LECTURE 2

Let N be a smooth n-manifold.
Key feature of DIFF: tangent vector bundle

§ DIFF: TANGENT BUNDLE

See Wall §1

def. A tangent vector at $p \in N$ is a derivation z_p of $\mathcal{F}_p := \{ f : U \xrightarrow{\text{DIFF}} \mathbb{R} : p \in U \subseteq M \text{ chart} \}$

This means: $z_p : \mathcal{F}_p \longrightarrow \mathbb{R}$ satisfies
(i) $z_p(a_1 f_1 + a_2 f_2) = a_1 z_p(f_1) + a_2 z_p(f_2)$, $a_i \in \mathbb{R}$, $f_i \in \mathcal{F}_p$
(ii) $z_p(f_1 \cdot f_2) = z_p(f_1) \cdot f_2(p) + f_1(p) \cdot z_p(f_2)$, $f_i \in \mathcal{F}_p$

NOTE: Tangent vectors at p form a vector space T_pN (have $a_1 z_p^1 + a_2 z_p^2$)

Example: Given $\gamma : \mathbb{R} \xrightarrow{\text{DIFF}} N$, $\gamma(0) = p$, have $f \circ \gamma : \mathbb{R} \longrightarrow \mathbb{R}$ so can define
$$z_p(f) := \frac{d}{dt}\Big|_{t=0} (f \circ \gamma).$$
Denote $z_p = \frac{d\gamma}{dt}\Big|_{t=0}$

Example: Given $\varphi : U \longrightarrow \mathbb{R}^n$ chart of N, $\varphi(p) = 0$, have $f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ so can define
$$z_p^i(f) := \frac{\partial}{\partial x_i}\Big|_{x=0} (f \circ \varphi^{-1}).$$
Denote $z_p^i = \frac{\partial}{\partial x_i}\Big|_{x=0}$

THM (1.2.4 in Wall) The tangent vectors $\frac{\partial}{\partial x_i}\Big|_{x=0}$ for $i = 1, 2, \dots, n$ form a basis of T_pN.

IDEA: tangent bundle $TN = \bigsqcup_{n \in N} TN_p \longrightarrow N$ is a smooth vector bundle

def. A map $\pi : E \to N$ is a smooth fibre bundle with a fibre F, for B, F smooth manifolds and structure group $G \leq \mathrm{Diff}(F)$ if there is a cover $B = \bigcup_\alpha U_\alpha$ s.t.

(i) $\exists \psi_\alpha$ s.t.
$$\begin{tikzcd} U_\alpha \times F \arrow[rr, "\psi_\alpha", "\approx"'] \arrow[dr, "pr_1"'] & & \pi^{-1}(U_\alpha) \arrow[dl, "\pi"] \\ & U_\alpha & \end{tikzcd}$$

(ii) $(U_\alpha \cap U_\beta) \times F \xrightarrow{\psi_\alpha|} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\psi_\beta|^{-1}} (U_\alpha \cap U_\beta) \times F$
is given by $\mathrm{Id} \times g_{\alpha\beta}(p, x) = (p, g_{\alpha\beta}(x))$ for $p \in U_\alpha \cap U_\beta$, $x \in F$, $g_{\alpha\beta} \in G$.

If we can choose ψ_α so that $g_{\alpha\beta} \in G' \leq G$ we say the structure group can be reduced to G'.
Such a choice is called a reduction.

$GL_n\mathbb{R}$ $n \times n$ matrices with $\det \neq 0$
$GL_n^+\mathbb{R}$ $n \times n$ matrices with $\det > 0$
O_n $n \times n$ matrices with $\det \in \{-1, 1\}$

Example: $E = B \times F$ the trivial bundle

Example: $F = \mathbb{R}^n$, $G \leq GL_n(\mathbb{R})$ $\rightsquigarrow$ π is a smooth vector bundle of rank n.

Example: $TN := \bigsqcup_{p \in N} T_pN$ and $\pi : TN \longrightarrow N$, $z_p \longmapsto p$

Exercise: show this is a smooth vector bundle. Call it the tangent bundle of N.

Exercise: The total space of a smooth fibre bundle with F an m-manifold and N an n-manifold is a smooth $(m+n)$-manifold.

def. A smooth vector bundle is orientable if can reduce to $GL_n^+(\mathbb{R}) \leq GL_n(\mathbb{R})$.
It admits a Riemannian metric if can reduce to $O(n) \leq GL_n(\mathbb{R})$.

def. For $F : M \xrightarrow{sm} N$ have $dF : TM \to TN$ a smooth map of v. bundles defined by $dF(z_p)(f) := z_p(f \circ F)$, and called the differential of F.

