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The s-cobordism Theorem, the Sphere Embedding Theorem, and the Poincaré Conjecture

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We start this chapter by finishing the proof of the s-cobordism theorem begun in Chapter 1. This will allow us to prove the category losing version of the 4-dimensional Poincaré conjecture (Theorem 20.3), that every smooth homotopy 4-sphere is homeomorphic to S^4.

Then we will prove an alternative statement of the disc embedding theorem [FQ90, Theorem 5.1B]. This version differs from the usual disc embedding theorem in that the intersection conditions are on the immersed discs $\{f_i\}$ rather than on the dual spheres $\{g_i\}$, and that the embedded discs $\{\overline{f}_i\}$ obtained as a consequence of the theorem are regularly homotopic to the original immersed discs $\{f_i\}$. We show that the two versions are logically equivalent. We directly obtain the *sphere embedding theorem* from the second version. The sphere embedding theorem is the key ingredient needed to deduce the exactness of the surgery sequence, as we show in Chapter 22. For the applications to surgery, it will be important that we have geometrically transverse spheres in the outcome of the sphere embedding theorem.

20.1 The s-cobordism Theorem

In this section, we state and prove the s-cobordism theorem for smooth s-cobordisms between closed 4-manifolds. As indicated in Chapter 1, the disc embedding theorem is a key ingredient in the proof.

Theorem 20.1 (**s-cobordism theorem**) *Let N be a smooth, 5-dimensional h-cobordism between closed 4-manifolds M_0 and M_1 with vanishing Whitehead torsion $\tau(N, M_0)$. Further, suppose that $\pi_1(N)$ is a good group. Then N is homeomorphic to the product $M_0 \times [0,1]$. In particular, M_0 and M_1 are homeomorphic.*

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Proof The first part of the proof was already explained in Chapter 1. We give the argument again, with a few more details, for the convenience of the reader. First, since N is smooth, we may choose a Morse function $F\colon N \to [0,1]$ with $F^{-1}(i) = M_i$ for $i = 0, 1$. This gives rise to a handle decomposition of N relative to M_0, as defined in Chapter 13. As usual, by transversality, we may assume that the handles are attached in increasing order of index. Since N is connected, we may assume, by handle cancellation, that this handle decomposition has no 0- or 5-handles.

Next, we use the process of *handle trading* to trade 1- and 4-handles for 3- and 2-handles, respectively, as follows. Let $N_2 \subseteq N$ denote the union of $M_0 \times [0,1]$ and the 1- and 2-handles of N. Let M_2 denote the new boundary, so $\partial N_2 = -M_0 \sqcup M_2$.

Consider the chain of inclusion-induced maps $\pi_1(M_0) \to \pi_1(N_2) \to \pi_1(N)$. Since N is built from N_2 by attaching handles of index strictly greater than 2, the second map is an isomorphism. The composition is an isomorphism by hypothesis. Thus the first map is an isomorphism.

Fix a 1-handle h^1 in N_2, with core arc α. We claim that there is an arc $\beta \subseteq M_0$ such that $\gamma := \alpha \cup \beta$ is a null-homotopic loop in N_2. To see this, first choose any arc $\beta' \subseteq M_0$ with the same endpoints as α. Then there is some loop $\delta \subseteq M_0$ with the same image in $\pi_1(N_2)$ as $\alpha \cup \beta'$, since the inclusion-induced map $\pi_1(M_0) \to \pi_1(N_2)$ is surjective. Any band sum of β' and δ^{-1} is the desired β. By transversality, we assume that γ is disjoint from the attaching circles of all the 1- and 2-handles of N_2 and then we push γ to the boundary M_2.

By turning handles upside down, we see that the inclusion-induced map $\pi_1(M_2) \to \pi_1(N_2)$ is an isomorphism. Thus γ bounds an immersed disc in M_2, since it is null-homotopic in N_2. By finger moves in the direction of γ, we see that γ bounds an embedded disc in M_2. Thicken this disc to produce a cancelling 2-/3-handle pair. More precisely, insert a collar of $M_2 \times [0,1]$ into the handle decomposition and thicken by pushing the interior of the disc into this collar. The result is the addition of a single cancelling 2-/3-handle pair compatible with the old handle decomposition. By the choice of γ the 2-handle cancels the 1-handle h^1, leaving the 3-handle behind. Iterating this process allows us to trade all the 1-handles in N for 3-handles, and the same argument for the dual handlebody of N built on M_1 trades all 4-handles for 2-handles.

