LECTURE 8

Lemma. $\mathcal{J}^{\text{ult}}$ can be modified to the identity matrix $Id_{(\mathbb{Z}\pi)^k}$ by the moves listed below, if and only if all the remaining handles can be put into alg. cancelling position.

MOVES: 1° interchange rows

2° add rows ($\pm \mathbb{Z}[\pi]$)

3° (de)stabilize

4° multiply a row by $g \in \pi$ (or $-g$)

proof. Show that each move on matrices can be realised by a move on handles. [Exercise]. □

def. The Whitehead group $Wh(\pi)$ is the set of equivalence classes under moves 1°–4° of invertible matrices of arbitrary size with entries in $\mathbb{Z}[\pi]$ with group structure $\mathcal{J} + \mathcal{J}' = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & \mathcal{J}' \end{pmatrix}$.

NOTE: Equivalently, $Wh(\pi) := GL(\mathbb{Z}\pi)^{ab} \big/ \langle [g], [-g] : g \in \pi \rangle$

where $GL(R) := \operatorname{colim}_{n\to\infty} GL_n(R)$ for a ring R,

and ab denotes abelianisation ($K_1(R) := GL(R)^{ab}$)

Examples. $Wh(\{1\}) = 0$ since $\mathbb{Z}\{1\} = \mathbb{Z}$ has Euclidean algorithm

$Wh(\pi) = 0$ for $\pi =$ free abelian group [Bass-Heller-Swan '64]

$Wh(\mathbb{Z}/_5\mathbb{Z}) = \mathbb{Z}$ generated by the unit $t + t^{-1} - 1 \in GL_1$.

Conjecture. $Wh(\pi) = 0$ iff π is torsion-free.

def. Whitehead torsion of $(W, \partial_0 W, \partial_1 W)$ is $\tau_W := [\mathcal{J}^{\text{ult}}] \in Wh(\pi_1 W)$.

Remark. $\tau_W = 0$ iff $\partial_i W \hookrightarrow W$ are simple homotopy equivalences.

gives the name to the s-cobordism Thm.

Step 3.

We now want to use Whitney moves to turn an algebraically cancelling pair of handles, into a geometrically cancelling pair.

– Whitney Trick Lemma –

based spheres: $\tilde{A} = A \cup w_A$, $\tilde{B} = B \cup w_B$

If $\dim N \geq 5$ and $\tilde{A}: S^{n_1} \hookrightarrow N$, $\tilde{B}: S^{n_2} \hookrightarrow N$ have $\tilde{I}(\tilde{A} \pitchfork \tilde{B}) = +1$, then there is an isotopy of $\tilde{A}$ such that $\tilde{A}' \pitchfork \tilde{B} = \{pt\}$.

$(n_1 + n_2 = n = \underbrace{\dim N}_{\geq 5})$

proof. Having $\tilde{I}(\tilde{A} \pitchfork \tilde{B}) = \sum_{p \in A \pitchfork B} \varepsilon_p g_p = +1 = (\overbrace{\varepsilon_p g_p + \varepsilon_q g_q}^{0}) + \ldots + \underbrace{\varepsilon_r}_{+1} \underbrace{g_r}_{1}$

implies that we can find pairs $p, q \in A \pitchfork B$ such that $\varepsilon_q g_q = -\varepsilon_p g_p$

$\Rightarrow \exists$ Whitney circle $\gamma_1 \cdot \gamma_2^{-1}$ through p and q, which is nullhomotopic in N

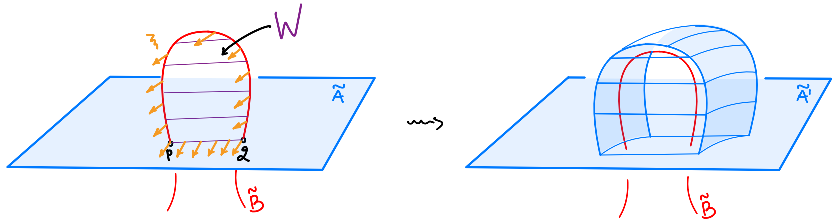
Since $n_i \leq n-3$, $i = 1,2$ $\Rightarrow \pi_1(N \setminus (A \cup B)) \cong \pi_1 N \Rightarrow \gamma_1 \gamma_2^{-1} \simeq *$ in $N \setminus A \cup B$

$\Rightarrow \gamma_1 \gamma_2^{-1}$ bounds an immersed disc in $N \setminus A \cup B$.

