LECTURE 4

Exercise. Use the Handlebody Decomposition Thm to prove the classification of compact surfaces.

Exercise. Relate surgery on a $(k-1)$-sphere and handle attachment of a k-handle.

§ HANDLE CALCULUS

See Wall §5.

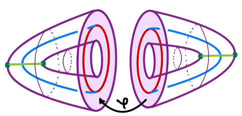
Cor of Amb Istpy Ext. **– Isotopy Lemma –** If $\varphi_i : S^{k-1}\times D^{n-k} \hookrightarrow \partial N$ are isotopic $i=1,2$, then $N\cup_{\varphi_i} h^k$ are diffeomorphic $i=1,2$.

– Unknot Lemma –

If $N := \mathbb{D}^n \cup_\varphi h^k$ and $A := \varphi|_{S^{k-1}\times 0} : S^{k-1} \hookrightarrow \partial \mathbb{D}^n$ bounds a disc, $A(S^{k-1}) = \partial\Delta^k$, then N is a D^{n-k}-bundle over a smooth manifold homeomorphic to S^n.

proof. Push the interior of Δ into $\mathbb{D}^n$, so $\Delta' \subseteq \mathbb{D}^n$, $\partial\Delta' = A(S^{k-1})$. Then $\mathbb{D}^n$ can be viewed as a tub nbhd $\nu_\Delta : \Delta' \times D^{n-k} \xrightarrow{\cong} \mathbb{D}^n$. Then: $N = \mathbb{D}^n \cup_\varphi h^k \cong (\Delta' \times D^{n-k}) \underset{S^{k-1}\times D^{n-k} \circlearrowleft \varphi}{\cup} (\mathbb{D}^k \times D^{n-k})$

Now, the projections $\mathbb{D}^k \times D^{n-k} \longrightarrow \mathbb{D}^k$ glue together along $(\partial\mathbb{D}^k)\times D^{n-k}$ to a well-defined map $N \longrightarrow \underbrace{(\mathbb{D}^k \cup_A \mathbb{D}^k)}_{\text{see Lemma below}}$ which is a fibre bundle with fibre D^{n-k}.



□

def. For a diffeomorphism $A : S^{k-1} \circlearrowleft \cong$ define the smooth manifold $S(A) := \mathbb{D}^k \cup_A \mathbb{D}^k$.

Lemma. $S(A)$ is always homeomorphic to S^k.

proof. Define a homeomorphism $\mathbb{D}^k \cup_A \mathbb{D}^k \xrightarrow[\approx]{\mathrm{id}\cup \bar{A}} \mathbb{D}^k \cup_{\mathrm{id}_{S^{k-1}}} \mathbb{D}^k = S^k$

where $\bar{A} : \mathbb{D}^k \longrightarrow \mathbb{D}^k$, $\bar{A}(\underset{[0,1]}{r}, \underset{S^{k-1}}{v}) = (r, A(v))$ is a homeomorphism extending A radially. □

NOTE: $S(A)$ is not diffeomorphic to S^k in general. It is called a "twisted sphere".
We will see:

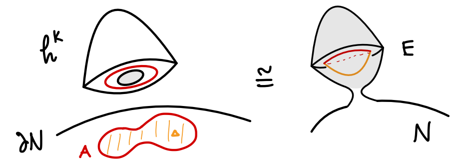
Smale's h-cobordism Thm $\Rightarrow$ Every exotic sphere of dim ≥ 5 is a twisted sphere.

Exercise. A twisted sphere $S(A) = \mathbb{D}^k \cup_A \mathbb{D}^k$ is diffeomorphic to S^k if and only if $A : S^{k-1} \to S^{k-1}$ extends to a diffeomorphism $\mathbb{D}^k \to \mathbb{D}^k$.