At this point, we have produced a handle decomposition of N, relative to M_0, consisting only of 2- and 3-handles, attached in that order. Let $N_{1/2}$ denote the 5-manifold consisting of M_0 and the 2-handles, and let $M_{1/2}$ denote the 4-manifold obtained as a result. That is, $\partial N_{1/2} = -M_0 \sqcup M_{1/2}$. Then the inclusion-induced map $\pi_1(M_{1/2}) \to \pi_1(N_{1/2})$ is an isomorphism, since $N_{1/2}$ is produced from $M_{1/2}$ by attaching only 3-handles. We also know that the inclusion-induced map $\pi_1(N_{1/2}) \to \pi_1(N)$ is an isomorphism, since N is produced from $N_{1/2}$ by adding only 3-handles. Thus, the inclusion-induced map $\pi_1(M_{1/2}) \to \pi_1(N)$ is an isomorphism and can be used to identify the two groups.

We obtain a chain complex corresponding to the handle decomposition of N constructed above of the form

$$0 \to C_3(\widetilde{N}, \widetilde{M_0}) \xrightarrow{\partial_3} C_2(\widetilde{N}, \widetilde{M_0}) \to 0,$$

where $\widetilde{N}$ is the universal cover of N and $\widetilde{M_0}$ is the universal cover of M_0. Each $C_i(\widetilde{N}, \widetilde{M_0})$ is a finitely generated, free $\mathbb{Z}[\pi_1(N)]$-module with basis elements corresponding to a choice of lift of the i-handles of N. The boundary map records the intersections, with $\mathbb{Z}[\pi_1(N)] = \mathbb{Z}[\pi_1(M_{1/2})]$ coefficients, between the belt spheres $\{S_1, \ldots, S_k\}$ of the 2-handles, corresponding to $\{0\} \times S^2 \subseteq D^2 \times D^3$, and the attaching spheres $\{T_1, \ldots, T_k\}$ of the 3-handles, corresponding to $S^2 \times \{0\} \subseteq D^3 \times D^2$. Each of the sets $\{S_i\}$ and $\{T_i\}$ is a collection of pairwise disjoint, framed, embedded 2-spheres in $M_{1/2}$.

The vanishing of the Whitehead torsion $\tau(N, M_0)$ implies that after possible stabilization (by adding cancelling pairs of 2- and 3-handles) and handle slides (corresponding to basis changes), the boundary map ∂_3 is represented by the identity matrix [Lüc02, Chapter 2]. In other words, we may assume that $\lambda(S_i, T_j) = \delta_{ij}$, measured in $\mathbb{Z}[\pi_1(M_{1/2})]$, for all i, j. As in the introduction, we wish to perform an isotopy of the family $\{T_i\}$ such that these intersection numbers are realized geometrically, so that we can cancel the 2-handles with the 3-handles.

Since the inclusion-induced map from $\pi_1(M_0)$ to $\pi_1(N)$ is an isomorphism, the 2-handles in N are attached to M_0 along homotopically trivial circles in M_0. These null homotopies, glued to the cores of the attached 2-handles (pushed to the boundary), produce a collection $\{S_i^\#\}$ of possibly unframed, immersed spheres in $M_{1/2}$, such that the collections $\{S_i\}$ and $\{S_i^\#\}$ are geometrically transverse. The identical argument applied to the dual handlebody obtained by turning the handles upside down produces the family $\{T_i^\#\}$ of possibly unframed, immersed spheres in $M_{1/2}$ geometrically transverse to $\{T_i\}$.

Note that we do not have any control over the intersections between the families $\{S_i^\#\}$ and $\{T_i\}$ or between the families $\{S_i\}$ and $\{T_i^\#\}$. There are also uncontrolled intersections within and between the families $\{S_i^\#\}$ and $\{T_i^\#\}$.

We will now arrange for framed, geometrically transverse spheres for $\{S_i\}$ that are disjoint from $\{T_i\}$, and for framed, geometrically transverse spheres for $\{T_i\}$ that are disjoint from $\{S_i\}$. This will deviate slightly from the proof sketched in Chapter 1, since we desire framed dual spheres rather than the unframed ones we produced there.

