15

Basic Geometric Constructions

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We have now described the finite and infinite iterated objects that will appear in the proof of the disc embedding theorem. In this chapter, we gather some tools that we will use to build them.

Throughout this chapter we work in an ambient smooth 4-manifold M.

15.1 The Clifford Torus

In a neighbourhood of a double point of immersed surfaces A and B (where possibly $A = B$) in M, lies an embedded torus, as shown in Figure 15.1. We call this torus the *Clifford torus* of the double point. More concretely, the local picture at the double point is modelled by the xy- and zt-planes intersecting at the origin in $\mathbb{R}^4$, and the Clifford torus corresponds to $S^1 \times S^1$, where the first S^1 factor is the unit circle in the xy-plane and the second is the unit circle in the zt-plane. The fundamental group of this torus is generated by meridians of the two surfaces. Clifford tori are equipped with a canonical framing by construction.

A finger move from an immersed surface A to an immersed surface B within the ambient 4-manifold M, where possibly $A = B$, changes the fundamental group of $M \smallsetminus (A \cup B)$ by adding the relation corresponding to the 2-cell of the Clifford torus of either of the newly introduced double points. The standard argument for this [Cas86] goes as follows. Let A' denote the surface A after the finger move. After the finger move, there is an embedded, framed Whitney disc D for the two new intersection points between A' and B, the Whitney move along which would undo the finger move. Observe that $\pi_1(M \smallsetminus (A \cup B)) \cong \pi_1(M \smallsetminus (A' \cup B \cup D))$. The Whitney disc D intersects the Clifford torus T in a single point, so there is a meridional disc of D that is also a disc in T. The boundary of this disc is a meridian for D and is freely homotopic in $M \smallsetminus (A' \cup B \cup D)$ to the attaching circle for the 2-cell of T. This attaching circle corresponds to a commutator $[\mu_A, \mu_B]$, where μ_A and μ_B are appropriately based meridians of A and B, respectively. Let μ_D be a (based) meridian of D.

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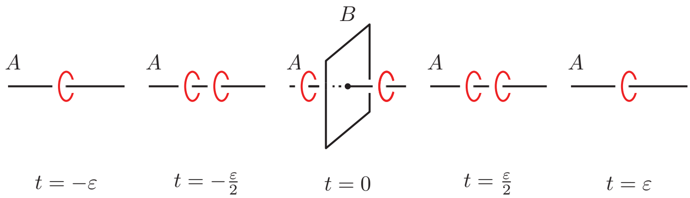

Figure 15.1 A Clifford torus at a double point of the surfaces A and B. As usual, the fourth coordinate is interpreted as time, t. The double point is seen at the $t = 0$ time slice. The red circles depicted trace out the Clifford torus as we move backwards and forwards in time. Note that each S^1 factor of the torus corresponds to a meridian of either A or B, and as such bounds a meridional disc that intersects the respective surface transversely at a single point. Compare with Figure 1.5.

Given a group G and $g \in G$, let $\langle\langle g \rangle\rangle$ denote the normal subgroup of G generated by g. We have that

$$\begin{aligned} \pi_1(M \smallsetminus (A' \cup B)) &\cong \pi_1(M \smallsetminus (A' \cup B \cup D))/\langle\langle \mu_D \rangle\rangle \\ &\cong \pi_1(M \smallsetminus (A' \cup B \cup D))/\langle\langle \gamma_T \rangle\rangle \\ &\cong \pi_1(M \smallsetminus (A \cup B))/\langle\langle [\mu_A, \mu_B] \rangle\rangle, \end{aligned}$$

as asserted.

15.2 Elementary Geometric Techniques

The following operations will be described on (immersed) surfaces. Where necessary, these operations can be extended to framed surfaces, and therefore to stages of capped or uncapped gropes or towers. However, we will not comment on this every time, unless there is extra particular care that needs to be taken.