Example: $\gamma : \mathbb{R} \to N$ $\rightsquigarrow$ $d\gamma : T\mathbb{R} \longrightarrow TN$ $\quad d\gamma\left(\frac{\partial}{\partial t}\right) = \frac{d\gamma}{dt}$
$f : N \to \mathbb{R}$ $\rightsquigarrow$ $df : TN \longrightarrow T\mathbb{R}$ $\quad df(z_p) = z_p(f) \frac{\partial}{\partial t}$
Locally have $dx_i \in (T_pN)^\vee = \mathrm{Hom}(T_pN, \mathbb{R})$ $\quad dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$

§ Submanifolds & Transversality

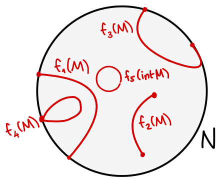
def. A smooth map $f: M \to N$ is an <u>immersion</u> if $Df|_x : TM_x \to TN_{f(x)}$ is injective for every $x \in M$. A <u>smooth embedding</u> is a top. embedding which is an immersion. A smooth embedding is <u>neat</u> if

1° $f(M) \cap \partial N = f(\partial M)$

2° $\forall p \in \partial N \ \exists (U, \varphi : U \hookrightarrow \mathbb{R}^n_+)$ s.t. $U \cap M = \varphi^{-1}(0 \times \dots \times 0 \times \mathbb{R}^{m-1} \times \mathbb{R}_+)$.

def. A (neat) <u>submanifold</u> is a closed subset $M \subseteq N$ s.t. the inclusion map is a (neat) smooth embedding. We define $\operatorname{codim}(M,N) := \dim N - \dim M$.

Examples.

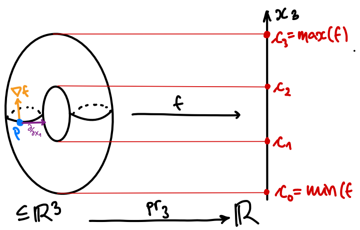


$M = \mathbb{D}^1$, $N = \mathbb{D}^2$
f_1, f_2, f_3 are smooth embeddings
f_1 is neat

def. Let $f: M \to N$ smooth. We call $y \in N$ a <u>regular value</u> if $df_x : TM_x \to TN_{f(x)}$ is surjective for all $x \in f^{-1}(y)$ and $df_x| : T(\partial M)_x \to TN_{f(x)}$ is surjective for all $x \in f^{-1}(y) \cap \partial M$.

def. Let $f: N \to \mathbb{R}$ smooth.
Points $p \in N$ for which $df_p = 0$ are <u>critical points</u>.
Image of a critical point is a <u>critical value</u>.

NOTE: Values of f that are not critical are regular.



Thm. [Cor. of Implicit Function Theorem]
If $y \in N \setminus \partial N$ is a regular value of a smooth map $f: M \to N$, and of $f|_{\partial M}$ then $f^{-1}(y)$ is a neat smooth submanifold of M.
Moreover, $\operatorname{codim}(f^{-1}(y), M) = \dim N$

def. An <u>isotopy</u> is a smooth map $f: M \times [0,1] \to N$ s.t. $\forall t \in [0,1]$ $f_t : M \xhookrightarrow{\text{nt}} N$ is a smooth embedding.
An <u>ambient isotopy</u> of N is a smooth map $F: N \times [0,1] \to N$ s.t. $\forall t \in [0,1]$ $F_t : N \xrightarrow{\cong} N$ is a diffeomorphism.
Given an isotopy $f_t : M \to N$ we say that $F_t : N \to N$ is an <u>ambient extension</u> of f_t if $\forall t \in [0,1]$ $F_t \circ f_0 = f_t$.

Example: Path trough regular values gives an isotopy of preimages.

Thm [Cerf 1961, Palais 1960] – Ambient Isotopy Extension – [Wall 2.4.2]
If M is compact, then any $f_t : M \to N$ admits an ambient extension.

NOTE: This is useful when we want to "move" not only a submanifold but also its tubular nbhd.

def. Two smooth maps $f: M_1 \to N$ and $g: M_2 \to N$ are <u>transverse</u>, $f \pitchfork g$, if $(\forall x_1 \in M_1, x_2 \in M_2)$ $f(x_1) = g(x_2) =: y \Rightarrow df(TM_1)_{x_1} + dg(TM_2)_{x_2} = TN_y$

In particular:
1° $\dim M_1 + \dim M_2 < \dim N$ then $f \pitchfork g$ iff $f(M_1) \cap g(M_2) = \emptyset$
2° $\dim M_1 + \dim M_2 = \dim N$ then $f \pitchfork g$ iff $f(x_1) = g(x_2) \Rightarrow df(TM_{x_1}) \oplus dg(TM_{x_2}) \cong TN_y$.
3° $\dim N = 2 \dim M$ then $f \pitchfork f$ iff $f(x_1) = f(x_2)$ & $x_1 \neq x_2 \Rightarrow df(TM_{x_1}) \oplus df(TM_{x_2}) = TN_y$
4° $g: \{y\} \hookrightarrow N$ then $f \pitchfork \{y\}$ iff y is a regular value of f.