NOTE: The Unknot Lemma is not true if $\Delta \subseteq \mathbb{D}^n$ instead of $\Delta \subseteq \partial\mathbb{D}^n$.
The condition $\Delta \subseteq \partial\mathbb{D}^n$ is equivalent to A being "unknotted" whereas $\Delta \subseteq \mathbb{D}^n$ is equivalent to A being "slice".
↖ A is isotopic to a small sphere (in a chart)

For example: $\exists$ many $A : S^1 \hookrightarrow S^3$ s.t. $A \not\simeq U$ but A is slice e.g.

Cor. If $A : S^{k-1} \hookrightarrow \partial N$ bounds a disc $A(S^{k-1}) = \partial\Delta^k$, then
$$N \cup_\varphi h^k \cong N \natural E$$
where $E \longrightarrow S(A)$ is a D^{n-k}-bundle.



– Upside Down Lemma – For every handle decomposition of $(W, \partial_0 W, \partial_1 W)$ there is an "upside-down" decomposition of $(W, \partial_1 W, \partial_0 W)$ with handles of index $n-k$ attached along the belt spheres of k-handles of the original decomposition.

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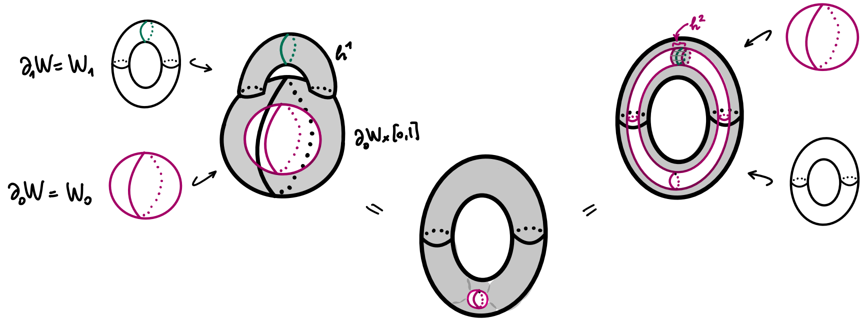
proof. FACT: Every handle decomposition corresponds to a Morse function, call it h.
Then $-h$ yields a decomposition of the upside-down cobordism.
We just observe that turning a k-handle upside-down turns its belt region into the attaching region.



– Reordering Lemma – If $k \le \ell$ then $(N \cup_{\varphi_1} h_1^\ell) \cup_{\varphi_2} h_2^k \cong (N \cup_{\varphi_2''} h_2^k) \cup_{\varphi_1'} h_1^\ell$

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for some isotopic attaching map $\varphi_2'' \cong \varphi_2$, with $\operatorname{im}\varphi_2 \subseteq \partial N$, and φ_1' has the same image as φ_1.

proof. Denote $A_2 :=$ the attaching sphere of h_2, $B_1 :=$ the belt sphere of h_1.

Thm [Thom] If $A : M \to N$ a smooth map and $B \subseteq N$ a compact submanifold then there is an ambient isotopy of N, taking A to A' such that $A' \pitchfork B$. Moreover, the isotopy can be assumed to be the identity outside of any open nbhd of B.

Assuming this, we have $A_2' \pitchfork B$ i.e. $dA_2'(TS^{k-1})_a + dB(TS^{n-\ell-1})_b = T\partial(N \cup h_1)_x$ for every a, b s.t. $A_2'(a) = B(b)$. However, since

$$\dim B_1 + \dim A_2 = n-\ell-1 + k-1 = (n-1) + (k-\ell) - 1 < n-1 = \dim \partial(N \cup h_1)$$

we must have $A_2' \cap B = \emptyset$. We can isotope further, so that $A_2'' \subseteq \partial N$ (i.e. the belt region) away from

By the Ambient Isotopy Extension Thm we have $\varphi_2''(S^{k-1} \times D^{n-k}) \subseteq \partial N$.

Thus, the two handles can be attached in any order (or simultaneously). $\square$

sketch proof of Thom's Thm:

Firstly, find a tubular nbhd U_B of B contained in the given open set $U \supseteq B$.