15.2.1 Tubing

Tubing was described in [FQ90, Sections 1.8 and 1.9]. Suppose we have an immersed connected surface, A. Let Σ and Σ' be two other immersed surfaces intersecting A transversely at points p and p', respectively, with $p \neq p'$. Let γ be a smooth, embedded arc in A joining p and p'. Consider a tubular neighbourhood of γ intersecting A and B in small discs about p and p'. Cut out these discs from $A \cup B$ and glue on the rest of the boundary of the neighbourhood of γ to $\Sigma \cup \Sigma'$. In other words, we are gluing in a meridional annulus for γ. This process is called *tubing* Σ *into* Σ' (*along* $\gamma \subseteq A$) or vice versa. (In [FQ90], the resulting surface is called the *sum* of Σ and Σ'.) We shall at times also refer to this process as tubing p into the surface Σ'. If Σ and Σ' are distinct framed immersed surfaces, then the result of tubing is itself immersed and inherits a framing coming from the framing of the annulus, of Σ, and of Σ'. If Σ and Σ' are the same immersed framed surface, we need additionally that the intersection points p and p' have opposite signs, otherwise we would obtain a

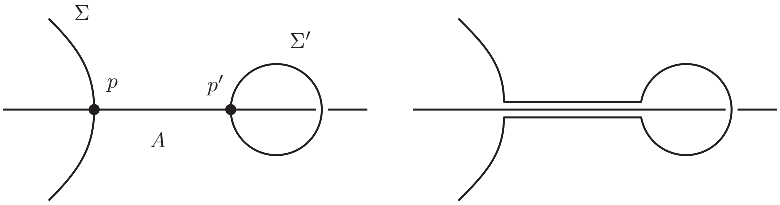

Figure 15.2 Tubing into a transverse sphere.

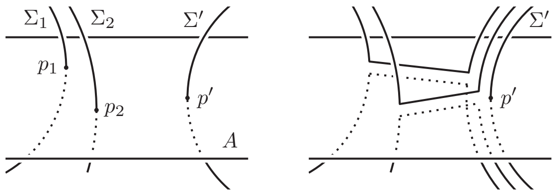

Figure 15.3 Tubing multiple points of intersection.

nonorientable surface. Note that the notion of 'opposite signs' is still meaningful even if the ambient space M is nonorientable, by using the arc γ to transport a local orientation at p to one at p'.

We will often tube surfaces into geometrically transverse spheres; that is, Σ' will be a framed sphere such that p' is its unique point of intersection with A. This situation is depicted in Figure 15.2. Note that this operation then decreases the number of points of intersection between A and Σ by one.

In most cases, we will tube into a parallel push-off of Σ' instead of Σ' itself. This will allow us to tube multiple times, as follows. Suppose that Σ' is a framed surface geometrically transverse to A; that is, A and Σ' intersect exactly once, at the point p'. Now suppose we have surfaces Σ_i intersecting A at the points p_i, for $i = 1, \ldots, k$. Then take k parallel push-offs of Σ', using distinct nonvanishing sections of the normal bundle (of the form $\Sigma' \times \{p\}$ in the given framing). These intersect A at k distinct points. Find pairwise disjoint, embedded arcs on A joining these points to the $\{p_i\}$ and use these to tube each Σ_i into a distinct parallel push-off of Σ'. This eliminates the intersection points $p_1, \ldots, p_k$, and also leaves the original Σ' unchanged for further tubing or other constructions. This process is described in Figure 15.3.

Note that if some surface B intersects Σ', then now Σ_i intersects B for each i. In particular, if Σ' has double points, then the tubing creates new points in $\Sigma_i \pitchfork \Sigma_j$, for all i, j, as well as in $\Sigma_i \pitchfork \Sigma'$.

15.2.2 Boundary Twisting

Boundary twisting was introduced in [FQ90, Section 1.3]. Suppose we have two immersed surfaces A and B in a 4-manifold M such that part of the boundary of B is embedded in A, as shown in Figure 15.4(a), and this part of ∂B lies in the interior of M. For us, this situation usually arises when B is a Whitney disc pairing intersection points of A with itself

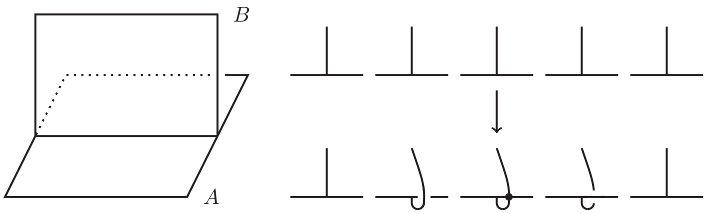

Figure 15.4 Boundary twisting. Left: Before boundary twisting. Right: Cross sections for the picture on the left, before and after boundary twisting. Each cross section shows a 3-dimensional time slice.