Let $\{S_i'\}$ and $\{T_i'\}$ denote parallel copies of $\{S_i\}$ and $\{T_i\}$, respectively. Then, by hypothesis, the collection $\{S_i\} \cup \{S_i'\}$ is pairwise disjoint, as is the collection $\{T_i\} \cup \{T_i'\}$ (recall that $\{S_i\}$ and $\{T_i\}$ are collections of mutually disjoint, framed embedded spheres). Additionally, $\lambda(S_i', T_j) = \lambda(S_i, T_j') = \delta_{ij}$ for all i, j. Thus all but one intersection point between S_i' and T_i, as well as between S_i and T_i', can be paired by immersed Whitney discs, for each i (Proposition 11.10). In addition, all the intersection points between S_i' and T_j, as well as between S_i and T_j', can be paired by immersed Whitney discs, for all $i \neq j$. Let $\{W_\ell^S\}$ and $\{W_\ell^T\}$ be the collections of these Whitney discs for the extraneous intersections between $\{S_i\}$ and $\{T_j'\}$, and between $\{S_i'\}$ and $\{T_j\}$, respectively.

We make the interiors of the $\{W_\ell^T\}$ disjoint from $\{T_i\}$ by tubing into the unframed, geometrically transverse spheres $\{T_i^\#\}$. This creates new intersections of the Whitney discs with $\{S_i\}$ but not with $\{T_i\}$, since $\{T_i^\#\}$ and $\{T_i\}$ are geometrically transverse. Then remove all intersections of the interiors with $\{S_i\}$ by (disjoint) finger moves in the direction of $\{T_i\}$. See Figure 20.1. Boundary twist, at the expense of new intersections with $\{S_i'\}$, to

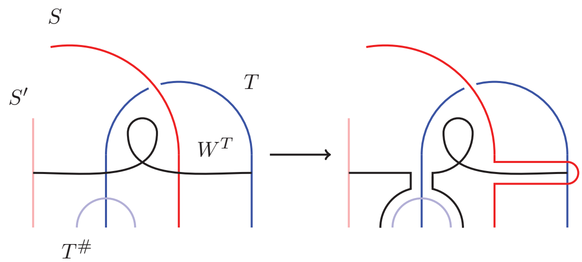

Figure 20.1 Obtaining a Whitney disc for intersections between $\{S_i'\}$ and $\{T_i\}$ with interior in the complement of $\bigcup\{S_i\} \cup \bigcup\{T_i\}$. We see a Whitney disc W^T (black) pairing intersection points between S' (light red) and T (blue). Remove intersections of W^T and T by tubing into the unframed, geometric dual $T^{\#}$ (light blue). Intersections of W^T and S, including any new ones created in the previous step, can be removed by a finger move in the direction of T.

correct the framing of the Whitney discs. We still call the resulting collection of Whitney discs $\{W_\ell^T\}$, but note that they are framed and their interiors lie in the complement of $\bigcup\{S_i\} \cup \bigcup\{T_i\}$. We do not control the intersections within the collection $\{W_\ell^T\}$. We have created some new algebraically cancelling intersections between $\{S_i\}$ and $\{T_i\}$ via the finger move. Instead of finger moving the $\{S_i\}$ over $\{T_i\}$, we could have finger moved T_i over S_i, with isotopic results. So we shift perspective and consider these finger moves to be an isotopy of $\{T_i\}$, instead.

A similar process makes the interiors of the $\{W_\ell^S\}$ disjoint from $\bigcup\{S_i\} \cup \bigcup\{T_i\}$. That is, remove intersections of the discs with $\{S_i\}$ by tubing into the unframed, transverse spheres $\{S_i^{\#}\}$, next remove all intersections with $\{T_i\}$ by disjoint finger moves in the direction of $\{S_i\}$, and then boundary twist at the expense of new intersections with $\{T_i'\}$, to frame the Whitney discs. We still call the resulting collection of framed, immersed Whitney discs $\{W_\ell^S\}$.

Perform the Whitney move on $\{S_i'\}$ along the Whitney discs $\{W_\ell^T\}$, and call the resulting spheres $\{\widehat{T}_i\}$. Note that the collections $\{T_i\}$ and $\{\widehat{T}_i\}$ are geometrically transverse, and, moreover, $S_i \cap \widehat{T}_j = \emptyset$ for all i, j. Then perform the Whitney move on $\{T_i'\}$ along the Whitney discs $\{W_\ell^S\}$ and call the resulting spheres $\{\widehat{S}_i\}$. As desired, the collections $\{S_i\}$ and $\{\widehat{S}_i\}$ are geometrically transverse, and $\widehat{S}_i \cap T_j = \emptyset$ for all i, j. Thus the collections $\bigcup\{S_i\} \cup \bigcup\{T_i\}$ and $\bigcup\{\widehat{S}_i\} \cup \bigcup\{\widehat{T}_i\}$ are geometrically transverse, and as a result the collection $\bigcup\{S_i\} \cup \bigcup\{T_i\}$ is π_1-negligible in $M_{1/2}$. We also remark that in the process of constructing $\bigcup\{\widehat{S}_i\} \cup \bigcup\{\widehat{T}_i\}$, we have moved the $\{T_i\}$ by an isotopy, the collection $\{S_i\}$ is unaffected, and we have created some new algebraically cancelling intersection points between the collections $\{S_i\}$ and $\{T_i\}$.