Since $n \geq 5 \Rightarrow \gamma_1 \gamma_2^{-1}$ bounds an embedded disc $W: D^2 \hookrightarrow N$ with $\operatorname{int} W \pitchfork A \cup B = \emptyset$

Since $\varepsilon_q = -\varepsilon_p$ and $n > 4 \Rightarrow W$ can be framed.

$\Rightarrow$ We can perform the Whitney move to remove p, q. Continue with other pairs, until precisely one intersection p with $\varepsilon_p g_p = +1$ left. □



Corollaries.

Thm – Top Poincaré Conjecture in dim ≥ 6 –

If N is a smooth homotopy n-sphere and $n \geq 6$,
then N is homeomorphic to S^n (i.e. N is an exotic n-sphere).

proof. Remove two small discs from N. The resulting manifold is a simply connected h-cobordism from S^{n-1} to itself, so by the h-cobordism theorem:

$$(N \setminus \mathbb{D}_1^n \cup \mathbb{D}_2^n, \partial \mathbb{D}_1^n, \partial \mathbb{D}_2^n) \cong (\partial \mathbb{D}_1^n \times [0,1], \partial \mathbb{D}_1^n \times \{0\}, \partial \mathbb{D}_1^n \times \{1\})$$

We can glue back $\mathbb{D}_1^n$ by $id_{\partial \mathbb{D}_1^n}$, but $\mathbb{D}_2^n$ has to be glued back by a homeomorphism extending the diffeomorphism $\partial \mathbb{D}_2^n \to \partial \mathbb{D}_1^n \times \{1\}$ (use the radial extension, see Lecture 3) $\square$.

Thm [Diff Schoenflies Conjecture in dim ≥ 6]

If $K : S^{n-1} \hookrightarrow S^n$ is a smooth embedding and $n \geq 6$,
then the closure of each component of $S^n \setminus K(S^{n-1})$ is diffeomorphic to $\mathbb{D}^n$.

proof. Since K has a tubular neighbourhood, we see that the closure of each component of $S^n \setminus K(S^{n-1})$ is a smooth manifold with boundary S^{n-1}.
It is simply connected by Seifert-van-Kampen Theorem.
Thus, if we remove from it a small disc we get a simply connected h-cobordism. By the h-cobordism Theorem this is diffeomorphic to $S^{n-1} \times [0,1]$, and we can put back the disc by the identity to get a diffeomorphism to $\mathbb{D}^n$. $\square$

Lemma. PC $\Rightarrow$ SC

proof. observe: Schönflies B^4 is contractible, has $\partial = S^3$.
So $\mathcal{B} \cup B^4$ is a homotopy S^4.
so $\mathcal{B} \cup B^4$ is standard S^4
but Palais implies B^4 is isotopic to northern hemisphere
so $\mathcal{B}$ is isotopic to the southern hemisphere. $\square$

IN FACT:

PC: $\forall$ homotopy S^4 is standard
$\iff$ $\forall$ homotopy B^4 is standard

$\Rightarrow$: $\mathcal{B} \cup B^4$ is a htpy S^4
so $\mathcal{B} \cup B^4 \cong S^4$
Palais: can isotope $B^4 \to$ northern hemisph.
then $\mathcal{B} \xrightarrow{\cong}$ southern hemisphere.

$\Leftarrow$: if $\mathcal{S}$ is a homotopy S^4
then $\mathcal{S} \setminus B^4$ is a homotopy B^4
so $\mathcal{S} \setminus B^4 \cong B^4$
any diffeo of S^3 extends over B^4, so $\mathcal{S} \cong S^4$.

$\Rightarrow$ SC: $\forall$ Schönflies B^4 is standard.