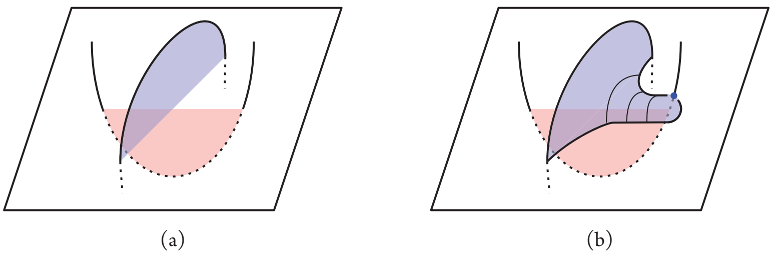

Figure 15.5 Boundary push off to ensure Whitney circles are disjoint. We perform a regular homotopy of one Whitney circle until it becomes disjoint from the other, introducing a new intersection point for the corresponding Whitney disc.

or some other surface. Another commonly occurring situation is that of a cap of a grope or tower attached to a lower stage. The operation of *boundary twisting* B about A consists of changing a collar of B near a point in its boundary on A, as depicted in Figure 15.4(b). Note that this creates a new point of intersection between A and B and changes the framing of B by a full twist.

15.2.3 Making Whitney Circles Disjoint

We saw in Chapter 11 that, given two immersed surfaces A and B, if $\lambda(A, B) = 0$, the intersection points between A and B can be paired up with Whitney discs (Proposition 11.10). *A priori,* the corresponding Whitney circles may intersect one another. Performing the Whitney trick with intersecting Whitney circles leads to new intersections, which we would like to avoid. We ensure that Whitney circles are disjoint by pushing one Whitney circle along the other, as shown in Figure 15.5. This is a regular homotopy, and as we see in the figure, leads to new intersections between a Whitney disc and either A or B.

To achieve this in general for surfaces A and B, enumerate the Whitney arcs, and then work on the arcs in order. For the ith arc, push other arcs with index greater than i off the ith arc, starting with one of the arcs closest to the endpoint, until the ith arc is disjoint from all other arcs. At the end of the process, all Whitney arcs are mutually disjoint. From now on, we always assume that Whitney circles are disjoint without comment or loss of generality.

Remark 15.1 The last two techniques show how to complete the proof of Proposition 11.10. In that proposition, we obtained immersed Whitney discs using algebraic topological considerations, due to the vanishing of the appropriate intersection and self-intersection numbers. These discs come from null homotopies and thus they may not *a priori* be framed. The boundary twisting operation allows us to ensure that the Whitney discs are framed, at the expense of adding new points of intersection between the Whitney discs and the original surfaces. Since such intersections are often already present, or at least cannot be assumed not to be present, this does not hurt us in practice. The boundary of the Whitney discs may not be embedded or mutually disjoint to begin with, but this can be ensured by the procedure of this section and we will occasionally assume this without comment.

15.2.4 Pushing Down Intersections

The technique of pushing down intersection points was introduced in [FQ90, Section 2.5]. Suppose we have two immersed surfaces A and B such that part of the boundary of B is embedded in A, as shown on the left of Figure 15.6 (note this is the same situation as in Figure 15.4). Then any intersection between B and some third surface C can be removed at the expense of adding two new intersections between C and A. This is shown on the right of Figure 15.6. For us, most often A will be part of a surface stage in a capped grope or tower, and B will be part of either a cap stage or a surface stage. We can then iteratively push down intersections with B to any surface stage including or below A. One advantage of doing this is that the new intersections appear in algebraically cancelling pairs and thus have associated Whitney discs. Alternatively, often a lower stage of a grope will have a geometrically transverse sphere, and so we can push down the intersection points and then tube the many new intersection points into the geometrically transverse sphere.

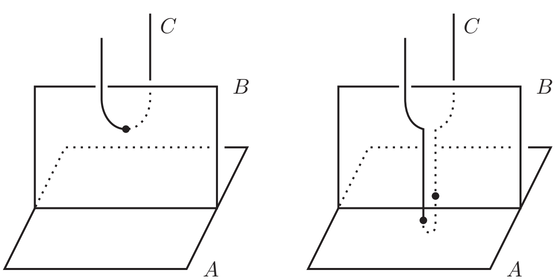

Figure 15.6 Pushing down an intersection point between C and B (left) leads to two new intersection points between C and A (right). Note that the two new intersection points are evidently paired by an embedded, framed Whitney disc. Only a single time slice is pictured. Note that we are performing a finger move, albeit one that pushes across the boundary and therefore does not preserve intersection numbers.