Now we return to our original problem, which is to perform a further isotopy of the $\{T_i\}$ so that the collections $\{S_i\}$ and $\{T_i\}$ become geometrically transverse. We have that $\lambda(S_i, T_j) = \delta_{ij}$ for all i, j. Thus all the unwanted double points between $\{S_i\}$ and $\{T_i\}$ can be paired up by framed, immersed Whitney discs $\{W_m\}$ in $M_{1/2}$ (Proposition 11.10). For each intersection of the interiors of $\{W_m\}$ with $\{S_i\}$, tube $\{W_m\}$ into the geometrically transverse spheres $\{\widehat{S}_i\}$. Similarly, for every intersection of the interiors of $\{W_m\}$ with $\{T_i\}$, tube $\{W_m\}$ into the geometrically transverse spheres $\{\widehat{T}_i\}$. Since $\{\widehat{S}_i\}$ and $\{\widehat{T}_i\}$ are framed, the collection $\{W_m\}$ remains framed and we have now arranged that the interior of $\{W_m\}$ lies in the complement of $\bigcup\{S_i\} \cup \bigcup\{T_i\}$.

Our goal is to apply the disc embedding theorem to the collection $\{W_m\}$ in this complement. With this in mind, we need to construct algebraically transverse spheres for $\{W_m\}$.

The desired spheres will arise from Clifford tori. Let Σ_m be the Clifford torus at one of the two double points paired by some W_m. For each m, the Clifford torus Σ_m intersects W_m exactly once, and the collection of such Clifford tori is embedded and pairwise disjoint. Cap each Σ_m with the meridional discs to $\{S_i\}$ and $\{T_i\}$ described in Figure 20.2. Each cap has a unique intersection with $\bigcup\{S_i\} \cup \bigcup\{T_i\}$, and none of them intersects $\{W_m\}$. Tube these intersections into parallel copies of the relevant members of the set of geometrically transverse spheres $\{\widehat{S}_i\} \cup \{\widehat{T}_i\}$. Contract these capped surfaces to produce algebraically transverse spheres $\{R_m\}$ for the discs $\{W_m\}$. Since the collection $\{R_m\}$ is produced by contraction of capped surfaces with mutually disjoint bodies, each element is framed and we see that $\lambda(R_m, R_{m'}) = 0 = \mu(R_m)$ for all m, m'.

The Whitney discs $\{W_m\}$, along with the collection of spheres $\{R_m\}$, now satisfy the hypotheses of the disc embedding theorem in the 4-manifold

$$M' := M_{1/2} \smallsetminus \left(\bigcup \nu S_i \cup \bigcup \nu T_i\right).$$

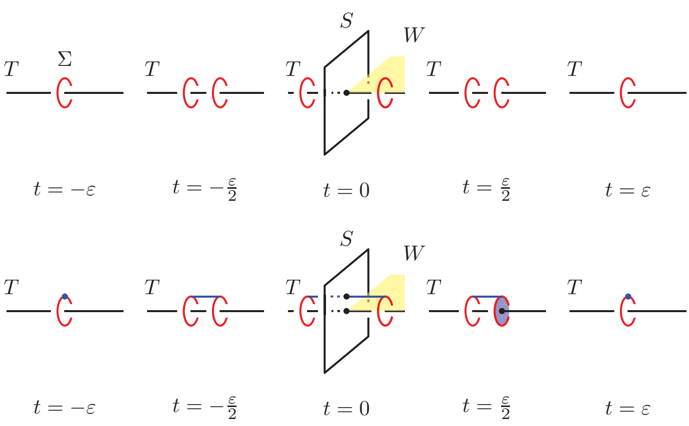

Figure 20.2 Obtaining a transverse sphere from a Clifford torus. Top: The Clifford torus Σ (red) at one of the two intersection points paired up by the Whitney disc W (yellow). The single point of intersection between T and W is shown in the central panel. Bottom: The two meridional discs are shown in blue. We see that each meridional disc intersects exactly one of $\{S, T\}$, exactly once.

Since $\{S_i\} \cup \{T_i\}$ is π_1-negligible in $M_{1/2}$, we see that $\pi_1(M') \cong \pi_1(M_{1/2}) \cong \pi_1(N)$, which is a good group by hypothesis. Additionally, the intersection numbers of $\{R_m\}$ vanish in M', since they vanish in $M_{1/2} \supseteq M'$, and the inclusion map induces a π_1-isomorphism.