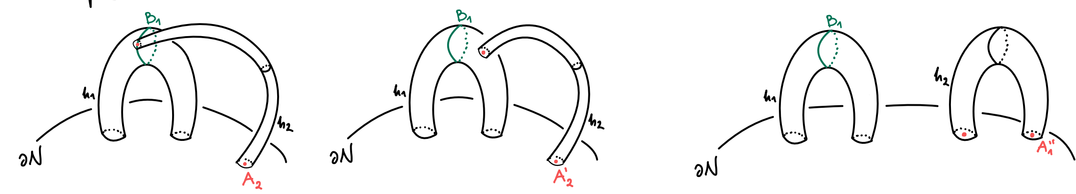
Then apply to $E = U_B \to B$ the following:

Lem. If $f : M \to E$ is smooth and $E \xrightarrow{\pi} N$ a smooth vector bundle, then there exist a section $s : N \to E$ such that $f \pitchfork s$.

Thus, there is an obvious isotopy from B to $s(B) \subseteq U_B \subseteq U$ and we can extend it by Id on $N \setminus U$.

To prove the Lemma, use Morse-Sard Thm to get the result for trivial bundles, and extend to all bundles using that all vector bundles have stable inverses. $\square$.

Example. $n=3$, $k=\ell=1$



– Cancellation Lemma – If $\underbrace{A_2 \pitchfork B_1 = \{p\}}$ then $(N \cup_{\varphi_1} h_1^k) \cup_{\varphi_2} h_2^{k+1} \cong N$.

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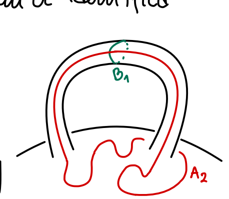
We say that h_1 and h_2 are in a geometrically cancelling position.
Or h_2 goes over h_1 geometrically once.

proof. Since A_2^{k-1} and B_1^{n-k-1} intersect transversely, and $\nu_{B_1 \subseteq N \cup_{\varphi_1} h_1}$ can be identified with the belt region $D^k \times \partial D^{n-k} \subseteq \partial h_1^k$, we can assume $A_2 \cap \partial h_1^k \cong D^k \times \{p\}$ (the fibre of $\nu_{B_1 \subseteq N \cup_{\varphi_1} h_1}$ at $p \in B_1$).



Then by Cor. of Unknot Lemma for

$$N' := N \cup_{\varphi_1} h_1^k \quad \text{and} \quad A := A_1 \quad \text{and} \quad \Delta := A_2 \cap \partial N$$

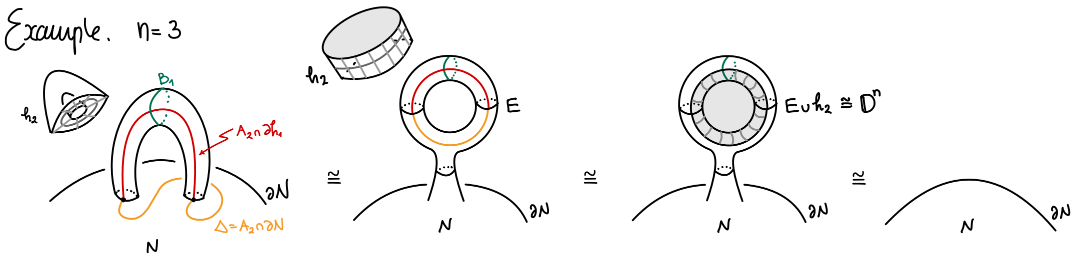
we have diffeos:

$$(N \cup_{\varphi_1} h_1^k) \cup_{\varphi_2} h_2^{k+1} \cong (N \natural E) \cup_{\varphi_2} h_2 \cong N \natural (E \cup_{\varphi_2} h_2) \cong N \natural D^n \cong N.$$

since $\operatorname{im}\varphi_2 \subseteq \partial E$

since $D^{n-k} \to E \to S(A_1)$ and h_2 glued onto $A_2 \subseteq \partial E$ which is a section of E. $\square$

Example. $n=3$



Note: We can reverse the argument to show that a cancelling pair can be added.