15.2.5 *Contraction and Subsequent Pushing Off*

Contraction and push-off was introduced in [FQ90, Section 2.3]. The (symmetric) contraction of a capped surface, depicted in Figure 15.7, converts a capped surface into an immersed disc. As shown by the figure, we start with a symplectic basis of curves on the surface and surger the surface using two copies each of framed immersed discs bounded by these curves joined by a square at the point of intersection of the curves. One could, alternatively, contract a capped surface by only surgering along one disc per dual pair, but this would not enable the pushing off procedure that we are about to describe in the next paragraph. Henceforth, whenever we talk about contraction, by default we will mean the symmetric contraction. Observe that the result of contracting a capped surface with embedded body has algebraically cancelling self-intersections.

After contracting a capped surface Σ^c with body Σ, any other surface A that intersected the caps of Σ^c can be pushed off the contracted surface, as we describe in Figure 15.8. The fact that we can perform the pushing off procedure, which is a regular homotopy, shows that the intersection number of the contracted surface with A is trivial. The push-off procedure reduces the number of intersection points between the contracted surface (an immersed disc) and the pushed off surfaces, so we gain some disjointness at the expense of converting a capped surface into an immersed disc. An additional cost is as follows. Suppose that a surface A intersects a cap of the capped surface, and a surface B intersects a dual cap. Then after pushing both A and B off the contraction, we obtain two intersection points between A and B. The contraction push-off operation is shown, via before and after pictures, in Figure 15.8.

A useful observation is that the homotopy class of the surface resulting from a contraction of a capped surface is independent of the choice of caps.

Lemma 15.2 *The homotopy class of the sphere or disc resulting from symmetric contraction of a fixed capped surface is independent of the choice of caps, provided the boundaries of the different choices of caps coincide.*

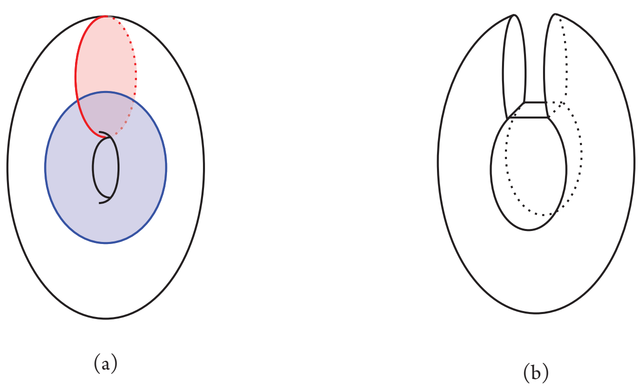

Figure 15.7 (Symmetric) contraction of a capped surface. Here we show the situation for embedded caps.

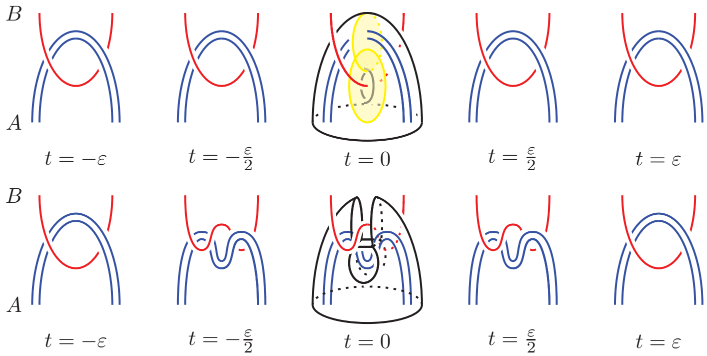

Figure 15.8 Top: Before contraction of a surface. Bottom: After contraction, with other surfaces pushed off the result of contraction. The capped surface being contracted is shown in the middle time slice. The surfaces A and B being pushed off the contraction are shown in blue and red, respectively. Note that the intersections of the pushed-off surfaces occur between diagrams one and two and between diagrams four and five in the bottom row of figures, namely one intersection in the past and one intersection in the future between each pair of surfaces that were pushed off dual caps.

Proof As explained in [FQ90, Section 2.3], an isotopy in the model capped surface induces a homotopy of the immersed copies. In the model, the symmetric contraction along dual caps $\{C, D\}$ is isotopic to the result of surgery along either cap; for example, C. This can be seen directly by isotoping across the region lying between the parallel copies of D used in the symmetric contraction (see Figure 15.7). Therefore, once the model is immersed in a 4-manifold, the symmetric contraction is homotopic to the result of surgery along one cap per dual pair.

