LECTURE 3

§ VECTOR FIELDS — see Wall §1.4 §1.5

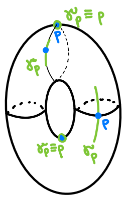
def. A (smooth) vector field on N is a (smooth) section of $TN \xrightarrow{\pi} N$,
that is, a (smooth) map $X: N \to TN$ with $\pi \circ X = id_N$. Write $X_p := X(p)$.

Exercise. In a chart (U, φ) around p: $X_p = \sum_{i=1}^{n} X^i(p) \frac{\partial}{\partial x_i}\Big|_p$,
where x_i are coordinates in $\varphi(U) \subseteq \mathbb{R}^n$ and $X^i: U \to \mathbb{R}$ smooth functions.

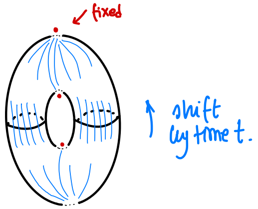
Thm [Integration of a vector field]
If N is a smooth n-manifold with $\partial N = \emptyset$ and $X: N \to TN$ is a smooth vector field,
then there is a continuous map $\varepsilon: N \to [0, \infty)$, $p \mapsto \varepsilon_p$
and a unique map
$$\gamma: \{(p,t) \in N \times \mathbb{R} : |t| < \varepsilon_p\} \longrightarrow N, \quad (p,t) \mapsto \gamma_p(t)$$
such that $\forall p \in N$ $\gamma_p: [-\varepsilon_p, \varepsilon_p] \longrightarrow N$
satisfies $\gamma_p(0) = p$ and $(d\gamma_p)_t(\partial/\partial t) = X_{\gamma_p(t)}$.

* Idea: solve a system of ordinary differential equations (ODEs)
and glue solutions together using a partition of unity.

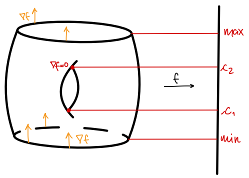
Example.



for some $t > 0$
have $\gamma(t): N \to N$



example with boundary:



in local coordinates:
$$\nabla f = \sum_{i,j} g_{ij} \frac{\partial f}{\partial x_j} \partial/\partial x_i$$
where $g = \sum_{i,j} g_{ij}\, dx_i\, dx_j$

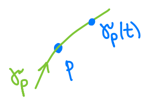
Exercise. Boundary of a smooth manifold admits a collar,
i.e. $\partial N \subseteq N$ has a nbhd diffeomorphic to $\partial N \times [0,1)$, with ∂N identified with $\partial N \times \{0\}$.
(Hint: Construct a vector field X on N s.t. $TN|_{\partial N} = T(\partial N) \oplus X$.)

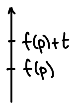
Thm [no critical points $\Rightarrow$ cylinder] *We call such $(N, \partial_0 N, \partial_1 N)$ a cobordism.*
Assume N compact and $\partial N = \partial_0 N \sqcup \partial_1 N$ for some smooth manifolds $\partial_i N$.
Let $f: N \to \mathbb{R}$ smooth with $f^{-1}(i) = \partial_i N$, for $i = 1, 2$.
If f has no critical points (i.e. $df_p \neq 0$ $\forall p \in N$)
then $N \cong \partial_0 N \times [0,1]$ (and in particular $\partial_0 N \cong \partial_1 N$).

Lemma [Wall 1.5.4]
$(N, \partial_0 N, \partial_1 N)$ a compact cobordism.
ξ a vector field on N s.t. ξ points inward on $\partial_0 N$,
ξ points outward on $\partial_1 N$
$f: N \to \mathbb{R}$ s.t. $\xi_p(f) > 0$ $\forall p \in N$
Then
$$N \cong \partial_0 N \times [0,1]$$

Apply this to $\xi = \nabla f$ defined as the v. field that corresponds to df
under $T^*N \overset{\cong}{\to} TN$ given by a Riem. metric.

* Idea:





in particular $p \in \partial_0 N$ have $f \circ \gamma_p(t) = t$

Therefore: the map $G: \partial_0 N \times [0,1] \longrightarrow N$, $(p,t) \mapsto \gamma_p(t)$ is well-defined and smooth,
and has a smooth inverse $p \longmapsto (\gamma_p(-f(p)), f(p))$.
$\square$

Q: What happens when there are critical points?
A: Morse theory: study how topology changes when passing through a critical point
$\rightsquigarrow$ a handle is attached, finitely many possibilities.

Key THM [Smale, Wallace, around 1960] — Handle Decomposition Theorem —

For any cobordism $(W, \partial_0 W, \partial_1 W)$ there exists a sequence of smooth cobordisms
$$\partial_0 W \times [0,1] = W_{-1} \subseteq W_0 \subseteq \dots \subseteq W_m = W$$
such that W_k is obtained from W_{k-1} by attaching a collection of k-handles to its top boundary: $W_k = W_{k-1} \cup h_1^k \cup \dots \cup h_{r_k}^k$

index of a handle

NOTE: Can take $\partial_0 W = \partial_1 W = \emptyset$ so W is a smooth manifold without boundary. Then theorem says W decomposes into a union of handles.