The disc embedding theorem replaces the Whitney discs $\{W_m\}$ by embedded discs with the same framed boundaries. (We also obtain geometrically transverse spheres for these embedded discs, but we will not need them here.) Perform Whitney moves on the $\{T_i\}$ using the framed, embedded Whitney discs to remove all the unwanted intersections. This is the desired isotopy of $\{T_i\}$, after which the collections $\{S_i\}$ and $\{T_i\}$ become geometrically transverse. Now the 2-handles and the 3-handles of the 5-manifold N can be cancelled in pairs. Since there are no remaining handles, N is homeomorphic to the product $M_0 \times [0,1]$, as desired. □

The h-cobordism theorem is an immediate corollary of the s-cobordism theorem, since the Whitehead torsion of a simply connected cobordism lies in the Whitehead group of the trivial group, which is trivial.

Theorem 20.2 (h-cobordism theorem) *Every smooth h-cobordism between simply connected, closed 4-manifolds M_0 and M_1 is homeomorphic to the product $M_0 \times [0,1]$.*

20.2 The Poincaré Conjecture

Possibly the most famous application of the disc embedding theorem is the 4-dimensional Poincaré conjecture.

Theorem 20.3 (Poincaré conjecture, category losing version) *Every closed, smooth 4-manifold homotopy equivalent to the 4-sphere S^4 is homeomorphic to S^4.*

The proof we give below uses 5-dimensional surgery and the h-cobordism theorem (Theorem 20.2). We discuss the category preserving Poincaré conjecture, that every topological homotopy 4-sphere is homeomorphic to S^4, in Section 21.6.2. As explained there, the known proofs require ingredients not proved in this book, such as the category preserving h-cobordism theorem.

Proof Let Σ be a closed, smooth 4-manifold homotopy equivalent to the 4-sphere S^4. The signature of Σ vanishes, since $H_2(\Sigma;\mathbb{Z}) = 0$. We claim that the tangent bundle of Σ is stably trivial. The obstructions to stably trivializing the tangent bundle of a smooth, oriented 4-manifold are the second Stiefel–Whitney class $w_2(T\Sigma)$ and the first Pontryagin class $p_1(T\Sigma)$. Since the cohomology of Σ is concentrated in degree four, $w_2(T\Sigma) = 0$, while $p_1(T\Sigma)$ vanishes because the signature vanishes, by the Hirzebruch signature formula $3\sigma(\Sigma) = \langle p_1(T\Sigma), [\Sigma]\rangle \in \mathbb{Z}$. Thus the tangent bundle is stably trivial, as claimed. It follows that Σ bounds a compact 5-manifold W with stably trivial tangent bundle, since the smooth, framed 4-dimensional cobordism group Ω_4^{fr} is trivial.

Construct a degree one normal map relative boundary $(f, \partial f)\colon (W, \Sigma) \to (D^5, S^4)$ by collapsing to a point the complement of an open tubular neighbourhood in W of some point

$x \in \Sigma$. More precisely, the map ∂f is a homotopy equivalence and f sends fundamental class to fundamental class. We use the fact that W is stably framed to construct the normal data required for the normal map. For more details on normal maps, see Section 22.1.4.

Perform 5-dimensional surgery on $(f, \partial f)$ to make f into a homotopy equivalence. This is possible, since the odd-dimensional surgery obstruction group $L_5(\mathbb{Z})$ of the trivial group is itself the trivial group. Thus Σ bounds a smooth, contractible 5-manifold W'. In this step we have used the main result of odd-dimensional surgery theory [Wal99, Theorem 6.4]; see also Section 22.1.6 below for the definition of the group $L_5(\mathbb{Z}) \cong L_5^s(\mathbb{Z})$. The proof so far shows that any smooth homology 4-sphere bounds a smooth, contractible 5-manifold.

Remove an open ball from the interior of W'. This produces a smooth h-cobordism from Σ to S^4. By the h-cobordism theorem, W' is homeomorphic to the product $S^4 \times [0,1]$. Consequently, Σ is homeomorphic to S^4. □

In the previous proof, the h-cobordism between Σ and S^4 could have been obtained using Wall's theorem [Wal64] showing that any two closed, smooth, simply connected 4-manifolds with isomorphic intersection forms are smoothly h-cobordant. The proof given here is more transparent and has the advantage that it applies, with a few modifications, to topological homotopy 4-spheres, as we describe in Section 21.6.2.