Now let $\{C_i, D_i\}_{i=1}^{g}$ and $\{C_i', D_i'\}_{i=1}^{g}$ be two sets of caps for a surface of genus g such that $\partial C_i = \partial C_i'$ and $\partial D_i = \partial D_i'$ form a dual pair of curves on the surface for each i. Then the result of contraction along $\{C_i, D_i\}_{i=1}^{g}$ is homotopic to the asymmetric contraction along $\{C_i\}$, which is homotopic to contraction along $\{C_i, D_i'\}_{i=1}^{g}$. This is, in turn, homotopic to the result of asymmetric contraction along $\{D_i'\}$, which finally is homotopic to the result of contraction on $\{C_i', D_i'\}_{i=1}^{g}$, as asserted. □

15.3 Replacing Algebraic Duals with Geometric Duals

We finish the chapter with two applications of the techniques introduced so far. The first lemma will be used repeatedly in the upcoming constructions. It allows us to improve algebraic duals into geometric duals, at the cost of introducing new self-intersections, and first appeared in this form in [Fre82a, Lemma 3.1]. Compare with the techniques described in Chapter 1 and see also [FQ90, Section 1.5].

Lemma 15.3 (Geometric Casson lemma) *Let M be a smooth 4-manifold. Let $\{f_i\}$ and $\{g_i\}$ be immersed finite collections of discs or spheres in M, transversely intersecting in their interiors in double points, with $\lambda(f_i, g_j) = \delta_{ij}$ for all i, j. Then there exist families $\{f_i'\}$ and $\{g_i'\}$ of immersed discs or spheres in M, again transversely intersecting only in their interiors in double points, such that:*

(i) for every i, f_i' is regularly homotopic to f_i;

(ii) for every i, g_i' is regularly homotopic to g_i;

(iii) the surfaces f_i' and g_i' intersect exactly once transversely;

(iv) the surfaces f_i' and g_j' are disjoint whenever $i \neq j$.

Similarly, given distinct families $\{f_i\}$ and $\{g_i\}$ of immersed discs or spheres such that $\lambda(f_i, g_j) = 0$ for all i, j, there exist pairwise disjoint families $\{f_i'\}$ and $\{g_i'\}$ such that, for each i, f_i' and g_i' are regularly homotopic to f_i and g_i, respectively.

Note that, since the lemma provides a regular homotopy, $\lambda(f_i, h) = \lambda(f_i', h)$, $\mu(f_i) = \mu(f_i')$, $\lambda(g_i, h) = \lambda(g_i', h)$, and $\mu(g_i) = \mu(g_i')$ for every i and for every immersed disc or sphere h in M.

Proof Since $\lambda(f_i, g_i) = 1$ for each i, we can pair up all but one of the points of intersection with framed, immersed Whitney discs with mutually disjoint boundaries (Proposition 11.10). Similarly, since $\lambda(f_i, g_j) = 0$ whenever $i \neq j$, all of the points of intersection between each f_i and each g_j are paired by Whitney discs. Note that each Whitney circle for the Whitney discs mentioned above has an arc lying in $\{f_i\}$ and an arc lying in $\{g_i\}$.

Push all the points of intersection between the Whitney discs and $\{f_i\}$ and $\{g_i\}$ off the Whitney discs. Do this so that the only new intersections are within each family. That is, push each intersection of a Whitney disc with an element of $\{f_i\}$ onto $\{f_i\}$ by pushing towards the arc of the Whitney circle lying in $\{f_i\}$. Do the same for the $\{g_i\}$, as shown in Figure 15.9.

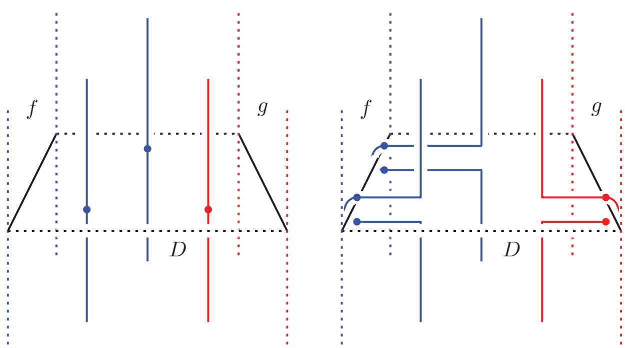

Figure 15.9 Proof of the geometric Casson lemma. Left: The Whitney disc D pairs two intersection points between the surfaces f (blue) and g (red). Only a small portion of the Whitney disc is shown. Right: Push off f intersections into f, and g intersections into g.