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§ NORMAL BUNDLES & TUB. NEIGHBOURHOODS
Wall §2.3 §2.5

Thm. For a smooth manifold N there is a neighbourhood U of $N \subseteq TN$ (zero-section) and a smooth map $\gamma: U \times [0,1] \to N$ s.t. for $v \in U$ and $s,t \in [0,1]$:

1) $\gamma_v(0) = \pi(v)$, $\dot{\gamma}_v(0) = v$ $\qquad \gamma_v : [0,1] \to N$

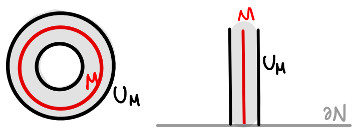
2) $\gamma_{sv}(t) = \gamma_v(st)$ $\qquad d\gamma_v\left(\frac{\partial}{\partial t}\right)_0 = v$

Moreover, if N is compact one can take $U = TN$.
We then define $\exp: U \to N$ by $\exp(v) = \gamma_v(1)$.

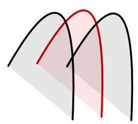
def. The normal bundle $\nu_{N \subseteq M}$ of a smooth submanifold $M \subseteq N$ is the quotient bundle $TN|_M / TM$.
If we fix a Riemannian metric on TN then $\nu_{N\subseteq M} = (TM)^\perp \subseteq TN$.

Thm. [Wall 2.3.8] There is a neighbourhood of $M \subseteq \nu_{M\subseteq N}$ on which $\exp: TN \to N$ is an embedding. Thus, $\exists\, U_M \underset{\text{open}}{\subseteq} N$, $M \subseteq U_M$ and $U_M \to M$ has a structure of a bundle with fibre $\mathbb{D}^{\dim N - \dim M}$, with associated v.bundle $\nu_{M\subseteq N}$, and $U_M \cap \partial N \to \partial M$ is a fibre bundle with associated v.bundle $\nu_{\partial M \subseteq \partial N}$

def. Such U_M is called a neat tubular neighbourhood.



NOTE: A normal vector field is a section of the bundle $\nu_{N\subseteq M} \to N$ and allows us to create "parallel push-offs" of submanifolds:



Exercise. We saw for $f: M \to N$, $y \in N$ regular value: $M^y := f^{-1}(y) \subseteq M$ submanifold. Show that $\nu_{M^y \subseteq M}$ is a trivial bundle and has a canonical trivialisation.

NOTE: - For $M \subseteq N$ if two out of $TM, \nu_{M\subseteq N}, TN$ are oriented, then is also the third, via:
$$TM \oplus \nu_{M\subseteq N} = TN|_M.$$

def. We say that two tub. nbhds $\varphi: U_M \hookrightarrow N$ and $\varphi': U'_M \hookrightarrow N$ are equivalent if there exists a map of fibre bundles

$$\begin{array}{ccc} U_M & \xrightarrow{\Psi} & U'_M \\ \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array}$$

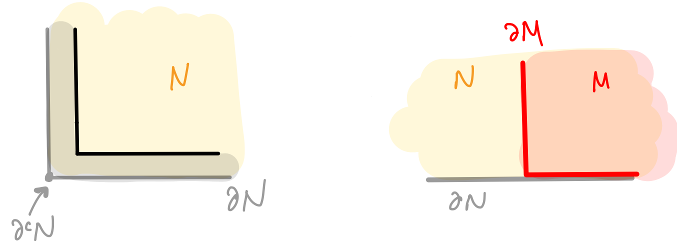
such that $\varphi' \circ \Psi$ is ambiently isotopic to φ, via an ambient isotopy that fixes $M \subseteq N$.

THM [Wall 2.5.5]
If N is a smooth manifold, and $M \subseteq N$ is a compact submanifold, then any two tub nbhds of M are equivalent.

THM [Palais] [Wall 2.5.6]
Any two $\mathbb{D}^n \hookrightarrow N$, either both orient. preserving or reversing, are ambiently isotopic

§ CORNERS
Wall §2.6.

Want: $\mathbb{D}^r \times \mathbb{D}^s \longrightarrow \mathbb{D}^{r+s}$
diffeomorphism away from the corner $(\partial\mathbb{D}^r) \times (\partial \mathbb{D}^s)$.



Whenever N has a corner, there is a straightening map: a homeomorphism to a smooth manifold with boundary that is a diffeomorphism away from the corners.