of Z_n, (taken along its principal branch defined as $\log(1)=0$). So

$$X_n = -\sum_{p\leq n}\log\left(1-\frac{Y_p}{\sqrt{p}}\right) = \sum_{p\leq n}\sum_k \frac{1}{k}\left(\frac{Y_p}{\sqrt{p}}\right)^k.$$

These sums clearly converge. Because of this explicit form we can calculate the characteristic functions. The calculations are done in [19, §3, Ex. 2] and this yields the following.

$$\varphi_n(t) = \mathbb{E}[\exp(it\cdot X_n)] = \prod_{p\leq n} {}_2F_1\left(\frac{1}{2}(it_1+t_2), \frac{1}{2}(it_1-t_2); 1; \frac{1}{p}\right),$$

where $t=(t_1,t_2)\in\mathbb{R}^2$, $t\cdot x = t_1x_1+t_2x_2$ is the inner product in $\mathbb{R}^2$ and ${}_2F_1$ denotes the Gauss hypergeometric function. Straightforward estimates (see [19] for details) then give

(1) $|\varphi_n(t)| \leq c(t)\exp(-\frac{1}{16}(\log\log n)|t|^2)$, where c is a non-decreasing function (in fact one can take a constant);
(2) $\varphi_n\left(\sqrt{\frac{2}{\log\log n}}t\right) \to \exp(-\frac{1}{2}|t|^2)$.

The conditions of Theorem 5 are fulfilled and hence we have

$$\frac{\log\log n}{2}\mathbb{P}[X_n\in B] \to \frac{1}{2\pi}m(B).$$

for any bounded Jordan measurable set $B\subset\mathbb{C}$.

3.9. **The Riemann zeta function on the critical line.** The results in this section are conjectural, but they are of interest to number theorists. By work of Selberg, the central limit theorem for $\log \zeta(1/2 + it)$ is known, after renormalizing by $\sqrt{\log \log T}$, see e.g. [10]. This is proved by asymptotic estimations of the moments, and there is no known bound for the corresponding characteristic functions. Thus, we cannot currently apply our theorems.

However, Keating and Snaith ([15], [16]) have proposed the following precise conjecture (based on links with Random Matrix Theory) concerning the characteristic function: for any $t = (t_1, t_2) \in \mathbb{R}^2$, they expect that

$$\frac{1}{T} \int_0^T \exp(it \cdot \log \zeta(1/2 + iu))du \sim \Phi(u) \exp\Big(-\frac{\log \log T}{4} |t|^2\Big)$$

as $T \to +\infty$, where the limiting function is the product of the corresponding factors for unitary matrices and for the "stochastic" version of ζ, described in the previous sections, i.e.,

$$\begin{aligned} \Phi(t_1, t_2) = & \frac{G(1 + \frac{it_1 - t_2}{2})G(1 + \frac{it_1 + t_2}{2})}{G(1 + it_1)} \\ & \times \prod_p {}_2F_1(\frac{1}{2}(it_1 + t_2), \frac{1}{2}(it_1 - t_2); 1; p^{-1}) \end{aligned}$$

(the normalization of $\log \zeta(1/2 + iu)$ is obtained by continuation of the argument from the value 0 for $\zeta(\sigma + iu)$ when σ real tends to infinity, except for the countable set of u which are ordinates of zeros of ζ.)

In [19, Cor. 9], it is shown that a suitable uniform version of this conjecture implies local limit theorems for

$$\frac{1}{T} m(u \in [0, T] \mid \log \zeta(1/2 + iu) \in B)$$

and, as a corollary, implies that the set of values of $\zeta(1/2 + iu)$, $u \in \mathbb{R}$, is *dense* in $\mathbb{C}$, which is an old and intriguing conjecture of Ramachandra.

The mod-φ framework allows us to show that a much weaker statement than the one considered in [19] is already sufficient to get the same local limit theorems. Indeed, we consider the following much stronger statement, which of course implies Ramachandra's conjecture, as being very likely to be true:

Conjecture 1 (Quantitative density of values of $\zeta(1/2 + it)$)**.** *For any bounded Jordan-measurable subset $B \subset \mathbb{C}$, we have*

$$\lim_{T \to +\infty} \frac{\frac{1}{2} \log \log T}{T} m(u \in [0, T] \mid \log \zeta(1/2 + iu) \in B) = \frac{m(B)}{2\pi}.$$

The point is that this follows using Theorem 5 from fairly weak decay estimates for the characteristic function of $\log \zeta(1/2 + it)$ (in comparison with what the Keating-Snaith conjecture suggests). For instance, if for all $k > 0$ there exists $C_k \geq 0$ such that

$$\Big| \frac{1}{T} \int_0^T \exp(it \cdot \log \zeta(1/2 + iu))du \Big| \leq \frac{C_k}{1 + |t|^4 (\log \log T)^2} \tag{16}$$

for all $T \geq 1$ and t with $|t| \leq k$, then Conjecture 1 is true. Indeed, in Theorem 5, we can take φ to be the characteristic function of a standard complex gaussian and X_n to be (for some arbitrary sequence T_n going to $+\infty$) a random variable with law given by the probability distribution of $\log \zeta(1/2 + iu)$ for u uniform on $[0, T_n]$. These satisfy **H1** trivially, and **H2** holds with

$$A_n(t_1, t_2) = A_n^*(t_1, t_2) = \sqrt{\frac{1}{2} \log \log T_n}(t_1, t_2),$$

because of Selberg's Central Limit Theorem. The hypothesis (16) states that, for any $k > 0$, we have

$$|\varphi_n(t)| \leq C_k h(A_n^* t)$$

for $|t| \leq k$, with

$$h(t_1, t_2) = \frac{1}{1 + 4|t|^4},$$

or equivalently

$$|\varphi(\Sigma_n^* t)| \leq C_k h(t)$$

for $|\Sigma_n^* t| \leq k$. Since $h \in L^1(\mathbb{R}^2)$, this gives (4), and we get the conjectured statement from the local limit theorem.

The significance of this remark is the fact that, for fixed $t \neq 0$, the decay rate of the characteristic function which is required is "only" of order $(\log \log T)^{-2}$, which is much weaker than what is suggested by the Keating-Snaith conjecture, and therefore might be more accessible.

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