Now we have framed, immersed Whitney discs with interiors lying in the complement of $\{f_i\} \cup \{g_i\}$, and we use them to perform the Whitney move. For each Whitney move, push f_i over the Whitney disc. This introduces new intersections among the $\{f_i\}$ coming from the intersection among the Whitney discs, but this is the price we agreed to pay. Note that surfaces or curves disjoint from the Whitney discs may be assumed to be unaffected by the construction, by doing everything in a small enough neighbourhood of the Whitney discs. By Proposition 11.1, proved next, the immersed Whitney move is a regular homotopy.

A virtually identical proof gives the second statement. □

More generally, the above argument can be used to ensure that the algebraic and geometric intersection numbers between finite collections $\{f_i\}$ and $\{g_i\}$ agree, at the expense of

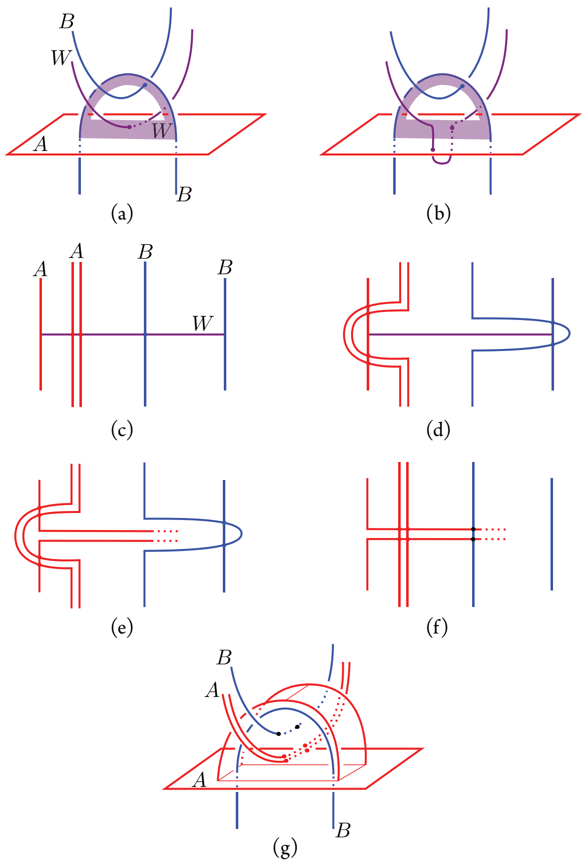

Figure 15.10 (a) A framed, immersed Whitney disc W (purple) pairing intersections between surfaces A (red) and B (blue). The disc W is not embedded, but an embedded collar of its boundary is shown. (b)–(f) A sequence of isotopies, finger moves, and Whitney moves along framed, embedded Whitney discs with interiors disjoint from $A \cup B$. (g) The result of the immersed Whitney move on (a).

increasing the number of geometric intersections within each family. On the other hand, we certainly may *not* use this argument to realize a self-intersection number $\mu(f) = 0$ geometrically.

Finally, we give a proof of Proposition 11.1, which we now recall.

Proposition 11.1 *A (framed) immersed Whitney move is a regular homotopy.*

Proof By the definition of regular homotopy, it suffices to show that a framed immersed Whitney move is a sequence of isotopies, finger moves, and Whitney moves along framed, embedded discs with interiors disjoint from the surfaces being homotoped. This is shown in Figure 15.10. First push down self-intersections of W to A (or B) along the collar of W to produce a framed, embedded Whitney disc, whose interior might still intersect $A \cup B$. Observe that this does not change A or B. Then, as in the proof of the geometric Casson lemma, push intersections of A with W towards A, and those of B with W towards B. Observe that these are finger moves on A and B, respectively. The result is a framed, embedded Whitney disc with interiors disjoint from $A \cup B$. Perform the Whitney move on A along this disc. Finally, reverse the effect of the previous finger moves on A and B to see that the result coincides with the result of the immersed Whitney move on the original Whitney disc and A and B. More precisely, this final reversal move consists of an isotopy on A and Whitney moves on B along framed, embedded Whitney discs with interiors disjoint from B. □