* Idea:
- Pick a function $f: W \to \mathbb{R}$, $f(\partial_i W) = i$ for $i=0,1$ and denote $W_{\le s} := f^{-1}((-\infty, s])$ for some $s \in \mathbb{R}$.
- We already saw: $W_{\le s} \cong W_{\le t}$ if f has no critical values in $[s,t] \subseteq \mathbb{R}$
- If $p \in f^{-1}([s,t])$ is a unique crit. point for some $[s,t] \subseteq \mathbb{R}$

WANT: p non-degenerate i.e. $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_p\right) \neq 0$ (Hessian of f at p is nondegenerate)

Step 1. There exists a Morse function $h: W \to \mathbb{R}$ with $h^{-1}(j) = W_j$, $j=0,1$.
meaning, critical values of h are distinct and critical points are non-degenerate, i.e. $\det\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\Big|_p\right)_{i,j} \neq 0$

def. $\mathrm{ind}_p(h)$:= the number of neg. eigenvalues of the Hessian.

Step 2. — Morse Lemma —

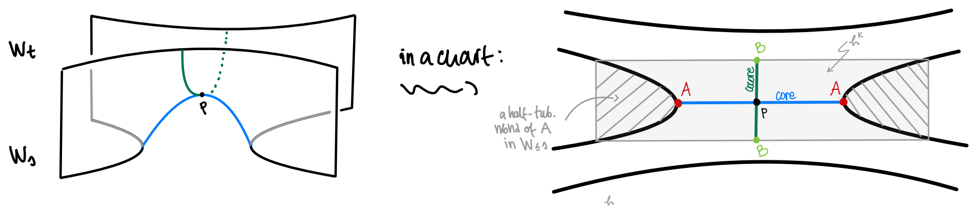
For a critical point $p \in W$ of h of index k, there exists a chart (U,φ) s.t. $h \circ \varphi^{-1}(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$.

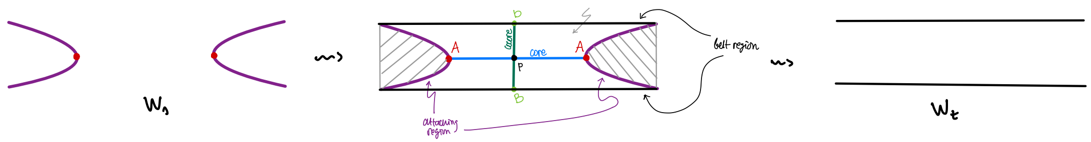
Step 3. — Passing a Critical Point Lemma —

If $W_{[s,t]}$ contains a single crit. point $p \in W$ of h, and $\mathrm{ind}_p(h) = k$, then $W_{\le t}$ is obtained from $W_{\le s}$ by attaching a handle of index k.

* Idea:

Can reparametrize h so that $W_{\le t} \setminus W_{\le s}$ contained in a chart $(U,\varphi) \ni p$





NOTE: W_t is obtained from W_s by surgery on the sphere A.

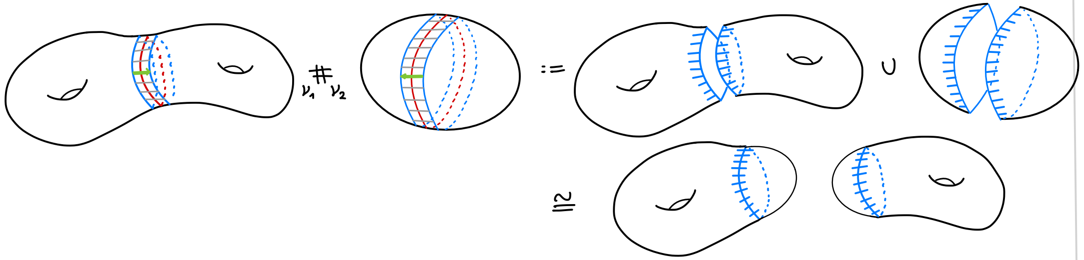
this means take out nbhd of A (given by $\varphi: S^{k-1} \times D^{n-k} \hookrightarrow W_s$)

and glue in $D^k \times S^{n-k-1}$ (the belt region).

§ How to (de)construct manifolds

def (GLUE) Given two smooth manifolds with $\dim N_1 = \dim N_2 = n$,
a rank $(m-n)$ bundle $E \longrightarrow Y$ over a smooth m-manifold, $m \le n$
and neat tub. nbhds $\nu_i : E \hookrightarrow N$ of neat submanifolds $\nu_i(Y)$.

Define:
$$N_1 \,{}_{\nu_1}\#_{\nu_2}\, N_2 := N_1 \setminus \nu_1(Y) \sqcup N_2 \setminus \nu_2(Y) \,/\, \nu_1(v) = \nu_2\left(\mathrm{rev}(|v|)\cdot \tfrac{v}{|v|}\right), \ \forall v \in E$$
where $\mathrm{rev} : (0,\infty) \longrightarrow (0,\infty)$ is an orientation reversing diffeomorphism.