20.3 The Sphere Embedding Theorem

We state and prove an alternative version of the disc embedding theorem, where the intersection assumptions are on the initial immersed discs rather than the dual spheres.

Theorem 20.4 ([FQ90, Theorem 5.1B]) *Let M be a smooth, connected 4-manifold with nonempty boundary and such that $\pi_1(M)$ is a good group. Let*

$$F = (f_1, \ldots, f_n) \colon (D^2 \sqcup \cdots \sqcup D^2, S^1 \sqcup \cdots \sqcup S^1) \looparrowright (M, \partial M)$$

be an immersed collection of discs in M with pairwise disjoint boundaries satisfying $\mu(f_i) = 0$ for all i and $\lambda(f_i, f_j) = 0$ for all $i \neq j$. Suppose, moreover, that there is an immersed collection

$$G = (g_1 \ldots, g_n) \colon S^2 \sqcup \cdots \sqcup S^2 \looparrowright M$$

of framed, algebraically dual spheres; that is, $\lambda(f_i, g_j) = \delta_{ij}$ for all $i, j = 1, \ldots, k$.

Then there exist mutually disjoint flat embeddings

$$\overline{F} = (\overline{f}_1, \ldots, \overline{f}_n) \colon (D^2 \sqcup \cdots \sqcup D^2, S^1 \sqcup \cdots \sqcup S^1) \hookrightarrow (M, \partial M)$$

with $\overline{f}_i$ regularly homotopic, relative boundary, to f_i for each i, together with an immersed collection of framed, geometrically dual spheres

$$\overline{G} = (\overline{g}_1 \ldots, \overline{g}_n) \colon S^2 \sqcup \cdots \sqcup S^2 \looparrowright M$$

such that, for each i, $\overline{g}_i$ is homotopic to g_i.

Proof Since $\lambda(f_i, g_j) = \delta_{ij}$, we may apply the geometric Casson lemma (Lemma 15.3) to arrange that $\{f_i\}$ and $\{g_i\}$ are geometrically transverse (note that the collection of transverse spheres $\{g_i\}$ may have any kind of intersections among themselves). This changes the collections by regular homotopy, and we continue to use the same notation. Since $\lambda(f_i, f_j) = 0$ for all $i \neq j$ and $\mu(f_i) = 0$ for all i, the intersections and self-intersections within $\{f_i\}$ are paired by framed, immersed Whitney discs (Proposition 11.10).

Consider one such Whitney disc D pairing up intersections between f_i and f_j, where possibly $i = j$. Such a disc may intersect itself, the collections $\{f_i\}$ and $\{g_i\}$, or other Whitney discs (see the left panel of Figure 20.3). For each intersection of D with f_ℓ, for some ℓ, tube D into a parallel push-off of the geometric dual g_ℓ, as shown in the right panel of Figure 20.3. This introduces potentially many new intersections, between D and anything that intersected g_ℓ (including g_ℓ itself), as well as new self-intersections of D coming from the self-intersections of g_ℓ. However, the interior of D no longer intersects any f_i, since g_ℓ intersects exactly one of the $\{f_i\}$, namely f_ℓ, at the intersection point we used for tubing. Do this for all the Whitney discs and their intersections with $\{f_i\}$. Now our Whitney discs are more complicated, but their interiors lie in the complement of $\bigcup\{f_i\}$. Call this collection of Whitney discs $\{D'_k\}$. These Whitney discs are framed, so if they were embedded we could perform the Whitney move along them to obtain the embedded discs we seek.

We wish to apply the disc embedding theorem to $N := M \smallsetminus \bigcup \nu f_i$. Since each f_i has a geometrically transverse sphere by construction, the collection $\{f_i\}$ is π_1-negligible, and so there is an isomorphism $\pi_1(N) \to \pi_1(M)$. Since $\pi_1(M)$ is good, we conclude that $\pi_1(N)$ is also good, as desired.

