LECTURE 10

§ 4-MANIFOLDS

We saw: Smale's h-cobordism theorem + Barden-Mazur-Stallings s-cobordism theorem apply to cobordisms W with $\dim W \geq 6$.
For $\dim W = 5$ we could prove the Normal Form Lemma, but could not proceed further since the Whitney trick fails.

key **Thm** [Freedman 1982] – s-cobordism Theorem in dim 5 –
If $(W, \partial_0 W, \partial_1 W)$ is an h-cobordism with $\dim W = 5$ and trivial Whitehead torsion $Wh(W, \partial_0 W) \in Wh(\pi_1 W)$ and $\pi_1 W$ is a good group,

e.g. all finite groups
all abelian groups

then W is topologically trivial,
i.e. there is a homeomorphism $(W, \partial_0 W, \partial_1 W) \cong (\partial_0 W \times [0,1], \partial_0 W \times \{0\}, \partial_0 W \times \{1\})$.

proof. As before (see Lecture 6): Step 0 Remove 0- and 5-handles Lemma.
Step 1 Normal Form Lemma
using Handle Trading Lemma

trade each 1-handle h^1 for a 3-handle, as follows:

note: we are in case $k=1$ which we saw works for $\dim W = 5$ as well:

Let $L \subseteq \partial h^1$ be a push-off of the core of h^1.
Then $\partial L \subseteq \partial_0 W$ bounds an arc $\alpha \subseteq \partial_0 W \setminus$ attaching regions of all other 1-handles and 2-handles (since $\partial_0 W \setminus (\bigsqcup S^0 \times D^4 \cup \bigsqcup S^1 \times D^3)$ is still connected. $\Longrightarrow$ A survives to $\partial_1 W^{\leq 2}$
$\Longrightarrow$ $A := L \cup \alpha : S^1 \xhookrightarrow{sm} \partial_1 W^{\leq 2}$ goes over h^1 geometrically once

Lemma: The arc α can be chosen so that $A := L \cup \alpha : S^1 \hookrightarrow \partial_1 W^{\leq 2}$ is nullhomotopic.

proof. $\pi_1 W^{\leq 2} \xleftrightarrow{\cong} \pi_1 W$ (since attaching higher cells does not change π_1)
$\pi_1 \partial_1 (W^{\leq 2}) \xleftrightarrow{\cong} \pi_1 W^{\leq 2}$ (turn $W^{\leq 2}$ upside down, handles are index $5-1 > 2$ and $5-2 > 2$)
$\pi_1 \partial_0 W \xleftrightarrow{\cong} \pi_1 W$. (by the h-cobordism assumption)
$\Longrightarrow$ $\pi_1 \partial_1 W^{\leq 2} \cong \pi_1 \partial_0 W$.

A might be nontrivial $[A] \neq 0 \in \pi_1 \partial_1 W^{\leq 2} \cong \pi_1 W^{\leq 2} \cong \pi_1 \partial_0 W$.
Let β be a loop in $\partial_0 W$ realizing this class, chosen so that it misses all att. spheres of 1- and 2-handles. Thus, β lives in $\partial_1 W^{\leq 2}$, and replacing α with $\alpha\beta^{-1}$ gives $A := L \cdot \alpha\beta^{-1} \simeq *$ in $\partial_1 W^{\leq 2}$. □

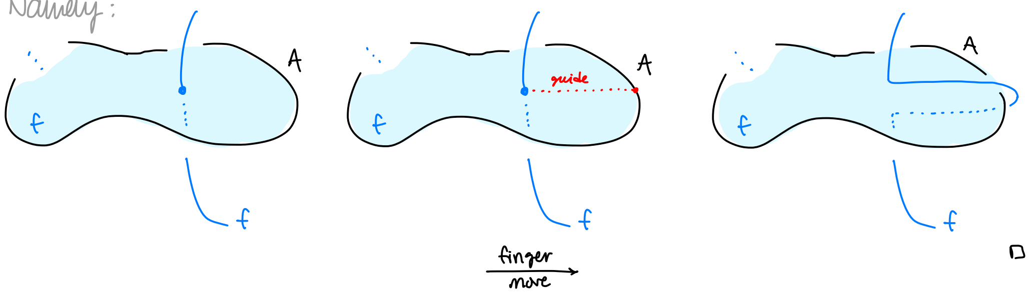
Cor. A bounds an embedded disc Δ in $\partial_1 W^{\leq 2}$.
proof. We saw $A \simeq *$ in the 4-manifold $\partial_1 W^{\leq 2}$.
Thom Transversality $\Longrightarrow$ A bounds an immersed disc $f: D^2 \looparrowright \partial_1 W^{\leq 2}$.

[Recall: **Thm** [Thom] If $A: M \to N$ a smooth map and $B \subseteq N$ a compact submanifold then there is an ambient isotopy of N, taking A to A' such that $A' \pitchfork B$. Moreover, the isotopy can be assumed to be the identity outside of any open nbhd of B.

Cor. If $D^2 \xrightarrow{f} N$ a smooth map s.t. $f(\partial D^2) = \alpha$ then $\exists$ amb. isotopy of N s.t. $f' \pitchfork f'$ and $f'(\partial D^2) = \alpha$.]

Do Finger Moves $\Longrightarrow$ A bounds an embedded disc $\Delta: D^2 \hookrightarrow \partial_1 W^{\leq 2}$.

Namely:



□

cont. of Handle Trading:
now we can thicken Δ into a "mushroom" = cancelling 2-/3-pair
$\rightsquigarrow$ cancel h^2 and h^1, so h^3 left.

□.