NOTE: For $\nu_i : E \hookrightarrow \partial N_i$ can still define $N_1 \,{}_{\nu_1}\#_{\nu_2}\, N_2$.
We can first glue the boundaries: $\partial N_1 \,{}_{\nu_1}\#_{\nu_2}\, \partial N_2$
But then we have to make sure we can "put N_1 and N_2 back in" and still have a smooth structure. We can do this using "half-tubular" neighbourhoods

THM. This operation yields a well-defined smooth manifold, which up to diffeomorphism does not depend on rev and depends only on ν_i up to isotopy.

Some special cases:

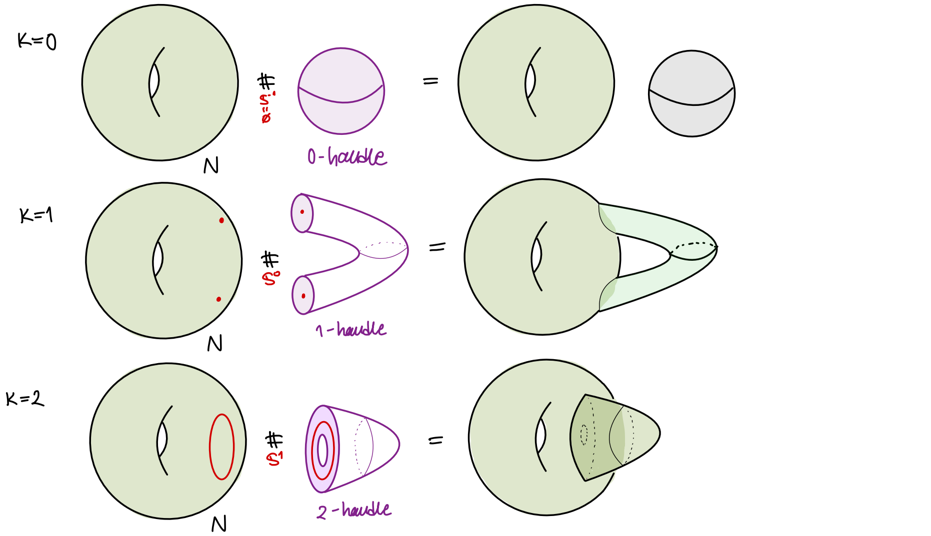
1° connected sum $N_1 \# N_2$ for $Y = pt$, $E = \mathbb{R}^n$, $\nu_i : \mathbb{R}^n \hookrightarrow \mathrm{int} N_i$

2° boundary connected sum $N_1 \natural N_2$ for $Y = pt$, $E = \mathbb{R}^{n-1}$, $\nu_i : \mathbb{R}^{n-1} \hookrightarrow \partial N_i$

THM [Palais]
Any two $D^n \hookrightarrow N$, either both orient. preserving or reversing, are ambiently isotopic.
Hence, connected & boundary connected sum are well-defined and indep. of all choices.

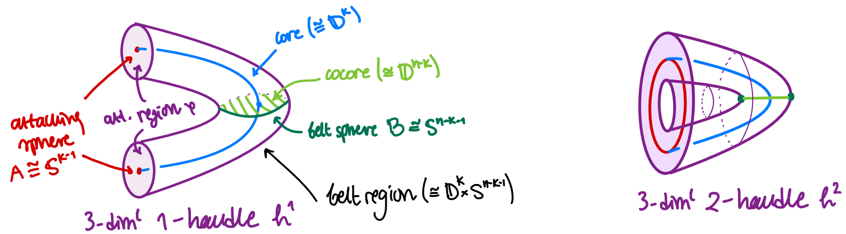
3° handle attachment: $N_1 = N$, $N_2 = D^n$, $Y = S^{k-1}$
$\nu_1 : S^{k-1} \times \mathbb{R}^{n-k} \hookrightarrow \partial N$, $\nu_2 : S^{k-1} \times \mathbb{R}^{n-k} \hookrightarrow S^{k-1} \times D^{n-k} \subseteq D^k \times D^{n-k} \approx D^n$.

Examples.



k=3 this is boundary connected sum with D^3 (so not possible for $N = S^1 \times D^2$)

NOTE: We simplify by thinking of handle attachment as just $N \cup_\varphi h^k \equiv N \,{}_{\nu_1}\#_{\nu_2}\, D^n$
where $h^k := D^k \times D^{n-k}$ is the handle of index k (k-handle)
$\varphi := \overline{\nu}_1 : S^{k-1} \times D^{n-k} \hookrightarrow \partial N$ is the attaching region



NOTE: The usual defⁿ of $N \cup_\varphi h^k$ as the gluing of top. spaces is apriori not a smooth manifold, but our defⁿ $N \,{}_{\nu_1}\#_{\nu_2}\, D^n$ is!

4° surgery along a sphere: $N_1 = N$, $N_2 = S^n$, $Y = S^{k-1}$, $\nu_1 : S^{k-1} \times \mathbb{R}^{n-k} \hookrightarrow \partial N$, $\nu_2 : S^{k-1} \times \mathbb{R}^{n-k} \underset{std}{\hookrightarrow} S^n$