Next, we find algebraically transverse spheres for the Whitney discs $\{D'_k\}$. As before, these will arise from Clifford tori. Let Σ_k be the Clifford torus at one of the two double points paired by some D'_k. As we saw earlier, the Clifford torus Σ_k intersects D'_k exactly once. Each torus is framed and embedded, and the collection of such Clifford tori is pairwise disjoint. Cap each Σ_k with meridional discs to $\{f_i\}$ (see Figure 20.2). Each cap has a unique intersection with $\{f_i\}$, and none intersects $\{D'_k\}$. Tube these intersections between the

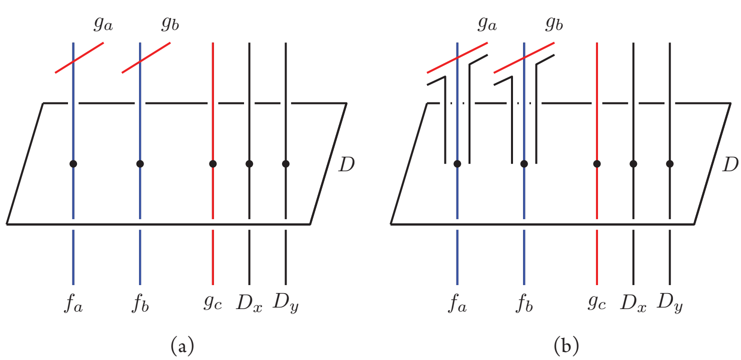

Figure 20.3 Left: A schematic picture of a piece of a Whitney disc D. It may intersect $\{f_i\}$, $\{g_i\}$, or other Whitney discs. Recall that $\{f_i\}$ and $\{g_i\}$ are geometrically transverse. Right: Remove intersections of D with $\{f_i\}$ by tubing into $\{g_i\}$.

caps and $\{f_i\}$ into the set of geometrically transverse spheres $\{g_i\}$ and contract the resulting capped surfaces in the complement of $\{f_i\}$ to produce algebraically transverse spheres $\{g'_k\}$ for the discs $\{D'_k\}$ lying in N. Since the collection $\{g'_k\}$ is produced by contraction of capped surfaces with disjoint bodies, each g'_k is framed and $\lambda(g'_k, g'_\ell) = 0 = \mu(g'_k)$ in N for all k, ℓ. Moreover, note that by Lemma 17.11, we have $[g'_k] = 0 \in \pi_2(M)$ for each k.

We may, therefore, apply the disc embedding theorem to replace the immersed Whitney discs $\{D'_k\}$ with topologically embedded Whitney discs $\{W_k\}$ with normal bundles that induce the right framing on the boundaries, and framed, geometrically transverse spheres $\{R_k\}$ in N, with R_k homotopic to g'_k for each k. For each intersection of some g_i with some W_k, tube that g_i into the geometrically transverse sphere R_k. This transforms the collection $\{g_i\}$ to a collection $\{\overline{g}_i\}$, the elements of which may have more intersections among themselves, but are still geometrically transverse to $\{f_i\}$. The $\{\overline{g}_i\}$ are still framed because the $\{R_k\}$ are. Since g'_k is null-homotopic in M for every j, so is R_k. It follows that $\overline{g}_i$ is homotopic to g_i for each i.

Moreover, we obtain embedded, flat, framed Whitney discs for the intersections among the $\{f_i\}$ in $M \setminus (\bigcup \nu f_i \cup \bigcup \nu \overline{g}_i)$ (see Figure 20.4). Perform the Whitney move on $\{f_i\}$ over the Whitney discs $\{W_k\}$ to obtain flat, embedded discs $\{\overline{f}_i\}$, regularly homotopic to the $\{f_i\}$ with the same framed boundary as the $\{f_i\}$ as well as framed, geometrically transverse spheres $\{\overline{g}_i\}$. □

Proposition 20.5 *Theorem 20.4 and the disc embedding theorem are equivalent.*

Proof Since we have already deduced Theorem 20.4 from the disc embedding theorem, it suffices to show the converse. Begin with immersed discs $\{f_i\}$ with algebraically transverse spheres $\{g_i\}$ with $\lambda(f_i, g_j) = \delta_{ij}$ and $\lambda(g_i, g_j) = \mu(g_i) = 0$ for all i, j. Tube each intersection and self-intersection within $\{f_i\}$ into $\{g_i\}$ using the unpaired intersection points between $\{f_i\}$ and $\{g_i\}$ (see Figure 16.2). This replaces $\{f_i\}$ with a collection of discs, which we still call $\{f_i\}$, with the same framed boundaries, and satisfying $\lambda(f_i, f_j) = \mu(f_i) = 0$ for all i, j. Moreover, we still have that $\lambda(f_i, g_j) = \delta_{ij}$. Apply Theorem 20.4 to achieve the conclusion of the disc embedding theorem. □

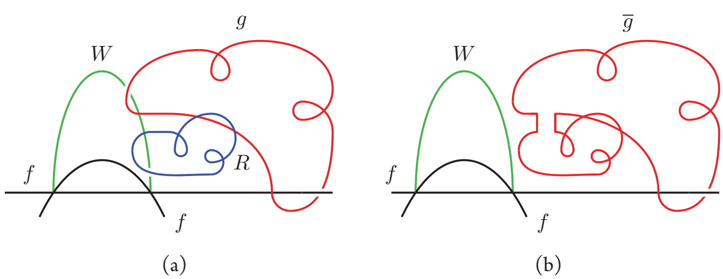

Figure 20.4 Left: An embedded Whitney disc (green) with a geometrically dual sphere (blue), both with interiors in the complement of $\{f_i\}$, has been produced.
Right: After tubing $\{g_i\}$ into the geometrically dual spheres, as needed, we have produced the spheres $\{\overline{g}_i\}$, which are geometrically dual to $\{f_i\}$.