Step 2 Algebraically cancelling pairs:

$$0 \to C_3^{\tilde{u}} \xrightarrow{\delta_3^{\tilde{u}}} C_2^{\tilde{u}} \to 0$$

with $\delta_3^{\tilde{u}}$ represented by the identity matrix

(using $Wh(W,\partial W)=0$ and Handle Slides)

$\Longrightarrow$ In the middle level $W_{1/2} := \partial_1(W^{\le 2})$ where $W^{\le 2} = \partial_0 W \times [0,1] \cup$ 2-handles

we have the belt spheres $B_1, \dots, B_r : S^2 \hookrightarrow W_{1/2}$ of 2-handles $(\{0\}\times S^2 \subseteq D^2 \times D^3)$

and the attaching spheres $A_1, \dots, A_r : S^2 \hookrightarrow W_{1/2}$ of 3-handles $(S^2 \times \{0\} \subseteq D^3 \times D^2)$

so that:

- each $\{B_i\}$ and $\{A_j\}$ is a collection of pairwise disjoint, framed, embedded spheres
- $\tilde{I}(A_j \pitchfork B_i) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \ne j \end{cases} \in \mathbb{Z}[\pi_1 W_{1/2}]$

WANT: Isotope A_j so that there intersection numbers are realised geometrically,

so that we can cancel each pair of handles, $i = 1, \dots, r$.

Lemma W. There exist framed immersed Whitney discs $W_m : D^2 \to W_{1/2}$, $m = 1, \dots, r'$

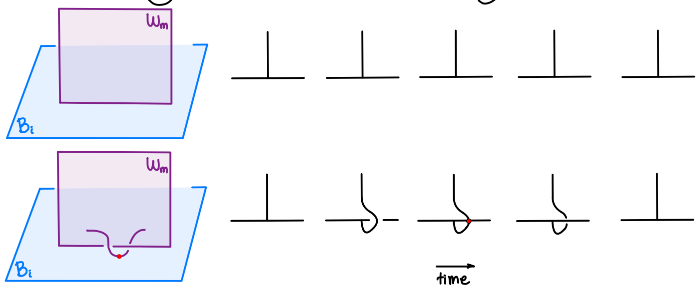
pairing up all unwanted intersections between A_j and B_i.

proof. As before, if intersection points have the same group element but opposite signs then there is a nullhomotopic Whitney circle between them.

By general position, there is an immersed Whitney disc.

If it is not framed, we can do boundary twists to it:

this corrects the framing at the expense of creating (more) intersections with B_i.



$\square$

Note that in general not only W_m are not embedded, but they also intersect A_j and B_i, so doing Whitney moves won't make A_j and B_i geom. cancelling. To remove W–A and W–B intersections we use geom. duals $\hat{A}_j$ and $\hat{B}_i$ constructed as follows.

Lemma #. There are collections of unframed immersed spheres $\{B_i^\#\}$ and $\{A_i^\#\}$ that are geometrically dual to the collections $\{B_i\}$ and $\{A_i\}$ respectively.

i.e. $B_i^\# \pitchfork B_j = A_i^\# \pitchfork A_j = \emptyset$ unless $i=j$ when they are each a point.

Lemma ^. After an isotopy of $\{A_i\}$,

There is a collection of framed immersed spheres $\{\hat{B}_i\} \cup \{\hat{A}_i\}$

that is geometrically dual to the collection $\{B_i\} \cup \{A_i\}$.

i.e. $\hat{C}_i \pitchfork D_j = \emptyset$ unless $i=j$ and $C=D$ when $=\{pt\}$, for $C, D \in \{A, B\}$

proofs of these next time.

Lemma W-improved. The discs W_m can be modified to have the interior disjoint from all A_j and B_i.

proof. We can tube each intersection of W_m with A_j into $\hat{A}_j$ and each intersection of W_m with B_i into $\hat{B}_i$.

Since $\hat{A}_j$ and $\hat{B}_i$ framed, discs W_m stay framed after the tubing.

Lemma G. There exist a collection of framed immersed spheres $\{G_m\}$ that is algebraically dual to the collection $\{W_m\}$.

i.e. $\tilde{I}(G_n \pitchfork W_m) = \delta_{nm}$.

Moreover, G_m are disjoint from all A_j and B_i.

key **Thm [Freedman 1982] – Disc Embedding Thm –**

If M is a smooth connected 4-manifold with $\partial M = \emptyset$ and $\pi_1 M$ a good group, and $W_m : (\mathbb{D}^2, \partial \mathbb{D}^2) \hookrightarrow (M, \partial M)$ is a framed immersed collection with emb. boundary which has a framed immersed collection $\{G_m\}$ of algebraic duals, then

there exists a locally flat embedded collection $\{\overline{W_m}\}$ with the same framed boundary as $\{W_m\}$

not needed in the current proof $\Big\{$ and with a framed immersed collection $\{\overline{G_m}\}$ of geometric duals with $\overline{G_m} \underset{htpc}{\simeq} G_m$.

PROOF VERY HARD.

We now apply Disc Emb. Thm: to W_m and G_m in $M := W_{1/2} \setminus (\bigcup \nu B_i \cup \bigcup \nu A_j)$.

Note: $\pi_1 M \cong \pi_1 W_{1/2}$ $(\cong \pi_1 W)$ since A_j and B_i have duals (so their meridians are nullhomotopic in M).

Therefore, we can perform Whitney moves on A_j along the framed loc. flat discs $\overline{W_m}$ to remove all unwanted intersections with B_i. This is a loc. flat isotopy of A_j, making it into a geometric dual of B_j, so that 2- and 3-handles geom. cancel. There are no other handles in $(W, \partial_0 W)$, so W is homeomorphic to $\partial_0 W \times [0,1]$.

$\square$