We can now prove the sphere embedding theorem, stated next, which we will apply in Chapter 22 to prove the exactness of the surgery sequence. The key difference from Theorem 20.4 is that we embed spheres instead of discs.

Sphere embedding theorem *Let M be a smooth, connected 4-manifold such that $\pi_1(M)$ is good. Suppose there exists an immersed collection*

$$F = (f_1, \dots, f_n)\colon S^2 \sqcup \cdots \sqcup S^2 \looparrowright M$$

of spheres with $\lambda(f_i, f_j) = 0$ for every $i \neq j$ and $\mu(f_i) = 0$ for all i. Suppose, moreover, that there is an immersed collection

$$G = (g_1, \dots, g_n)\colon S^2 \sqcup \cdots \sqcup S^2 \looparrowright M$$

of framed, algebraically dual spheres; that is, $\lambda(f_i, g_j) = \delta_{ij}$ for all i, j.

Then there exists an embedding,

$$\overline{F} = (\overline{f}_1, \dots, \overline{f}_n)\colon S^2 \sqcup \cdots \sqcup S^2 \hookrightarrow M,$$

of a collection of spheres in M, with each $\overline{f}_i$ regularly homotopic to f_i, together with framed geometrically dual spheres,

$$G = (\overline{g}_1, \dots, \overline{g}_n)\colon S^2 \sqcup \cdots \sqcup S^2 \looparrowright M,$$

with $\overline{g}_i$ homotopic to g_i for each i.

The sphere embedding theorem is summarized in Figure 20.5. Note that the assumption $\mu(f_i) = 0$ implies that all the self-intersections of f_i can be paired up with Whitney discs, but it does not imply that f_i has trivial normal bundle. Since $\overline{f}_i$ is regularly homotopic to

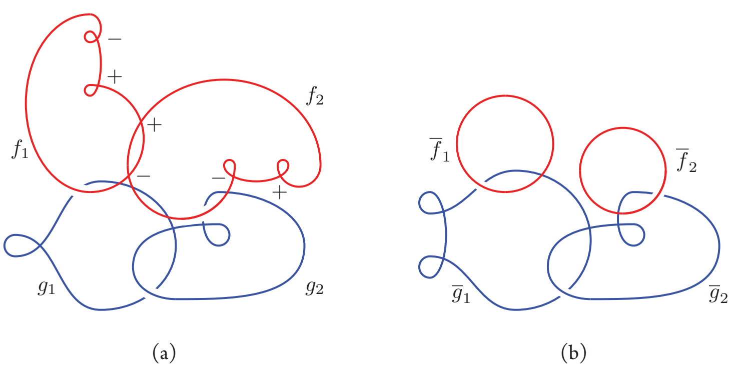

Figure 20.5 Summary of the sphere embedding theorem. We start with the situation in (a) and produce the situation in (b).

f_i, and f_i has a normal bundle, albeit not a trivial one, we deduce that so does $\overline{f}_i$. Moreover, the Euler numbers of the normal bundles of f_i and $\overline{f}_i$ coincide.

Proof For each i, find a point on f_i away from all intersections and self-intersections of $\{f_i\}$. Choose a small open ball around this point. Use embedded 1-handles in M disjoint from $\bigcup\{f_i\} \cup \bigcup\{g_i\}$ to connect these small balls into one large open ball B. Let $N := M \smallsetminus B$. Since $\pi_1(N) \cong \pi_1(M)$ and removing B does not change any intersection and self-intersection numbers, we may apply Theorem 20.4 to N and the discs $\{f_i \smallsetminus (f_i \cap B)\}$. This replaces the discs with regularly homotopic, disjointly embedded, flat discs equipped with an immersed collection of framed, geometrically transverse spheres in N. Gluing together B and N as well as $\{f_i \cap B\}$ and the embedded discs just constructed produces the desired embedded spheres $\{\overline{f}_i\}$ in M. □